World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:12, 2011

On a discrete-time $GI^X/Geo/1/N$ queue with single working vacation and partial batch rejection

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Abstract—This paper treats a discrete-time finite buffer batch arrival queue with a single working vacation and partial batch rejection in which the inter-arrival and service times are, respectively, arbitrary and geometrically distributed. The queue is analyzed by using the supplementary variable and the imbedded Markov-chain techniques. We obtain steady-state system length distributions at pre-arrival, arbitrary and outside observer's observation epochs. We also present probability generation function (p.g.f.) of actual waiting-time distribution in the system and some performance measures.

Keywords—Discrete-time; Finite buffer; Single working vacation; Batch arrival; Partial rejection

I. Introduction

QUEUEING systems with vacations have been widely used in the performance analysis of computer communication systems, manufacturing and production systems, in which the server utilizes the idle time for different purpose. Past work may be divided into two categories: (i) the case of classical vacation, i.e., the server completely stops service in the vacation period and (ii) the case of working vacation, i.e., the server renders service to the customers with a lower service rate. In the case of server vacation, the readers are referred to the survey paper [4,5,15,16,17,23] and references therein. In the case of working vacations, the server provides service to the queue with a lower service rate during the vacation period. Two common working vacation policies are multiple working vacations and sing working vacation. Multiple working vacations policy has the following properties: when a vacation ends and the system is not empty, a busy period begins with the normal service rate. Otherwise, if the system is empty, the server takes another vacation. Under the single working vacation policy, the server enters into vacation when there are no customers and takes service at a lower rate during the vacation period. Meanwhile, he only takes one vacation each time, and must come back to the normal working level no matter whether there are customers at the vacation ending instant. If there are customers when the vacation ends, the server begins to serve one customer at the normal rate immediately; otherwise, he will stay in an idle period. [14] first examined an M/M/1 queue with multiple working vacations (Such model is denoted by M/M/1/WV queue) and modeled a wavelength division multiplexing (WDM) optical access network using multiple wavelengths which can be reconfigured. The work of [14] is rooted in performance analysis of gateway router in fiber communication networks. On the other hand, working vacation policy has practical application background in optimal design of the system. When the number of customers in the system is relatively few, we set a lower speed operating period in order to economize operating cost and energy consumption. Furthermore, [14] obtained the stochastic decomposition structures of the system indices in the $M/M/1$ queue with working vacations. Later [8,12,19] generalized results in [14] to an $M/G/1$ queue with working vacations. [20] presented a $Geo/G/1$ queue with disasters and multiple working vacations and [9] analyzed a $Geo^X/G/1$ queue with working vacations. [1] and [11] respectively discussed a continuous-time $GI/M/1$ and a discrete-time $GI/Geo/1$ queue with multiple working vacations. From the literatures listed above, we discovered that all research efforts focus on the multiple working vacation (MWV) policy, however, the literatures on single working vacation queues, readers may refer to [3] and [10].

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Though the working vacation queueing models with infinite buffer size have been studied extensively in the past years, many a time there is also need for finite buffer size. Queues with finite buffer space are more realistic in real life situations than queues with infinite buffer space as it is used to store arrived customers if server is busy. To the best of our knowledge, the only work about general input working vacation queueing model with finite buffer size can be found in [2,6,7,21,22], where [2] discussed the $GI/M/1/N$ queue with multiple working vacations, [21] presented the $GI^{[x]}/M^b/1/L$ queue with multiple working vacations and partial batch rejection, [6] analyzed the $GI^{[x]}/M/1/N$ queue with single working vacations and partial batch rejection, a finite buffer size discrete-time multiple working vacation queue was considered by [7], [22] have introduced changeover time into the working vacation. However, there is no work that deals with discrete-time renewal input bulk arrival queue with single working vacations. This motivates us to investigate such queueing system in this work, denoted by $GI^X/Geo/1/N/SWV$. Such a discrete-time queue with working vacation policy has many significant applications. Firstly, discrete-time queueing systems have wide applications in design and control of manufacturing and telecommunication systems, and in modeling and analyzing of computer communication networks. Moreover, the discrete-time system can be used to approximate the continuous systems. Secondly, batch arrival queues have more extensive applications in the computer networks and communication systems because the cells arrive in batch. Thirdly, the single working vacation policy is more reasonable for employees under management structures and in optimal design of the system.

The rest of this paper is organized as follows. In Section 2, we give the discrete queueing model. In Section 3, we analyze the model and obtain steady state distributions at arbitrary, pre-arrival and outside observer's observation epochs. Various performance measures are presented in Section 4.

II. Model Description

Thereinafter, we denote $\bar{x} = 1 - x$ for any real number $x \in (0, 1)$. We consider a finite buffer $GI^X/Geo/1$ queue of size N(including the one in service) where the time axis is divided into equal intervals called slots and all queueing activities occur at the slot boundaries. Let the time axis be marked by $0, 1, \cdots, n, \cdots$. Here, we consider the early arrival system (EAS), that is, a potential arrival can only take place in (n, n^+) and a potential departure can only take place in (n^-, n). We assume that the beginning and ending of vacations occurs at division point n. Arriving customers are queued according to the first-come, first-served (FCFS) discipline. The server can serve only one customer at a time. Various stochastic processes involved in the system are independent of each other.

The detailed description of the model is given as follows:
(1). Customers arrive in batches of random size X with probability mass function (p.m.f.) $P(X = j) = \chi_j, j = 1, 2, \cdots$, and mean $E[X] = \overline{\chi}$.
(2). The inter-arrival times A of two successive arrivals are independently identically distributed (i.i.d.) random variables with common p.m.f. $P(A = i) = a_i,\ i \geq 1$, corresponding p.g.f. $\widetilde{A}(z) = \sum_{i=1}^{\infty} a_i z^i$ and mean λ^{-1}.
(3). The service time $\{S_k^b, k \geq 1\}$ in a regular busy period is geometrically distributed with p.m.f. $P(S_k^b = i) = \mu\overline{\mu}^{i-1}, i \geq 1, 0 < \mu < 1$.
(4). The working vacation is an operating period in a lower rate, the service time $\{S_k^v, k \geq 1\}$ in a working vacation period has a geometric distribution with p.m.f. $P(S_k^v = i) = \nu\overline{\eta}^{i-1}, i \geq 1, 0 < \nu < \mu < 1$.
(5). The server commences a vacation of random length at the epoch when the system becomes empty after all the customers being served by the normal service rate μ. The distribution of vacation time V is geometrically distributed with rate θ $(0 < \theta < 1)$, i.e., $P(V = i) = \theta\bar{\theta}^{i-1}, i \geq 1$. If the customers arrive during a vacation period the server will serve them at a lower service rate ν. The server is permitted to take one vacation each time and will come back to the normal working level no matter whether there are customers. More specifically, on return from a working vacation if the server finds the system nonempty he will change to the normal service rate, and the service interrupted at the end of vacation restarts from the beginning; otherwise, if there is no customer when a vacation ends, the server remains dormant until the next customer arrives.
(6). Partial batch rejection policy is adopted. Since the buffer space is finite, if a batch upon arrival doesn't find enough space in the buffer then a part of customers fills the vacant spaces and the rest is rejected. This is known as partial batch rejection. The state of the system at time n is described by the following random variables:

$N_s(n)$: the number of customers in the system (including the one in service);
$U(n)$: remaining inter-arrival time for the group going to enter into the system;

$$J(n) = \begin{cases} 0, & \text{if the server is on working vacation,} \\ 1, & \text{if the server is in busy period or dormant period.} \end{cases}$$

Then we have a trivariate Markov process $\{N_s(n), J(n), U(n)\}$. Now we define the following probabilities as

$$P_{0,0}(u,n) = P(N_s(n) = 0, J(n) = 0, U(n) = u), u \geq 0,$$
$$P_{k,i}(u,n) = P(N_s(n) = k, J(n) = i, U(n) = u),, u \geq 0,$$
$$1 \leq k \leq N,\ i = 0, 1.$$

In steady-state, the above probabilities are denoted, respectively, as $P_{k,i}(u),\ 0 \leq k \leq N,\ i = 0, 1, u \geq 0$.

III. Analysis of the Model

In the following subsections, using different analysis methods: supplementary variable and embedded Markov chain technique, we will consider three kinds of system length at different time epochs, that is, the steady state system length distribution at arbitrary, pre-arrival and an outside observer's observation epoch.

A. Steady-state distribution at arbitrary epoch

To obtain the system length distribution at arbitrary epoch and performance measures of the system, we develop the difference equations using the remaining inter-arrival time as supplementary variable. Observing the state of the system at two consecutive time epochs n and $(n + 1)$, and using probability argument, we get a set of difference equations: for $u \geq 1$,

$$P_{0,0}(u-1) = \bar{\theta}P_{0,0}(u) + \bar{\theta}\nu P_{1,0}(u) + \bar{\theta}\nu a_u\chi_1 P_{0,0}(0) + \mu P_{1,1}(u) + \mu a_u \chi_1 P_{0,1}(0), \quad (1)$$

$$P_{i,0}(u-1) = \overline{\theta}\overline{\nu}P_{i,0}(u) + \overline{\theta}\nu P_{i+1,0}(u) + \overline{\theta}\nu a_u \sum_{k=1}^{i+1} \chi_k P_{i+1-k,0}(0) + \overline{\theta}\overline{\nu} a_u \sum_{k=1}^{i} \chi_k P_{i-k,0}(0), 1 \leq i \leq N-2, \quad (2)$$

$$P_{N-1,0}(u-1) = \overline{\theta}\overline{\nu}P_{N-1,0}(u) + \overline{\theta}\nu P_{N,0}(u) + \overline{\theta}\nu a_u \times \sum_{k=0}^{N} P_{k,0}(0) \sum_{m=N-k}^{\infty} \chi_m + \overline{\theta}\overline{\nu} a_u \sum_{k=1}^{N-1} \chi_k P_{N-1-k,0}(0), \quad (3)$$

$$P_{N,0}(u-1) = \overline{\theta}\overline{\nu}P_{N,0}(u) + \overline{\theta}\overline{\nu} a_u \sum_{k=0}^{N} P_{k,0}(0) \sum_{m=N-k}^{\infty} \chi_m, \quad (4)$$

$$P_{0,1}(u-1)$$
$$= P_{0,1}(u) + \theta P_{0,0}(u) + \theta\nu P_{1,0}(u) + \theta\nu a_u \chi_1 P_{0,0}(0), \quad (5)$$
$$P_{i,1}(u-1)$$
$$= \overline{\mu}\big(P_{i,1}(u) + a_u\chi_i P_{0,1}(0) + a_u \sum_{k=1}^{i-1}\chi_{i-k}P_{k,1}(0)\big)$$
$$+ \mu\big(P_{i+1,1}(u) + a_u\chi_{i+1}P_{0,1}(0) + a_u\sum_{k=1}^{i}\chi_{i-k+1}P_{k,1}(0)\big)$$
$$+ \theta\nu\big(P_{i+1,0}(u) + a_u\sum_{k=0}^{i}\chi_{i-k+1}P_{k,0}(0)\big)$$
$$+ \theta\overline{\nu}\big(P_{i,0}(u) + a_u\sum_{k=0}^{i-1}\chi_{i-k}P_{k,0}(0)\big), 1 \le i \le N-2, \quad (6)$$
$$P_{N-1,1}(u-1)$$
$$= \overline{\mu}\Big(P_{N-1,1}(u) + a_u\chi_{N-1}P_{0,1}(0) + a_u\sum_{k=1}^{N-2}\chi_{N-1-k}P_{k,1}(0)\Big)$$
$$+ \mu\Big(P_{N,1}(u) + a_u\sum_{k=N}^{\infty}\chi_k P_{0,1}(0) + a_u\sum_{k=1}^{N}P_{k,1}(0)\sum_{m=N-k}^{\infty}\chi_m\Big)$$
$$+ \theta\overline{\nu}\big(P_{N-1,0}(u) + a_u\sum_{k=1}^{N-1}\chi_k P_{N-1-k,0}(0)\big)$$
$$+ \theta\nu\big(P_{N,0}(u) + a_u\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m\big), \quad (7)$$
$$P_{N,1}(u-1)$$
$$= \overline{\mu}\Big(P_{N,1}(u) + a_u\sum_{k=N}^{\infty}\chi_k P_{0,1}(0) + a_u\sum_{k=1}^{N}P_{k,1}(0)\sum_{m=N-k}^{\infty}\chi_m\Big)$$
$$+ \theta\overline{\nu}\Big(P_{N,0}(u) + a_u\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m\Big). \quad (8)$$

(Note that $\chi_0 = 0, \sum_a^b = 0$, if $a > b$.)

We introduce the following z-transforms

$$\widetilde{P}_{i,j}(z) = \sum_{u=0}^{\infty}P_{i,j}(u)z^u, j = 0,1, i = 0,1,2,\cdots,N.$$

So that $\widetilde{P}_{i,j}(1) = P_{i,j}, j = 0,1, i = 0,1,2,\cdots,N$. Our objective is to obtain $P_{i,j}, j = 0,1, i = 0,1,2,\cdots,N$.

Multiplying (1) to (8) by z^u and summing over u from 1 to ∞, we obtain

$$(z-\overline{\theta})\widetilde{P}_{0,0}(z)$$
$$= \overline{\theta}\nu[\widetilde{P}_{1,0}(z) - P_{1,0}(0)] + \overline{\theta}[\nu\widetilde{A}(z)\chi_1 - 1]P_{0,0}(0)$$
$$+ \mu[\widetilde{P}_{1,1}(z) + \widetilde{A}(z)\chi_1 P_{0,1}(0) - P_{1,1}(0)], \quad (9)$$

$$(z-\overline{\theta}\overline{\nu})\widetilde{P}_{i,0}(z)$$
$$= \overline{\theta}\nu\left[\widetilde{P}_{i+1,0}(z) - P_{i+1,0}(0) + \widetilde{A}(z)\sum_{k=1}^{i+1}P_{i+1-k,0}(0)\chi_k\right]$$
$$+ \overline{\theta}\overline{\nu}\Big(\widetilde{A}(z)\sum_{k=1}^{i}P_{i-k,0}(0)\chi_k - P_{i,0}(0)\Big), 1 \le i \le N-2, \quad (10)$$

$$(z-\overline{\theta}\overline{\nu})\widetilde{P}_{N-1,0}(z)$$
$$= \overline{\theta}\nu\left[\widetilde{P}_{N,0}(z) - P_{N,0}(0) + \widetilde{A}(z)\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m\right]$$
$$+ \overline{\theta}\overline{\nu}\left[\widetilde{A}(z)\sum_{k=1}^{N-1}P_{N-1-k,0}(0)\chi_k - P_{N-1,0}(0)\right], \quad (11)$$

$$(z-\overline{\theta}\overline{\nu})\widetilde{P}_{N,0}(z)$$
$$= \overline{\theta}\overline{\nu}\left[\widetilde{A}(z)\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m - P_{N,0}(0)\right], \quad (12)$$

$$(z-1)\widetilde{P}_{0,1}(z)$$
$$= \theta\nu\left[\widetilde{P}_{1,0}(z) - P_{1,0}(0) + \widetilde{A}(z)\chi_1 P_{0,0}(0)\right]$$
$$+ \theta\left[\widetilde{P}_{0,0}(z) - P_{0,0}(0)\right] - P_{0,1}(0), \quad (13)$$

$$(z-\overline{\mu})\widetilde{P}_{i,1}(z)$$
$$= \overline{\mu}\left[\widetilde{A}(z)\left(\chi_i P_{0,1}(0) + \sum_{k=1}^{i-1}\chi_{i-k}P_{k,1}(0)\right) - P_{i,1}(0)\right]$$
$$+ \mu\widetilde{A}(z)\left(\sum_{k=1}^{i}\chi_k P_{i-k+1,1}(0) + \chi_{i+1}P_{0,1}(0)\right)$$
$$+ \mu\left(\widetilde{P}_{i+1,1}(z) - P_{i+1,1}(0)\right) + \theta\nu\left(\widetilde{P}_{i+1,0}(z) - P_{i+1,0}(0)\right)$$
$$+ \theta\overline{\nu}\left(\widetilde{P}_{i,0}(z) - P_{i,0}(0) + \widetilde{A}(z)\sum_{k=0}^{i-1}\chi_{i-k}P_{k,0}(0)\right)$$
$$+ \theta\nu\widetilde{A}(z)\sum_{k=0}^{i}\chi_{i-k+1}P_{k,0}(0), 1 \le i \le N-2, \quad (14)$$

$$(z-\overline{\mu})\widetilde{P}_{N-1,1}(z)$$
$$= \overline{\mu}\left[\widetilde{A}(z)\left(\chi_{N-1}P_{0,1}(0) + \sum_{k=1}^{N-1}\chi_k P_{N-1-k,1}(0)\right)\right]$$
$$+ \mu\widetilde{A}(z)\left(\sum_{k=1}^{N}P_{k,1}(0)\chi_{N-k} + \sum_{k=N}^{\infty}\chi_k P_{0,1}(0)\right)$$
$$+ \mu\left(\widetilde{P}_{N,1}(z) - P_{N,1}(0)\right) - \overline{\mu}P_{N-1,1}(0)$$
$$+ \theta\overline{\nu}\left(\widetilde{P}_{N-1,0}(z) - P_{N-1,0}(0) + \widetilde{A}(z)\sum_{k=1}^{N-1}\chi_k P_{N-1-k,0}(0)\right)$$
$$+ \theta\nu\left(\widetilde{P}_{N,0}(z) - P_{N,0}(0) + \widetilde{A}(z)\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m\right), \quad (15)$$

$$(z-\overline{\mu})\widetilde{P}_{N,1}(z)$$
$$=\overline{\mu}\left[\widetilde{A}(z)\left(\sum_{k=N}^{\infty}\chi_k P_{0,1}(0)+\sum_{k=1}^{N}P_{k,1}(0)_{m=N-k}^{\infty}\chi_m\right)\right]$$
$$+\theta\overline{\nu}\left(\widetilde{P}_{N,0}(z)-P_{N,0}(0)+\widetilde{A}(z)\sum_{k=0}^{N}P_{k,0}(0)\sum_{m=N-k}^{\infty}\chi_m\right)$$
$$-\overline{\mu}P_{N,1}(0) \tag{16}$$

Summing all Eqs.(9)-(16) yields

$$\sum_{i=0}^{N}\left(\widetilde{P}_{i,0}(z)+\widetilde{P}_{i,1}(z)\right)=\frac{\widetilde{A}(z)-1}{z-1}\sum_{i=0}^{N}\left(P_{i,0}(0)+P_{i,1}(0)\right).$$

Taking the limit as $z\to 1$ in the above equation and using the normalization condition $\sum_{i=0}^{N}\left(\widetilde{P}_{i,0}(1)+\widetilde{P}_{i,1}(1)\right)=1$, one can obtain that

$$\sum_{i=0}^{N}\left(P_{i,0}(0)+P_{i,1}(0)\right)=\lambda. \tag{17}$$

.

Let $P_{i,j}^{-}(j=0,1,i=0,1,2,\cdots,N)$ be pre-arrival epoch probabilities, namely, an arrival sees i customers in the system at an arrival epoch when the server is in state j. Applying Bayes¡¯ theorem and using (17), we obtain

$$P_{i,j}^{-}=\frac{P_{i,j}(0)}{\sum_{i=0}^{N}(P_{i,0}(0)+P_{i,1}(0))}=\frac{P_{i,j}(0)}{\lambda},$$
$$j=0,1;0\le i\le N. \tag{18}$$

Setting $z=1$ in Eqs. (9)-(12),(14)-(16) and using (18), after simplification, we can get relations between pre-arrival and arbitrary epoch probabilities

$$P_{N,0}=\frac{\lambda\overline{\theta}\overline{\nu}}{1-\overline{\theta}\overline{\nu}}\sum_{k=0}^{N-1}P_{k,0}^{-}\sum_{m=N-k}^{\infty}\chi_m, \tag{19}$$

$$P_{N-1,0}=\frac{\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\left[\nu\left(P_{N,0}+\lambda\sum_{k=0}^{N-1}P_{k,0}^{-}\sum_{m=N-k}^{\infty}\chi_m\right)\right.$$
$$\left.+\lambda\overline{\nu}\left(\sum_{k=1}^{N-1}P_{N-1-k,0}^{-}\chi_k-P_{N-1,0}^{-}\right)\right], \tag{20}$$

$$P_{i,0}=\frac{\overline{\theta}\nu}{1-\overline{\theta}\overline{\nu}}\left(P_{i+1,0}-\lambda P_{i+1,0}^{-}+\lambda\sum_{k=1}^{i+1}P_{i+1-k,0}^{-}\chi_k\right)$$
$$+\frac{\lambda\overline{\theta}\overline{\nu}}{1-\overline{\theta}\overline{\nu}}\left(\sum_{k=1}^{i}\chi_k P_{i-k,0}^{-}-P_{i,0}^{-}\right),1\le i\le N-2, \tag{21}$$

$$P_{N,1}=\frac{\theta\overline{\nu}}{\mu}\left(P_{N,0}+\lambda\sum_{k=0}^{N-1}P_{k,0}^{-}\sum_{m=N-k}^{\infty}\chi_m\right)$$
$$+\frac{\lambda\overline{\mu}}{\mu}\left(P_{0,1}^{-}\sum_{k=N}^{\infty}\chi_k+\sum_{k=1}^{N-1}P_{k,1}^{-}\sum_{m=N-k}^{\infty}\chi_m\right), \tag{22}$$

$$P_{N-1,1}=P_{N,1}+\lambda\left(P_{0,1}^{-}\sum_{k=N}^{\infty}\chi_k+\sum_{k=1}^{N-1}P_{k,1}^{-}\sum_{m=N-k}^{\infty}\chi_m\right)$$
$$+\frac{\lambda\overline{\mu}}{\mu}\left(P_{0,1}^{-}\chi_{N-1}+\sum_{k=1}^{N-2}P_{N-1-k,1}^{-}\chi_k-P_{N-1,1}^{-}\right)$$
$$+\frac{\theta\overline{\nu}}{\mu}\left(P_{N-1,0}-\lambda P_{N-1,0}^{-}+\lambda\sum_{k=1}^{N-1}P_{N-1-k,0}^{-}\chi_k\right)$$
$$+\frac{\theta\nu}{\mu}\left(P_{N,0}+\lambda\sum_{k=0}^{N-1}P_{k,0}^{-}\sum_{m=N-k}^{\infty}\chi_m\right), \tag{23}$$

$$P_{i,1}=P_{i+1,1}+\frac{\theta\overline{\nu}}{\mu}\left(P_{i,0}+\lambda\left(\sum_{k=1}^{i}P_{i-k,0}^{-}\chi_k-P_{i,0}^{-}\right)\right)$$
$$+\frac{\theta\nu}{\mu}\left(P_{i+1,0}^{-}+\lambda\left(\sum_{k=1}^{i+1}P_{i+1-k,0}^{-}\chi_k-P_{i+1,0}^{-}\right)\right)$$
$$+\lambda\left(P_{0,1}^{-}\chi_{i+1}+\sum_{k=1}^{i}P_{i+1-k,1}^{-}\chi_k-P_{i+1,1}^{-}\right)$$
$$+\frac{\lambda\overline{\mu}}{\mu}\left(P_{0,1}^{-}\chi_i+\sum_{k=1}^{i-1}P_{i-k,1}^{-}\chi_k-P_{i,1}^{-}\right),$$
$$1\le i\le N-2, \tag{24}$$

$$P_{0,0}=\theta^{-1}\Big(\overline{\theta}\nu(P_{1,0}-\lambda P_{1,0}^{-})+\lambda\overline{\theta}(\nu\chi_1-1)P_{0,0}^{-}$$
$$+\mu(P_{1,1}-\lambda P_{1,1}^{-}+\lambda\chi_1 P_{0,1}^{-})\Big), \tag{25}$$

$$P_{0,2}=1-\sum_{i=0}^{N}P_{i,0}+\sum_{i=1}^{N}P_{i,1}. \tag{26}$$

One may note that through computing the pre-arrival epoch probabilities, the arbitrary time epoch probabilities can be obtained from Eqs. (19)-(26). So, in the next subsection, using the embedded Markov chain technique, we will investigate the system length distribution at a pre-arrival epoch.

B. Steady-state distribution at Pre-arrival Epoch

Let customer batches arrive at time epochs $T_1,T_2,\cdots$ and the inter-arrival times $\tau_n=T_n-T_{n-1},(n=1,\cdots;T_0=0)$ be mutually independent and identically distributed random variables with common p.m.f. $\{a_u,u\ge 1\}$. Let the state of the system at pre-arrival epoch of the n-th batch be defined as $\{(N_n^{-},J_n^{-}),n\ge 1\}$, where N_n^{-} denotes the number of customers in the system, that is, suppose that the n-th batch arrives at the system in (k,k^{+}), then $N_n^{-}=N_s(k),J_n^{-}=J(k)$, and

$$J_n^{-}=\begin{cases}0, & \text{the n-th batch arrival occurs during a working vacation period,}\\ 1, & \text{the n-th batch arrival occurs during a busy or idle period.}\end{cases}$$

Then $\{(N_n^{-},J_n^{-}),n\ge 1\}$ is an embedded two-dimensional Markov chain with the state space $\{(i,j),j=0,1;j\le i\ge N\}$.

In limiting case, $P_{i,j}^{-}=\lim_{n\to\infty}P(N_n^{-}=i,J_n^{-}=j),j=0,1,i=0,1,2,\cdots,N$.

Fof $i, j = 0, 1, 2, \cdots, N$, define

$$P_{(i,k)(j,m)} = P\{N_{n+1}^- = j, J_{n+1}^- = m | N_n^- = i, J_n^- = k\}$$
$$:= \begin{cases} \mathbf{A}_{i,j}, & k = m = 0, \\ \mathbf{B}_{i,j}, & k = 0, m = 1, \\ \mathbf{C}_{i,j}, & k = 1, m = 0, \\ \mathbf{D}_{i,j}, & k = m = 1. \end{cases}$$

Now we develop the transition probabilities of the two-dimensional Markov chain $\{(N_n^-, J_n^-), n \geq 0\}$.

Observing the state of the system at two consecutive embeded-ded points, we have the one step transition probability matrix (TPM) $\mathscr{P}$ of dimension $(2N+2) \times (2N+2)$ having four block matrices of the form:

$$\mathscr{P} = \begin{bmatrix} \mathbf{A}_{(N+1)\times(N+1)} & \mathbf{B}_{(N+1)\times(N+1)} \\ \mathbf{C}_{(N+1)\times(N+1)} & \mathbf{D}_{(N+1)\times(N+1)} \end{bmatrix}.$$

Based on the single working vacation and partial rejection policy, through probabilistic arguments, some entries $P_{(i,k)(j,m)}$ of the TPM $\mathscr{P}$ are given as follows:
(1) For $0 \leq i \leq N, j = 1, 2, \cdots, N$, the transition from $(i, 0)$ to $(j, 0)$ occurs only if the residual vacation time is greater than an inter-arrival time and there are $\min\{i + X - j, N - j\}$ service completions during an inter-arrival time. So, we have

$$\begin{aligned} \mathbf{A}_{i,j} = & \sum_{k=\max\{j-i,1\}}^{N-i} \chi_k \sum_{u=\max\{1,i+k-j\}}^{\infty} a_u \times \\ & \binom{u}{i+k-j} \nu^{i+k-j} \overline{\nu}^{u-(i+k-j)} \overline{\theta}^u \\ & + \sum_{k=N-i+1}^{\infty} \chi_k \sum_{u=\max\{1,N-j\}}^{\infty} a_u \times \\ & \binom{u}{N-j} \nu^{N-j} \overline{\nu}^{u-(N-j)} \overline{\theta}^u \\ \triangleq & \sum_{k=\max\{j-i,1\}}^{N-i} \chi_k c_{i+k-j} + c_{N-j} \sum_{k=N-i+1\}}^{\infty} \chi_k. \end{aligned}$$

where $c_m = \sum_{u=\max\{1,m\}}^{\infty} a_u \binom{u}{m} \nu^m \overline{\nu}^{u-m} \overline{\theta}^u, m \geq 0$.

(2) For $0 \leq i \leq N$, there are two possible cases to cause the transition from $(i, 0)$ to $(0, 0)$. Case 1: if the residual vacation time is longer than one inter-arrival time, more than $\min\{i + X, N\}$ customers can be served during the inter-arrival time. Case 2: if the residual vacation time is not longer than one inter-arrival time, all $\min\{i + X, N\}$ customers are served during the inter-arrival time and the next arrival occurs during another vacation period–$j (j = 0, 1, \cdots, \min\{i + X, N\} - 1)$ completions before the working vacation ends and $\min\{i + X, N\} - j$ service completions during a busy period after the working vacation ends. Therefore,

$$\begin{aligned} & \mathbf{A}_{i,0} \\ & = \sum_{l=1}^{N-i} \chi_l \Big[\sum_{u=i+l}^{\infty} a_u \overline{\theta}^u \sum_{k=i+l}^{u} \binom{k-1}{i+l-1} \nu^m \overline{\nu}^{k-(i+l)} + \\ & \sum_{u=k}^{\infty} \sum_{j=0}^{i+l-1} \sum_{k=\max\{1,j\}}^{u-(i+l-j)} a_u \theta \overline{\theta}^{k-1} \binom{k}{j} \nu^j \overline{\nu}^{k-j} \times \\ & \quad \sum_{n=i+l-j}^{u-k} \binom{n-1}{i+l-j-1} \mu^{i+l-j} \overline{\mu}^{n-(i+l-j)} \overline{\theta}^{u-k-n} \Big] \\ & + \sum_{l=N-i+1}^{\infty} \chi_l \Big[\sum_{u=N}^{\infty} a_u \overline{\theta}^u \sum_{k=N}^{u} \binom{k-1}{N-1} \nu^m \overline{\nu}^{k-(N)} \\ & + \sum_{u=k}^{\infty} \sum_{j=0}^{N-1} \sum_{k=\max\{1,j\}}^{u-(N-j)} a_u \theta \overline{\theta}^{k-1} \binom{k}{j} \nu^j \overline{\nu}^{k-j} \times \\ & \quad \sum_{n=N-j}^{u-k} \binom{n-1}{N-j-1} \mu^{N-j} \overline{\mu}^{n-(N-j)} \overline{\theta}^{u-k-n} \Big] \\ & \triangleq \sum_{l=1}^{N-i} \chi_l (\beta_{i+l} + \gamma_{i+l}) + (\beta_N + \gamma_N) \sum_{k=N-i+1}^{\infty} \chi_k, \end{aligned}$$

where

$$\beta_m = \sum_{u=m}^{\infty} a_u \overline{\theta}^u \sum_{k=m}^{u} \binom{k-1}{m-1} \nu^m \overline{\nu}^{k-m}, m \geq 1,$$

$$\begin{aligned} \gamma_m = & \sum_{u=k}^{\infty} a_u \sum_{j=0}^{m-1} \sum_{k=\max\{1,j\}}^{u-(m-j)} \theta \overline{\theta}^{k-1} \binom{k}{j} \nu^j \overline{\nu}^{k-j} \times \\ & \sum_{n=m-j}^{u-k} \binom{n-1}{m-j-1} \mu^{m-j} \overline{\mu}^{n-(m-j)} \overline{\theta}^{u-k-n}, m \geq 1. \end{aligned}$$

(3) For $0 \leq i \leq N, j = 1, 2, \cdots, N$, the transition from $(i, 0)$ to $(j, 1)$ occurs if if the working vacation time is not longer than an inter-arrival time and there are $\min\{i+X, N\} - j$ service completions during one inter-arrival time given that working vacation ends and the busy period is going on: there are $m (m = 0, 1, 2, \cdots, \min\{i + X, N\} - j)$ service completions during a working vacation time and $\min\{i + X, N\} - j - m$ service completions during a normal busy period. Then, we have

$$\begin{aligned} & \mathbf{B}_{i,j} \\ & = \sum_{l=\max\{1,j-i\}}^{N-i} \chi_l \sum_{u=\max\{1,i+l-j\}}^{\infty} a_u \sum_{k=0}^{i+l-j} \sum_{m=\max\{1,k\}}^{u} \theta \overline{\theta}^{m-1} \\ & \times \binom{m}{k} \nu^k \overline{\nu}^{m-k} \binom{u-m}{i+l-j-k} \mu^{i+l-j-k} \overline{\mu}^{u-m-(i+l-j-k)} \\ & + \sum_{k=N-i+1}^{\infty} \chi_k \sum_{u=\max\{1,N-j\}}^{\infty} a_u \sum_{k=0}^{N-j} \sum_{m=\max\{1,k\}}^{u} \theta \overline{\theta}^{m-1} \times \\ & \binom{m}{k} \nu^k \overline{\nu}^{m-k} \binom{u-m}{N-j-k} \mu^{N-j-k} \overline{\mu}^{u-m-(N-j-k)} \\ & \triangleq \sum_{l=\max\{1,j-i\}}^{N-i} \chi_l d_{i+l-j} + d_{N-j} \sum_{l=N-i+1}^{\infty} \chi_l, \end{aligned}$$

where

$$d_m=\sum_{u=\max\{1,m\}}^{\infty}a_u\sum_{j=0}^{m}\sum_{k=\max\{1,j\}}^{u}\theta\overline{\theta}^{k-1}\binom{k}{j}\nu^j\overline{\nu}^{k-j}\times\binom{u-k}{m-1}\mu^{m-j}\overline{\mu}^{u-k-(m-j)},m\geq 0,$$

Using the sum of all elements of each row of the TPM, we have

$$B_{i,0}=1-\sum_{j=0}^{N}\mathbf{A}_{i,j}-\sum_{j=1}^{N}\mathbf{B}_{i,j}.$$

(4) For $1\leq i\leq N$, the transition from $(i,1)$ to $(0,0)$ occurs only if during an inter-arrival time all $\min\{i+X,N\}$ customers are served by the normal rate μ and then the newly vacation doesn't end. Then, we have:

$$\mathbf{C}_{i,0}=\sum_{l=1}^{N-i}\chi_l\sum_{u=i+l}^{\infty}a_u\sum_{k=i+l}^{u}\binom{k-1}{i+l-1}\mu^{i+l}\overline{\mu}^{k-(i+l)}\overline{\theta}^{u-k}+\sum_{l=N-i+1}^{\infty}\chi_l\sum_{u=N}^{\infty}a_u\sum_{k=N}^{u}\binom{k-1}{N-1}\mu^N\overline{\mu}^{k-N}\overline{\theta}^{u-k}\triangleq\sum_{l=1}^{N-i}\chi_l\alpha_{i+l}+\alpha_N\sum_{l=N-i+1}^{\infty}\chi_l,$$

where

$$\alpha_m=\sum_{u=m}^{\infty}a_u\sum_{k=m}^{u}\binom{k-1}{m-1}\mu^m\overline{\mu}^{k-m}\overline{\theta}^{u-k},m\geq 1.$$

(5) We should note that for $0\leq i\leq N,1\leq j\leq N$, the transition from $(i,1)$ to $(j,0)$ is an impossible event, then

$$\mathbf{C}_{i,j}=0.$$

(6) For $0\leq i\leq N,1\leq j\leq N$, the transition from $(i,1)$ to $(j,1)$ occurs only if during an inter-arrival time there are $\min\{i+X,N\}-j$ service completions by the normal service rate μ. Then, we have:

$$\mathbf{D}_{i,j}=\sum_{l=\max\{1,j-i\}}^{N-i}\chi_l\sum_{u=\max\{1,i+l-j\}}^{\infty}a_u\binom{u}{i+l-j}\times\mu^{i+l-j}\overline{\mu}^{u-(i+l-j)}+\sum_{l=N-i+1}^{\infty}\chi_l\sum_{u=N}^{\infty}a_u\binom{u}{N}\mu^N\overline{\mu}^{u-N}\triangleq\sum_{l=\max\{1,j-i\}}^{N-i}\chi_l b_{i+l-j}+b_{N-j}\sum_{l=N-i+1}^{\infty}\chi_l,$$

where

$$b_m=\sum_{u=\max\{1,m\}}^{\infty}a_u\binom{u}{m}\mu^m\overline{\mu}^{u-m},m\geq 0.$$

Utilizing the sum of all elements of each row of the TPM leads to

$$D_{i,0}=1-\mathbf{C}_{i,0}-\sum_{j=1}^{N}\mathbf{D}_{i,j}.$$

Let $\mathbf{P}^-=(P^-_{0,0},P^-_{1,0},P^-_{2,0},\cdots,P^-_{N,0},P^-_{1,1},P^-_{2,1},\cdots,P^-_{N,1})$ be the row vector of the pre-arrival epoch probabilities which can be obtained by solving $\mathbf{P}^-\mathscr{P}=\mathbf{P}^-$. The system of equations has been solved using the algorithm of GTH (Grassmann,Taksar and Heyman) given in Latouche and Ramaswami (1999).

C. Distribution of system size at outside observer's observation epoch

The distribution of system size at outside observer's observation epoch is needed to evaluate average sojourn time in the system using Little's rule. In an early arrival system, the outside observer's observation point falls in a time interval after a potential arrival and before a potential departure. Let $P^o_{i,0}(0\leq i\leq N)$ and $P^o_{i,1}(1\leq i\leq N)$ denote the probabilities that outside observer sees i customers in the system and the server on working vacation and in the busy period, respectively. By observing arbitrary and outside observer's observation epochs presented in Figure 1, we have

$$P_{0,0}=\overline{\theta}P^o_{0,0}+\overline{\theta}\nu P^o_{1,0}+\mu P^o_{1,1},$$
$$P_{i,0}=\overline{\theta}\overline{\nu}P^o_{i,0}+\overline{\theta}\nu P^o_{i+1,0},1\leq i\leq N-1,$$
$$P_{N,0}=\overline{\theta}\overline{\nu}P^o_{N,0},$$
$$P_{0,1}=\theta\nu P^o_{1,0}+\theta P^o_{0,0}+P^o_{0,1},$$
$$P_{i,1}=\overline{\mu}P^o_{i,1}+\mu P^o_{i+1,1}+\theta\overline{\nu}P^o_{i,0}+\theta\nu P^o_{i+1,0},1\leq i\leq N-1,$$
$$P_{N,1}=\overline{\mu}P^o_{N,1}+\theta\overline{\nu}P^o_{N,0}.$$

From the above equations, we can obtain

$$P^o_{N,0}=\frac{1}{\overline{\theta}\overline{\nu}}P_{N,0},$$
$$P^o_{i,0}=\frac{1}{\overline{\theta}\overline{\nu}}(P_{i,0}-\theta\nu P^o_{i+1,0}),1\leq i\leq N-1,$$
$$P^o_{N,1}=\frac{1}{\overline{\mu}}(P_{N,1}-\theta\overline{\nu}P_{N,0}),$$
$$P^o_{i,1}=\frac{1}{\overline{\mu}}\left(P_{i,1}-\mu P^o_{i+1,1}-\theta\overline{\nu}P^o_{i,0}-\theta\nu P^o_{i+1,0}\right),\ 1\leq i\leq N-1,$$
$$P^o_{0,0}=\frac{1}{\overline{\theta}}(P_{0,0}-\overline{\theta}\nu P^o_{1,0}-\mu P^o_{1,1}),$$
$$P^o_{0,1}=P_{0,1}-\theta\nu P^o_{1,0}-\theta P^o_{0,0}.$$

IV. Performance Measures

As steady-state probabilities at various epochs are known, various performance measures can easily be computed.

A. Blocking probabilities

An important performance measure of a finite buffer batch arrival single server queueing system is the blocking probability of the first-, an arbitrary- and the last- customer of an arriving batch. Denoting the three kinds of blocking probabilities as P_{BF},P_{BA} and P_{BL} respectively, then we can determine the three kinds of probabilities from section 3.2 as follows:

$$P_{BF} = P^-_{N,0} + P^-_{N,1},$$
$$P_{BA} = \sum_{i=0}^{N}(P^-_{i,0} + P^-_{i,1}) \sum_{j=N-i}^{\infty} r_{j+1},$$
$$P_{BL} = \sum_{i=0}^{N}(P^-_{i,0} + P^-_{i,1}) \sum_{j=N-i+1}^{\infty} \chi_j,$$

where $r_j = P(X \geq j)/\overline{\chi}$ is the probability that an arbitrary customer C occupies position j in its batch, $j = 1, 2, \cdots$.

B. Mean system length at outside observer¡¯s observation epoch

Three kinds of mean system length at outside observer's observation epoch are given respectively as follows:
(1) The average system length:

$$L^o_s = \sum_{i=1}^{N} iP^o_{i,0} + \sum_{i=1}^{N} iP^o_{i,1}.$$

(2)The average system length when the server is in normal busy period:

$$L^o_1 = \sum_{i=1}^{N} iP^o_{i,1}.$$

(3) The average system length when the server is on working vacation:

$$L^o_0 = \sum_{i=1}^{N} iP^o_{i,0}.$$

Let W_{SA} denote the average sojourn time in the system of an arbitrary customer in a batch which is accepted upon arrival. Then by Little's rule $W_{SA}|_{Little} = L^o_s/\lambda'$, where $\lambda' = \lambda\overline{\chi}(1 - P_{BA})$, $W_{SA}|_{Little}$ denotes the average sojourn time W_{SA} evaluated through Little's rule.

C. Sojourn time analysis

In this subsection, we discuss sojourn time in the system of the first customer (respectively, an arbitrary customer and the last customer) that is put in the queue of an arriving batch which is accepted under the FCFS service discipline and partially rejected policy. The sojourn time of the first customer (respectively, an arbitrary customer and the last customer) that is put in the queue of an arriving batch is given by the interval from the instant at which it enters in the system to the instant when it departs the system after its service completion and is denoted by T_{SF} (respectively, T_{SA} and T_{SL}). The mean values and the z-transforms of the random variables T_{SF}, T_{SA} and T_{SA} are, respectively, represented as W_{SF} and $\widetilde{W}_{SF}(z)$, W_{SA} and $\widetilde{W}_{SA}(z)$, W_{SL} and $\widetilde{W}_{SL}(z)$.

Now we introduce one lemma without proof, which will be used during the derivations of our main results.

Lemma1 If $\{S^v_k, k \geq 1\}$ and V are defined as in Sec.2 and independent mutually, for $k \geq 1$, we have the following results:

$$P\left(\sum_{j=1}^{k} S^v_j \leq V\right) = \frac{1}{\overline{\theta}}\left(\frac{\nu\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\right)^k,$$
$$E\left[z^{\sum_{j=1}^{k} S^v_j}\middle|\sum_{j=1}^{k} S^v_j \leq V\right] = \left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^k.$$

Lemma2 If $\{S^v_k, k \geq 1\}$ and V are defined as in Sec.2 and independent mutually, for $k \geq 0$, we have the following results:

$$P\left(\sum_{j=1}^{k} S^v_j \leq V < \sum_{j=1}^{k+1} S^v_j\right) = \begin{cases} \frac{\theta\overline{\nu}}{1-\overline{\theta}\overline{\nu}}, & k = 0, \\ \frac{1}{\overline{\theta}}\frac{\theta}{1-\overline{\theta}\overline{\nu}}\left(\frac{\nu\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\right)^k, & k \geq 1. \end{cases}$$
$$E\left[z^V\middle|\sum_{j=1}^{k} S^v_j \leq V < \sum_{j=1}^{k+1} S^v_j\right] = \begin{cases} \frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}, & k = 0, \\ \frac{1-\overline{\theta}\overline{\nu}}{1-\overline{\theta}\overline{\nu}z}\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^k, & k \geq 1. \end{cases}$$

Considering various possible cases and using the above two lemmas, the expressions of $\widetilde{W}_{SF}(z)$, $\widetilde{W}_{SA}(z)$ and $\widetilde{W}_{SL}(z)$ are given below:

$$\begin{aligned}
&\widetilde{W}_{SF}(z) \\
&= \frac{1}{1-P_{BF}}\left\{\sum_{i=0}^{N-1} P^-_{i,0}\left[P\left(\sum_{j=1}^{i+1} S^v_j \leq V\right)\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+1}\right.\right. \\
&\quad + P\left(V < S^v_1\right)\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1} \\
&+ \sum_{k=1}^{i} P\left(\sum_{j=1}^{k} S^v_j \leq V < \sum_{j=1}^{k+1} S^v_j\right) \\
&\quad \left.\times \frac{(1-\overline{\theta}\overline{\nu})}{1-\overline{\theta}\overline{\nu}z}\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^k\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1-k}\right] \\
&\quad \left. + \sum_{i=0}^{N-1} P^-_{i,1}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1}\right\} \\
&= \frac{1}{1-P_{BF}}\left\{\sum_{i=0}^{N-1} P^-_{i,0}\left[\frac{1}{\overline{\theta}}\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+1} + \frac{\theta\overline{\nu}z}{1-\overline{\theta}\overline{\nu}z}\times\right.\right. \\
&\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1} + \left.\sum_{k=1}^{i}\frac{1}{\overline{\theta}}\frac{\theta}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^k\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1-k}\right] \\
&\quad \left. + \sum_{i=0}^{N-1} P^-_{i,1}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+1}\right\},
\end{aligned}$$

$$
\begin{aligned}
&\widetilde{W}_{SA}(z)\\
&= \frac{1}{1-P_{BF}}\left\{\sum_{i=0}^{N-1} P_{i,0}^{-} \sum_{m=0}^{N-1-i} r_{m+1}\times\right.\\
&\left[P\left(\sum_{j=1}^{i+m+1} S_j^v \le V\right)\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+m+1}\right.\\
&+P\left(V<S_1^v\right)\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1}\\
&+\sum_{k=1}^{i+m} P\left(\sum_{j=1}^{k} S_j^v \le V < \sum_{j=1}^{k+1} S_j^v\right)\\
&\left.\times\frac{(1-\overline{\theta}\overline{\nu})}{1-\overline{\theta}\overline{\nu}z}\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^{k}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1-k}\right]\\
&\left.+\sum_{i=0}^{N-1} P_{i,1}^{-}\sum_{m=0}^{N-1-i} r_{m+1}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1}\right\}\\
&=\frac{1}{1-P_{BF}}\left\{\sum_{i=0}^{N-1} P_{i,0}^{-}\sum_{m=0}^{N-1-i} r_{m+1}\left[\frac{1}{\overline{\theta}}\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+m+1}\right.\right.\\
&+\frac{\theta\overline{\nu}z}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1}+\sum_{k=1}^{i+m}\frac{1}{\overline{\theta}}\frac{\theta}{1-\overline{\theta}\overline{\nu}z}\\
&\left.\times\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^{k}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1-k}\right]\\
&\left.+\sum_{i=0}^{N-1} P_{i,1}^{-}\sum_{m=0}^{N-1-i} r_{m+1}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m+1}\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\widetilde{W}_{SL}(z)=\frac{1}{1-P_{BL}}\times\\
&\left\{\sum_{i=0}^{N-1}P_{i,0}^{-}\sum_{m=1}^{N-i}\chi_m\left[P\left(\sum_{j=1}^{i+m}S_j^v\le V\right)\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+m}\right.\right.\\
&+P\left(V<S_1^v\right)\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m}\\
&+\sum_{k=1}^{i+m-1}P\left(\sum_{j=1}^{k}S_j^v\le V<\sum_{j=1}^{k+1}S_j^v\right)\\
&\left.\times\frac{(1-\overline{\theta}\overline{\nu})}{1-\overline{\theta}\overline{\nu}z}\left(\frac{(1-\overline{\theta}\overline{\nu})z}{1-\overline{\theta}\overline{\nu}z}\right)^{k}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m-k}\right]\\
&\left.+\sum_{i=0}^{N-1}P_{i,1}^{-}\sum_{m=0}^{N-i}\chi_m\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m}\right\}\\
&=\frac{1}{1-P_{BF}}\left\{\sum_{i=0}^{N-1}P_{i,0}^{-}\sum_{m=1}^{N-i}\chi_m\left[\frac{1}{\overline{\theta}}\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^{i+m}\right.\right.\\
&+\frac{\theta\overline{\nu}z}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m}+\sum_{k=1}^{i+m-1}\frac{1}{\overline{\theta}}\frac{\theta}{1-\overline{\theta}\overline{\nu}z}\left(\frac{\nu\overline{\theta}z}{1-\overline{\theta}\overline{\nu}z}\right)^{k}\\
&\left.\left.\times\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m-k}\right]+\sum_{i=0}^{N-1}P_{i,1}^{-}\sum_{m=1}^{N-i}\chi_m\left(\frac{\mu z}{1-\overline{\mu}z}\right)^{i+m}\right\},
\end{aligned}
$$

Now the mean sojourn times W_{SF}, W_{SA} and W_{SL} can be easily obtained and are given, respectively, as follows:

$$
\begin{aligned}
&W_{SF}=\frac{1}{1-P_{BF}}\times\\
&\left\{\sum_{i=0}^{N-1}P_{i,0}^{-}\left[\frac{i+1}{\nu\overline{\theta}^2}\left(\frac{\nu\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\right)^{i+2}+\frac{\theta\overline{\nu}[(i+1)(1-\overline{\theta}\overline{\nu})+\mu]}{\mu(1-\overline{\theta}\overline{\nu})^2}\right.\right.\\
&+\sum_{k=1}^{i}\left(\frac{\theta\overline{\theta}^{k-1}\nu^k[(i+1)(1-\overline{\theta}\overline{\nu})+(k+1)\overline{\theta}\overline{\nu}]}{(1-\overline{\theta}\overline{\nu})^{k+2}}\right.\\
&\left.\left.\left.+\frac{(i+1-k)\theta\overline{\theta}^{k-1}\nu^k\overline{\mu}}{\mu(1-\overline{\theta}\overline{\nu})^{k+1}}\right)\right]+\sum_{i=0}^{N-1}P_{i,1}^{-}\frac{i+1}{\mu}\right\},\\
&W_{SA}=\frac{1}{1-P_{BA}}\times\\
&\left\{\sum_{i=0}^{N-1}P_{i,0}^{-}\sum_{m=0}^{N-1-i}r_{m+1}\left[\frac{i+m+1}{\nu\overline{\theta}^2}\left(\frac{\nu\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\right)^{i+m+2}\right.\right.\\
&+\frac{\theta\overline{\nu}[(i+m+1)(1-\overline{\theta}\overline{\nu})+\mu]}{\mu(1-\overline{\theta}\overline{\nu})^2}\\
&+\sum_{k=1}^{i+m}\left(\frac{\theta\overline{\theta}^{k-1}\nu^k[(i+m+1)(1-\overline{\theta}\overline{\nu})+(k+1)\overline{\theta}\overline{\nu}]}{(1-\overline{\theta}\overline{\nu})^{k+2}}\right.\\
&\left.\left.+\frac{(i+m+1-k)\theta\overline{\theta}^{k-1}\nu^k\overline{\mu}}{\mu(1-\overline{\theta}\overline{\nu})^{k+1}}\right)\right]\\
&\left.+\frac{1}{\mu}\sum_{i=1}^{N-1}P_{i,1}^{-}\sum_{m=0}^{N-1-i}r_{m+1}(i+m+1)\right\},
\end{aligned}
\qquad (27)
$$

$$
\begin{aligned}
&W_{SL}=\frac{1}{1-P_{BL}}\times\\
&\left\{\sum_{i=0}^{N-1}P_{k,0}^{-}\sum_{m=1}^{N-i}\chi_m\left[\frac{i+m}{\nu\overline{\theta}^2}\left(\frac{\nu\overline{\theta}}{1-\overline{\theta}\overline{\nu}}\right)^{i+m+1}+\right.\right.\\
&\frac{\theta\overline{\nu}[(i+m+1)(1-\overline{\theta}\overline{\nu})+\mu]}{\mu(1-\overline{\theta}\overline{\nu})^2}\\
&+\sum_{k=1}^{i+m-1}\left(\frac{\theta\overline{\theta}^{k-1}\nu^k[(i+m)(1-\overline{\theta}\overline{\nu})+(k+1)\overline{\theta}\overline{\nu}]}{(1-\overline{\theta}\overline{\nu})^{k+2}}\right.\\
&\left.\left.+\frac{(i+m-k)\theta\overline{\theta}^{k-1}\nu^k\overline{\mu}}{\mu(1-\overline{\theta}\overline{\nu})^{k+1}}\right)\right]\\
&\left.+\frac{1}{\mu}\sum_{i=1}^{N-1}P_{k,1}^{-}\sum_{m=1}^{N-k}\chi_m(i+m)\right\}.
\end{aligned}
$$

Remark It may be noted here that the numerical value of the mean sojourn time in the system of an accepted arbitrary customer of an arriving batch evaluated through (27) matches exactly with the one obtained earlier using Little's rule, as it should be.

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