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4-Transitivity and 6-Figures in Finite Klingenberg Planes of Parameters (p^{2k-1}, p)

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Abstract**—**In this paper, we carry over some of the results which are valid on a certain class of Moufang-Klingenberg planes $\mathbf{M}(\mathcal{A})$ coordinatized by an local alternative ring $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ of dual numbers to finite projective Klingenberg plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring $\mathbf{Z}_q$ (where prime power $q = p^k$) instead of $\mathbf{A}$. So, we show that the collineation group of $\mathbf{M}(\mathcal{A})$ acts transitively on 4-gons, and that any 6-figure corresponds to only one inversible $m \in \mathcal{A}$.

Keywords**—**finite Klingenberg plane, projective collineation, 4-transitivity, 6-figures.

I. Introduction

Projective Klingenberg and Hjelmslev planes (more briefly: PK-planes and PH-planes, resp.) are generalizations of ordinary projective planes. These structures were introduced by Klingenberg in [14], [15]. As for finite PK-planes, these structures introduced by Drake and Lenz in [8] have been studied in detail by Bacon in [2].

In our previous paper [6] we studied a certain class (which we will denote by $\mathbf{M}(\mathcal{A})$) of Moufang-Klingenberg (briefly, MK) planes coordinatized by an local alternative ring

$$\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$$

of dual numbers (an alternative ring $\mathbf{A}$, $\varepsilon \notin \mathbf{A}$ and $\varepsilon^2 = 0$) introduced by Blunck in [5]. We showed that its collineation group is transitive on quadrangles and the coordinatization of these Moufang-Klingenberg planes is independent of the choice of the coordinatization quadrangle. By extending the concepts of $6-$figure to these Moufang - Klingenberg planes, we examined some properties of $6-$figures.

In the present paper we deal with finite PK-plane $\mathbf{M}(\mathcal{A})$ obtained by taking local ring $\mathbf{Z}_q$ (where q is a prime power) instead of $\mathbf{A}$. So, we will carry the results that are well-known for MK-planes from [6] $\mathbf{M}(\mathcal{A})$ to the finite PK-plane $\mathbf{M}(\mathcal{A})$.

II. Preliminaries

Let $\mathbf{M} = (\mathbf{P}, \mathbf{L}, \in, \sim)$ consist of an incidence structure $(\mathbf{P}, \mathbf{L}, \in)$(points, lines, incidence) and an equivalence relation '$\sim$' (neighbour relation) on $\mathbf{P}$ and on $\mathbf{L}$. Then $\mathbf{M}$ is called a *projective Klingenberg plane* (PK-plane), if it satisfies the following axioms:

(PK1) If P, Q are two non-neighbour points, then there is a unique line PQ through P and Q.

(PK2) If g, h are two non-neighbour lines, then there is a unique point $g \wedge h$ on both g and h.

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(PK3) There is a projective plane $\mathbf{M}^* = (\mathbf{P}^*, \mathbf{L}^*, \in)$ and incidence structure epimorphism $\Psi : \mathbf{M} \to \mathbf{M}^*$, such that the conditions

$$\Psi(P) = \Psi(Q) \Leftrightarrow P \sim Q,\ \Psi(g) = \Psi(h) \Leftrightarrow g \sim h$$

hold for all $P, Q \in \mathbf{P}$, $g, h \in \mathbf{L}$.

PK-plane $\mathbf{M}$ is called a *projective Hjelmslev plane* (PH-plane) If $\mathbf{M}$ furthermore provides the following axioms:

(PH1) If P, Q are two neighbour points, then there are at least two lines through P and Q.

(PH2) If g, h are two neighbour lines, then there are at least two points on both g and h.

A *Moufang-Klingenberg plane* (MK-plane) is a PK-plane $\mathbf{M}$ that generalizes a Moufang plane, and for which $\mathbf{M}^*$ is a Moufang plane (for the details see [1]).

A point $P \in \mathbf{P}$ is called *near* a line $g \in \mathbf{L}$ iff there exists a line h such that $P \in h$ for some line $h \sim g$.

An incidence structure automorphism preserving and reflecting the neighbour relation is called a *collineation* of $\mathbf{M}$.

Now we give the definition of an n-gon, which is meaningful when $n \geq 3$: An n-tuple of pairwise non-neighbour points is called an (ordered) *n-gon* if no three of its elements are on neighbour lines [6].

An *alternative ring* (*field*) $\mathbf{R}$ is a not necessarily associative ring (field) that satisfies the alternative laws $a(ab) = a^2b, (ba)a = ba^2,\ \forall a, b \in \mathbf{R}$. An alternative ring $\mathbf{R}$ with identity element 1 is called *local* if the set $\mathbf{I}$ of its non-unit elements is an ideal.

We summarize some basic concepts about the coordinatization of MK-planes from [3].

Let $\mathbf{R}$ be a local alternative ring. Then

$$\mathbf{M}(\mathbf{R}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$$

is the incidence structure with neighbor relation defined as follows:

$$\begin{aligned} \mathbf{P} &= \{(x,y,1) : x, y \in \mathbf{R}\} \cup \{(1,y,z) :\ y \in \mathbf{R},\ z \in \mathbf{I}\} \\ &\quad \cup \{(w,1,z) : w, z \in \mathbf{I}\} \\ \mathbf{L} &= \{[m,1,p] : m, p \in \mathbf{R}\} \cup \{[1,n,p] : p \in \mathbf{R},\ n \in \mathbf{I}\} \\ &\quad \cup \{[q,n,1] : q, n \in \mathbf{I}\} \end{aligned}$$

$$\begin{aligned}
[m,1,p] &= \{(x, xm+p, 1) : x \in \mathbf{R}\} \\
&\quad \cup \{(1, zp+m, z) : z \in \mathbf{I}\} \\
[1,n,p] &= \{(yn+p, y, 1) : y \in \mathbf{R}\} \\
&\quad \cup \{(zp+n, 1, z) : z \in \mathbf{I}\} \\
[q,n,1] &= \{(1, y, yn+q) : y \in \mathbf{R}\} \\
&\quad \cup \{(w, 1, wq+n) : w \in \mathbf{I}\}
\end{aligned}$$

and also

$$\begin{aligned}
P &= (x_1,x_2,x_3) \sim (y_1,y_2,y_3) = Q \\
&\Leftrightarrow x_i - y_i \in \mathbf{I}\ (i=1,2,3)), \forall P,Q \in \mathbf{P}
\end{aligned}$$

$$\begin{aligned}
g &= [x_1,x_2,x_3] \sim [y_1,y_2,y_3] = h \\
&\Leftrightarrow x_i - y_i \in \mathbf{I}\ (i=1,2,3)), \forall g,h \in \mathbf{L}.
\end{aligned}$$

Baker *et al.* [1] use $(O=(0,0,1), U=(1,0,0), V=(0,1,0), E=(1,1,1))$ as a coordinatization 4-gon. We stick to this notation throughout this paper. For more detailed information about the coordinatization see [1] and [3].

Now it is time to give the following theorem from [1].

Theorem 2.1: $\mathbf{M}(\mathbf{R})$ is an MK-plane, and each MK-plane is isomorphic to some $\mathbf{M}(\mathbf{R})$.

Let $\mathbf{A}$ be an alternative field and $\varepsilon \notin \mathbf{A}$. Consider $\mathcal{A} := \mathbf{A}(\varepsilon) = \mathbf{A} + \mathbf{A}\varepsilon$ with componentwise addition and multiplication as follows:

$$(a_1 + a_2\varepsilon)(b_1 + b_2\varepsilon) = a_1b_1 + (a_1b_2 + a_2b_1)\varepsilon,$$

where $a_i, b_i \in \mathbf{A},\ i=1,2$. Then $\mathcal{A}$ is an alternative ring with ideal $\mathbf{I} = \mathbf{A}\varepsilon$ of non-units. For more detailed information about $\mathcal{A}$ see the papers of [4], [5].

Theorem 2.2: If $\mathbf{R}$ is a (not necessarily commutative) local ring then $\mathbf{M}(\mathbf{R})$ is a PK-plane (cf. [15] or [9, Theorem 4.1]).

Drake and Lenz [8, Proposition 2.5] or [12, Theorem 1.2] observed that the following corollary is true for PK-planes. This corollary is a generalization of results which are given for PH-planes by Kleinfeld [13, Theorem 1] and Lüneburg [16, Satz 2.11].

Corollary 2.3: Let $\mathbf{M}(\mathbf{R})$ be PK-plane. Then there are natural numbers t and r which are called the parametres of $\mathbf{M}(\mathbf{R})$ and they are uniquely determined by incidence structure of a finite PK-plane [8, Proposition 2.7], with

1) *every point (line) has t^2 neighbours;*
2) *given a point P and a line l with $P \in l$, there exist exactly t points on l which are neighbours to P and exactly t lines through P which are neighbours to l;*
3) Let r be order of the projective plane $\mathbf{M}^*$. If $t \neq 1$ we have $r \leq t$ (then $\mathbf{M}$ is called *proper*; we have $t=1$ iff $\mathbf{M}$ is an ordinary projective plane)
4) *every point (line) is incident with $t(r+1)$ lines (points);*
5) $|\mathbf{P}| = |\mathbf{L}| = t^2(r^2+r+1)$.

Now consider ring $\mathbf{Z}_q$ where prime power $q = p^k$. We can state the elements of $\mathbf{Z}_q$ as $\mathbf{Z}_q = U' \cup I$ where U' is the set of units of $\mathbf{Z}_q$ and I is the set of non-units of $\mathbf{Z}_q$. Here it is clear that

$$I = \{0p, 1p, 2p, \cdots, (p^{k-1}-1)p\}$$

and so $|I| = p^{k-1}$. Let $\varepsilon \notin \mathbf{Z}_q$. Then $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q\varepsilon$ with componentwise addition and multiplication above is a local ring with ideal $\mathbf{I} := I + \mathbf{Z}_q\varepsilon$ of non-units, $|\mathbf{I}| = (p^{k-1})p^k$. Note that the set of units of $\mathcal{A}$ is $\mathbf{U} := U' + \mathbf{Z}_q\varepsilon$ and

$$|\mathbf{U}| = (p^k - p^{k-1})p^k = (p-1)p^{2k-1}.$$

Since $\mathcal{A}$ is a proper local ring and $\mathcal{A}/\mathbf{I} = \mathbf{Z}_p$, Ψ induces an incidence structure epimorphism from finite PK-plane $\mathbf{M}(\mathcal{A})$ onto the Desarguesian projective plane (with order p) coordinatized by the field $\mathbf{Z}_p$ [9, page 169, above Theorem 4.1]. Because of this, $\mathbf{M}(\mathcal{A})$ is called as Desarguesian PK-plane.

So, we have the following

Corollary 2.4: For finite PK-plane $\mathbf{M}(\mathcal{A})$, the parameters t and r in Corollary 2.3 are equal to p^{2k-1} and p, respectively.

A local ring $\mathbf{R}$ is called a *Hjelmslev ring* (briefly, H-ring) if it satisfies the following two conditions:

(HR1) $\mathbf{I}$ consists of two-sided zero divisor.

(HR2) For $a, b \in \mathbf{I}$, one has $a \in b\mathbf{R}$ or $b \in a\mathbf{R}$, and also $a \in \mathbf{R}b$ or $b \in \mathbf{R}a$.

By the last definition, we can say that $\mathcal{A}$ is not a H-ring. For example, for elements $a = 3 + 3\varepsilon$ and $b = \varepsilon$ of the ideal $\mathbf{I}$ of local ring $\mathcal{A} = \mathbf{Z}_{3^2} + \mathbf{Z}_{3^2}(\varepsilon)$, (HR2) is not valid.

From now on we restrict ourselves to PK-plane $\mathbf{M}(\mathcal{A}) = (\mathbf{P}, \mathbf{L}, \in, \sim)$ coordinatized by the local ring $\mathcal{A} := \mathbf{Z}_q + \mathbf{Z}_q\varepsilon$, with neighbour relation defined above.

III. 4-Transitivity and 6-Figures in $\mathbf{M}(\mathcal{A})$.

In the final section, first of all, from [6] we start by giving some collineations on $\mathbf{M}(\mathcal{A})$ where $w, z, q, n \in \mathbf{I}$ as follows:

For any $a, b \in \mathcal{A}$, the collineation $\mathrm{T}_{a,b}$ transforms points and lines as follows:

$$\begin{aligned}
(x,y,1) &\rightarrow (x+a, y+b, 1) \\
(1,y,z) &\rightarrow (1, y+z(b-ay), z) \\
(w,1,z) &\rightarrow (w+za, 1, z)
\end{aligned}$$

and

$$\begin{aligned}
[m,1,k] &\rightarrow [m,1,k+b-am] \\
[1,n,p] &\rightarrow [1,n,p+a-bn] \\
[q,n,1] &\rightarrow [q,n,1].
\end{aligned}$$

For any $\alpha, \beta \notin \mathbf{I}$, the collineation $\mathrm{S}_{\alpha,\beta}$ (here, it is enough to give $\mathrm{S}_{\alpha,\beta}$ instead of the collineations L_a and F_a in [6]) transforms points and lines as follows:

$$\begin{aligned}
(x,y,1) &\rightarrow (\beta x, \alpha y, 1) \\
(1,y,z) &\rightarrow (1, \alpha\beta^{-1}y, \beta^{-1}z) \\
(w,1,z) &\rightarrow (\alpha^{-1}\beta w, 1, \alpha^{-1}z)
\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow \left[\alpha\beta^{-1}m,1,\alpha k\right]\\ [1,n,p] &\rightarrow \left[1,\alpha^{-1}\beta n,\beta p\right]\\ [q,n,1] &\rightarrow \left[\beta^{-1}q,\alpha^{-1}n,1\right].\end{aligned}$$

The collineation I_1 transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\rightarrow \left(x^{-1},x^{-1}y,1\right) \quad if \quad x\notin\mathbf{I}\\ (x,y,1) &\rightarrow (1,y,x) \quad if \quad x\in\mathbf{I}\\ (1,y,z) &\rightarrow (z,y,1)\\ (w,1,z) &\rightarrow (z,1,w)\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow [k,1,m]\\ [1,n,p] &\rightarrow [p,n,1] \quad if \quad p\in\mathbf{I}\\ [1,n,p] &\rightarrow \left[1,-np^{-1},p^{-1}\right] \quad if \quad p\notin\mathbf{I}\\ [q,n,1] &\rightarrow [1,n,q].\end{aligned}$$

The collineation F transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\rightarrow (y,x,1)\\ (1,y,z) &\rightarrow (y,1,z) \quad if \quad y\in\mathbf{I}\\ (1,y,z) &\rightarrow \left(1,y^{-1},y^{-1}z\right) \quad if \quad y\notin\mathbf{I}\\ (w,1,z) &\rightarrow (1,w,z)\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow [1,m,k] \quad if \quad m\in\mathbf{I}\\ [m,1,k] &\rightarrow \left[m^{-1},1,-km^{-1}\right] \quad if \quad m\notin\mathbf{I}\\ [1,n,p] &\rightarrow [n,1,p]\\ [q,n,1] &\rightarrow [n,q,1].\end{aligned}$$

For any $s\in\mathcal{A}$, the collineation G_s transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\rightarrow (x,y-xs,1)\\ (1,y,z) &\rightarrow (1,y-s,z)\\ (w,1,z) &\rightarrow (w,1,z)\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow [m-s,1,k]\\ [1,n,p] &\rightarrow [1,n,p+psn]\\ [q,n,1] &\rightarrow [q+sn,n,1].\end{aligned}$$

The collineation I_2 transforms points and lines as follows:

$$\begin{aligned}(x,y,1) &\rightarrow \left(y^{-1}x,y^{-1},1\right) \quad if \quad y\notin\mathbf{I}\\ (x,y,1) &\rightarrow \left(1,x^{-1},x^{-1}y\right) \quad if \quad y\in\mathbf{I}\wedge x\notin\mathbf{I}\\ (x,y,1) &\rightarrow (x,1,y) \quad if \quad y\in\mathbf{I}\wedge x\in\mathbf{I}\\ (1,y,z) &\rightarrow \left(y^{-1},y^{-1}z,1\right) \quad if \quad y\notin\mathbf{I}\\ (1,y,z) &\rightarrow (1,z,y) \quad if \quad y\in\mathbf{I}\\ (w,1,z) &\longrightarrow (w,z,1)\end{aligned}$$

and

$$\begin{aligned}[m,1,k] &\rightarrow \left[-mk^{-1},1,k^{-1}\right] \quad if \quad k\notin\mathbf{I}\\ [m,1,k] &\rightarrow \left[1,-km^{-1},m^{-1}\right] \quad if \quad k\in\mathbf{I}\wedge m\notin\mathbf{I}\\ [m,1,k] &\rightarrow [m,k,1] \quad if \quad k\in\mathbf{I}\wedge m\in\mathbf{I}\\ [1,n,p] &\rightarrow \left[p^{-1},1,-np^{-1}\right] \quad if \quad p\notin\mathbf{I}\\ [1,n,p] &\rightarrow [1,p,n] \quad if \quad p\in\mathbf{I}\\ [q,n,1] &\rightarrow [q,1,n].\end{aligned}$$

So, we can give the following theorem without proof. For, its proof is same to Theorem 2 of [6]. Furthermore, this theorem is proved by Lemma 4.15 in [11].

Theorem 3.1: The group $\mathcal{G}$ of collineations of $\mathbf{M}(\mathcal{A})$ acts transitively on 3-gons.

Now, we can state the analogue of the result given by [2, Proposition 5.2.10 in Vol.I]. For the case of uniform H-rings (for the definition of uniform see [10]), the result is also in [7, Theorem 17]. Here, it is possible to give the proof of the following theorem, as more shortly than the proof of Theorem 3 in [6].

Theorem 3.2: $\mathcal{G}$ acts transitively on 4-gons of $\mathbf{M}(\mathcal{A})$.

Proof: Let (P,Q,R,S) be a 4-gon in $\mathbf{M}(\mathcal{A})$. It suffices to show that the points P,Q,R,S can be transformed by an element of $\mathcal{G}$ to $U,V,(1,1,1),O$, respectively. From Theorem 3.1, there exists a collineation σ which transforms P,Q,R to $U,\ V,\ (0,1,1)$, respectively. Let E denote the intersection point of the lines QR and PS. Then, since $\sigma(E)$ is non-neighbour to the points $\sigma(P),\ \sigma(Q),\ \sigma(R)$, it has the form $(0,b,1)$, where $b-1\notin\mathbf{I}$, and so $\sigma(S)$ has the form $(a,b,1)$, where $a\notin\mathbf{I}$. Therefore σ transforms P,Q,R,S to

$$(1,0,0),\ (0,1,0),\ (0,1,1),\ (a,b,1),$$

respectively. Then the mapping $\mathrm{T}_{-a,-b}$ transforms these points to

$$(1,0,0),\ (0,1,0),\ (-a,1-b,1),\ (0,0,1),$$

respectively and $\mathrm{S}_{(1-b)^{-1},-a^{-1}}$ transforms these points to

$$(1,0,0),\ (0,1,0),\ (1,1,1),\ (0,0,1),$$

respectively. ■

The following corollary is an obvious result of the last theorem:

Corollary 3.3: The coordinatization of $\mathbf{M}(\mathcal{A})$ is independent of the choice of the coordinatization base.

From now on, we carry over some concepts related to 6-figures to the $\mathbf{M}(\mathcal{A})$, in view of the paper of [6].

A *6-figure* is a sequence of six non-neighbour points $(ABC,A_1B_1C_1)$ such that (A,B,C) is 3-gon, and $A_1\in$

$BC, B_1 \in CA, C_1 \in AB$. The points A, B, C, A_1, B_1, C_1 are called vertices of this 6-figure. The 6-figures $(ABC, A_1B_1C_1)$ and $(DEF, D_1E_1F_1)$ are *equivalent* if there exists a collineation of $\mathbf{M}(\mathcal{A})$ which transforms A, B, C, A_1, B_1, C_1 to D, E, F, D_1, E_1, F_1 respectively. Now, we give a theorem from [6].

Theorem 3.4: Let $\mu = (ABC, A_1B_1C_1)$ be a 6-figure in $\mathbf{M}(\mathcal{A})$. Then, there is an $m \in \mathbf{U}$ such that μ is equivalent to $(UVO, (0,1,1)(1,0,1)(1,m,0))$ where $U = (1,0,0)$,$V = (0,1,0)$,$O = (0,0,1)$ are elements of the coordinatization basis of $\mathbf{M}(\mathcal{A})$.

We again give a theorem from [6]. Note that the proof of this theorem is more shorter.

Theorem 3.5: The 6-figures

$$(ABC, A_1B_1C_1), (BCA, B_1C_1A_1), (CAB, C_1A_1B_1)$$

are equivalent.

Proof: By Theorem 3.4 we may without loss of generality take $(UVO, U_1V_1O_1)$ instead of $(ABC, A_1B_1C_1)$, where

$$U_1 = (0,1,1), V_1 = (1,0,1), O_1 = (1,m,0)$$

with $m \in \mathbf{U}$. The collineation

$$h := S_{m,1} \circ I_2 \circ I_1$$

transforms $(UVO, U_1V_1O_1)$ to $(VOU, V_1O_1U_1)$ and also $(VOU, V_1O_1U_1)$ to $(OUV, O_1U_1V_1)$. ■

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