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Synchronization of Chaos in a Food Web in Ecological Systems

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Abstract**—**The three-species food web model proposed and investigated by Gakkhar and Naji is known to have chaotic behaviour for a choice of parameters. An attempt has been made to synchronize the chaos in the model using bidirectional coupling. Numerical simulations are presented to demonstrate the effectiveness and feasibility of the analytical results. Numerical results show that for higher value of coupling strength, chaotic synchronization is achieved. Chaos can be controlled to achieve stable synchronization in natural systems.

Keywords—Lyapunov Exponent, Bidirectional Coupling, Chaos Synchronization, Synchronization Manifold

I. INTRODUCTION

SYNCHRONIZATION is a ubiquitous phenomenon characteristic of many processes in natural systems and nonlinear science. It is an area of intensive research and is today considered as one of the basic nonlinear phenomena studied in mathematics, physics, engineering or life sciences [1]. Synchronizing of two dynamical systems generally means that one system somehow traces the motion of another. It is well known that many coupled oscillators have the ability to adjust to common behavior due to weak interaction between them. This gives rise to a situation in which synchronization-like phenomenon takes place [2]. The idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [3]. According to them two chaotic systems could be synchronized by coupling them. In ecological aspect, asynchrony allows for the global persistence of a population through rescue effects, even there are local extinctions. Indeed in certain circumstances, the highly synchronizing effects of chaos have been shown to enhance the global persistence of model populations even though large population swings often (but not always) associated with chaotic dynamics increase the chance of local extinction.

In this paper, Gakkhar and Naji model system [4] has been taken for chaos synchronization. They obtained several complex behaviors including limit cycles and chaos for different ranges of parameters. The bi-directional coupling technique is used to synchronize two model systems [5]. In this paper same technique has been applied in an ecological context. The phenomenon of synchronization through this mechanism in two model systems is investigated. It is seen that the synchronizing effects sensitively depends on the values of the coupling strength *K*. Then numerical simulations are done to verify the effectiveness of present method.

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II. GAKKHAR & NAJI FOOD WEB MODEL SYSTEM

The dynamics of the three-species system consisting of two independent preys and one predator is governed by the following system of differential equations [4]:

$$\begin{aligned}
\frac{dX_1}{dT} &= r_1 X_1\left(1-\frac{X_1}{K_1}\right) - F_1(X_1,X_2)X_3,\\
\frac{dX_2}{dT} &= r_2 X_2\left(1-\frac{X_2}{K_2}\right) - F_2(X_1,X_2)X_3, \qquad (1)\\
\frac{dX_3}{dT} &= e_1F_1(X_1,X_2)X_3 + e_2F_2(X_1,X_2)X_3 - dX_3;\\
F_i\left(X_1,X_2\right) &= \frac{A_i\,X_i}{\left(1+B_1\,X_1+B_2\,X_2\right)};\ i=1,2.
\end{aligned}$$

The model parameters r_i, K_i, e_i and d assume only positive values. The state variables X_1 and X_2 are the densities of two prey species, while X_3 is the density of a common predator. The predator consumes the prey X_i according to the functional response $F_i\left(X_1,X_2\right)$, where A_i is the search rate of a predator for the prey X_i and $B_i = h_i A_i$, h_i being the expected handling time spent with the prey X_i. The constant $e_i, i=1,2$ is conversion rate of prey X_i to predator X_3. Clearly, for very small values of B_1 and B_2 the nonlinear functional response approaches to linear type functional response. However, it approaches to hyperbolic Holling type-II functional response when only one of B_i tends to zero. Since the amount of food consumed by each predator per unit of time depends on the available food sources (the two preys X_1 and X_2), a variation in Holling type-II functional response is assumed as given in (1).

The following non dimensional parameters are chosen to non-dimensionalize the system (1)

$$y_i = \frac{X_i}{K_i},\ i=1,2,\ y_3 = \frac{X_3}{K_1},\ t = r_1 T,\ w_2 = \frac{A_1 K_1}{r_1},$$

$$w_3 = B_1 K_1, w_4 = B_2 K_2,\ w_5 = \frac{r_2}{r_1}, w_7 = \frac{A_2 K_1}{r_1},$$

$$w_8 = e_1 w_2,\ w_9 = \left(\frac{e_2 K_2}{K_1}\right) w_7,\ w_{10} = \frac{d}{r_1}$$

The transformed system is

$$\begin{aligned}
\frac{dy_1}{dt} &= y_1(1-y_1) - \frac{w_2 y_1 y_3}{1+w_3 y_1 + w_4 y_2} \\
\frac{dy_2}{dt} &= w_5(1-y_2) - \frac{w_7 y_2 y_3}{1+w_3 y_1 + w_4 y_2} \\
\frac{dy_3}{dt} &= y_3\left[\frac{w_8 y_1 + w_9 y_2}{1+w_3 y_1 + w_4 y_2} - w_{10}\right]
\end{aligned} \tag{2}$$

Numerical simulations of the system (2) were carried by Gakkhar and Naji [4] and it was observed that the system has chaotic behavior for the following set of parameter values:

$$w_2 = 3.0,\ w_3 = 1.5,\ w_4 = 2.0,\ w_5 = 1.135,$$
$$w_7 = 3.5,\ w_8 = 1.35,\ w_9 = 1.925,\ w_{10} = 0.05$$

As it is well known, the Lyapunov exponents measure the mean rate of divergence or convergence of nearby trajectories on to another. By using computational method of Lyapunov exponents for the continuous system [6], Lyapunov exponents of the system (2) are calculated as

$$\lambda_1 = 0.05782,\ \lambda_2 = 0.0001,\ \lambda_3 = -0.83685$$

As one exponent is positive, one is negative and one is almost zero, Lyapunov dimension is fractal, it also guarantees the chaos in the system (3).

The Lyapunov dimension of chaotic attractor obtained in model system (2) is computed by the formula of Kaplan and Yorke [7] as

$$D_x^{(L)} = j + \frac{1}{|\lambda_{j+1}|}\sum_{i=1}^{j}\lambda_i = 2.0012$$

Thus, the dynamics is of fractal dimension. It means that the nearby trajectories soon diverge and follow totally different paths in the attractor. This also confirms chaos in the model system (2).

III. Bidirectional Coupling

Many biological and physical systems consist of bi-directionally interacting elements or components which act as a controlled feed backing. In bi-directional (mutual) coupling, both drive and response subsystems are connected in such a way that they mutually influence each other's behavior. For this, two copies of the system (2) are taken and two way coupling is added through a linear constant term $K > 0$ in first two equations in both copies as follows:

$$\begin{cases}
\dfrac{dy_1}{dt} = y_1(1-y_1) - \dfrac{w_2 y_1 y_3}{1+w_3 y_1 + w_4 y_2} + K(y_1' - y_1) \\
\dfrac{dy_2}{dt} = w_5(1-y_2) - \dfrac{w_7 y_2 y_3}{1+w_3 y_1 + w_4 y_2} + K(y_2' - y_2) \\
\dfrac{dy_3}{dt} = y_3\left[\dfrac{w_8 y_1 + w_9 y_2}{1+w_3 y_1 + w_4 y_2} - w_{10}\right]
\end{cases} \tag{3a}$$

$$\begin{cases}
\dfrac{dy_1'}{dt} = y_1'(1-y_1') - \dfrac{w_2 y_1' y_3'}{1+w_3 y_1' + w_4 y_2'} + K(y_1 - y_1') \\
\dfrac{dy_2'}{dt} = w_5(1-y_2') - \dfrac{w_7 y_2' y_3'}{1+w_3 y_1' + w_4 y_2'} + K(y_2 - y_2') \\
\dfrac{dy_3'}{dt} = y_3'\left[\dfrac{w_8 y_1' + w_9 y_2'}{1+w_3 y_1' + w_4 y_2'} - w_{10}\right]
\end{cases} \tag{3b}$$

For $K = 0$ the two subsystems are uncoupled and for $K > 0$ both subsystems are bi-directionally coupled.

Let Y and Y' denote the column vectors of state variables and Γ is the matrix of coupling coefficients. The two coupled systems (3a) and (3b) can be written as:

$$\begin{cases}
\dot{Y} = F + K\Gamma(Y' - Y) \\
\dot{Y}' = F' + K\Gamma(Y - Y')
\end{cases} \tag{4}$$

where $\Gamma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

The discrete version of above equation is

$$\dot{Y}_j = F_j + K\Gamma(Y_{j+1} - Y_j), \quad j = 1,2 \tag{5}$$

The Synchronous state reside on a synchronization manifold (dimension 3) defined by $M = \{Y_1 = Y_2 = s(t)\}$ where the chaotic solution $s(t)$ satisfies $\dot{s} = F(s)$ [8].

The variational equation corresponding to (5), is

$$\dot{\xi}_j = DF(s)\xi_j + K\Gamma(\xi_{j+1} - \xi_j) \quad (6)$$

The collection of variations is $\xi = \{\xi_1, \xi_2, \xi_3, \xi_4\ \xi_5, \xi_6\}$. and $DF(s)$ is the Jacobian of F on $s(t)$, see [8]-[9]. Equation (6) can be written in the form

$$\dot{\xi} = [\mathbf{1} \otimes DF + K\mathbf{G} \otimes \Gamma].\xi \quad (7)$$

Where $\mathbf{1}$ is an $N \times N$ unit matrix and $\mathbf{G}$ is given by

$$\mathbf{G} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

Lyapunov exponents are calculated from master stability function of (7) [10]. The computed exponents are called as transverse Lyapunov exponents (TLEs) w. r. to synchronization manifold M. The necessary condition for stability of synchronization manifold is that the largest TLE is negative.

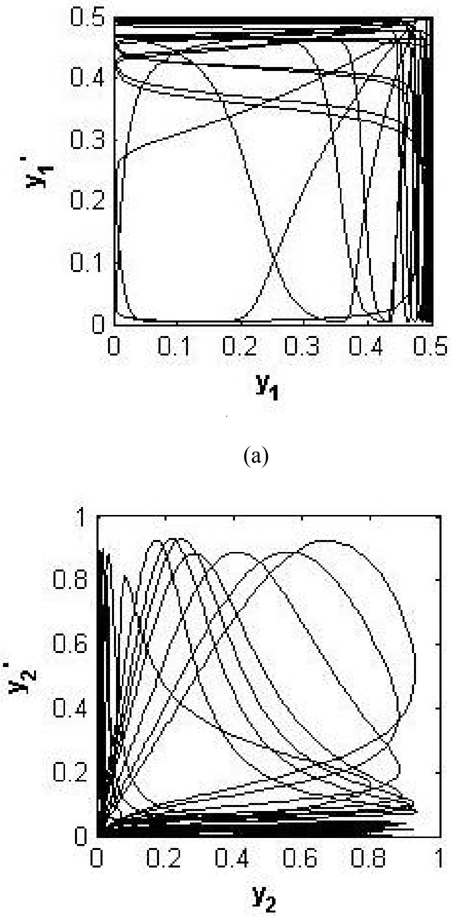
By using Equation (7), TLE are computed for same set of parameters, given in section 2 at $K = 0.05$ as follows:

$$\lambda_1 = -0.27341,\ \lambda_2 = -0.62559,\ \lambda_3 = -0.01385...$$

All transverse Lyapunov exponents are negative which ensures the stable synchronization manifold. It can be concluded that the systems are synchronized. Further, the bi-directional coupling may control the occurrence of chaos in the system (3).

IV. Numerical Simulations

In this section, simulation results have been shown for verifying the effectiveness the bi-directional coupling method. By using chosen parameter given in section 2, it has been demonstrated the model has chaotic behavior. For occurrence of synchronization phenomenon, the model system is explored for various values of critical parameter K. The system (3) is numerically integrated for various values of K. The solution of both systems y_1 versus y_1' and y_2 versus y_2' are plotted.



(a)

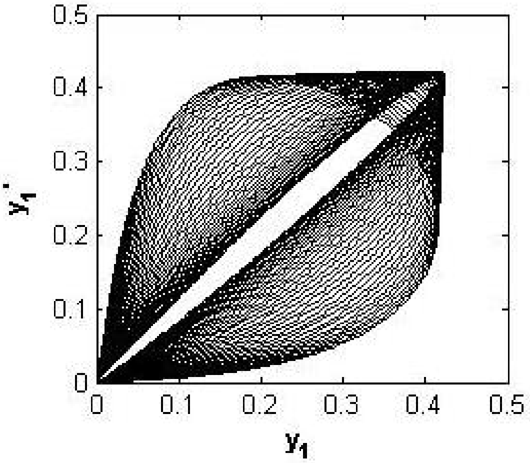
(b)

Fig. 1 Illustration of no synchronization when $K = 0.0004$

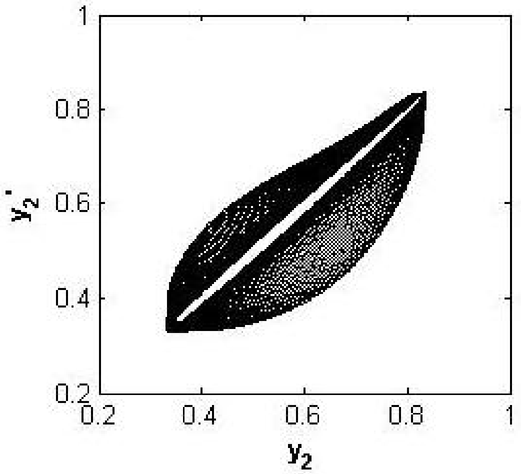
No synchronization is achieved for smaller value of K. For example, when $K = 0.0004$ synchronization of the system does not occur, as is evident from Fig. 1.

Both Systems are heading towards synchronization as K is increased. A typical result is shown for $K = 0.001$ in Fig. 2.

With further increase in K synchronization is improved. Fig. 3 displays the synchronization phenomenon in systems at $K = 0.05$. As the value of K is increased, the synchronization effect will be better.



(a)



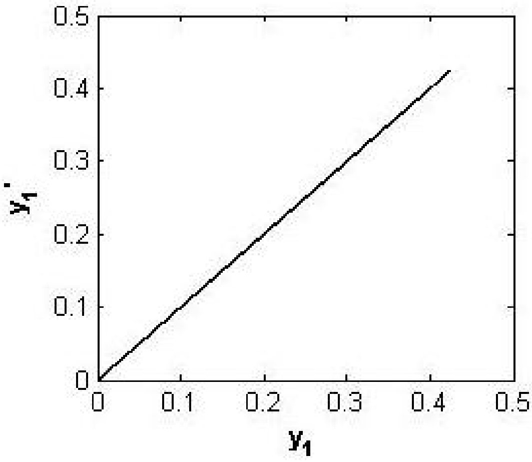
(b)

Fig. 2 Illustration of onset of synchronization of system for $K = 0.001$

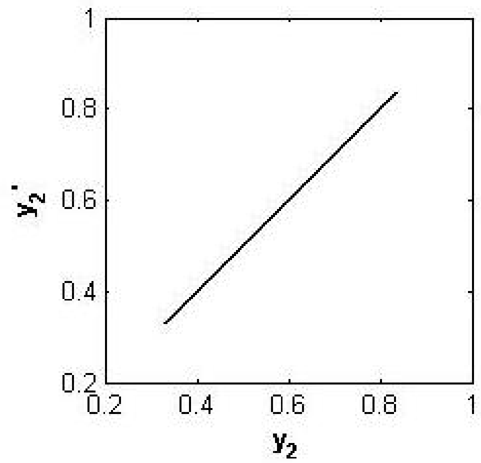
V. Conclusions

In this paper, chaos synchronization of a three species food web model system applying the bidirectional coupled method is discussed. From the numerical results, it is noticed that for sufficiently weak coupling K synchronous phenomenon does not occur. When the value of K is increased the two systems start to synchronize.

It is showed that when two different systems are coupled with sufficiently strong coupling strength, a general synchronous relation between their states could exist. As synchronization might play an important role in enhancing population persistence, the embarrassing situation caused by chaos can be changed and food web system can reach up to a dynamic balance.



(a)



(b)

. Fig. 3 Illustration of onset of synchronization of system for $K = 0.05$

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