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A Fuzzy Mixed Integer Multi-Scenario Portfolio Optimization Model

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***Abstract*—**In this paper, we propose a multiple objective optimization model with respect to portfolio selection problem for investors looking forward to diversify their equity investments in a number of equity markets. Based on Markowitz's M-V model we developed a Fuzzy Mixed Integer Multi-Objective Nonlinear Programming Problem (FMIMONLP) to maximize the investors' future gains on equity markets, reach the optimal proportion of the budget to be invested in different equities. A numerical example with a comprehensive analysis on artificial data from several equity markets is presented in order to illustrate the proposed model and its solution method. The model performed well compared with the deterministic version of the model.

***Keywords*—**Equity Markets, Future Scenarios, Portfolio Selection, Multiple Criteria Fuzzy Optimization

I. Introduction

PORTFOLIO Selection is how to configure a variety of equities positions to best meet the decision makers (DMs) risk and return trade-off. In 1952, Markowitz the founder of modern portfolio theory assumed DMs are risk averse, and variance is a measure for investment risk; a Mean-Variance portfolio selection MV-Model is established by Markowitz [1] who assumed it is necessary to calculate the covariance between the risky assets, giving the model to calculate the actual operation has brought difficulties.

DMs can efficiently allocate their capitals through potential portfolio diversification to include a number of multi-national risky equities that have several fuzzy returns or short-term holding periods[1]. The fuzzy returns result in conflicting future alternatives.If equity returns related to one scenario are one of those conflicting alternatives, then a multi criteria mathematical portfolio program can be developed which considers different scenarios in order to maximize the portfolio future returns and arrive at the net capital gain. This is to be achieved through an equity portfolio aiming to reach ultimate goal of preserving and generating wealth from a number of equity markets that the DM selects from.

II. Problem Definition

After the financial crisis, all equity markets collapsed during the year 2008 and rebounded in 2009. However some of the equity markets' investors are still a very far from understanding the logic of the markets, if markets commonly have bubbles, then investors can efficiently allocate their capitals through potential portfolio diversification to include a number of multi-national risky equities that have several fuzzy returns. Moreover, to mitigate the risks of equity allocation within a portfolio, they shall diversify their holdings in several markets.

Possibility portfolio models were initially proposed in Tanaka and Guo [2],[3] where portfolio models are based on exponential possibility distributions, rather than the mean-variance form in Markowitz's model that regards the portfolio selection as a probability phenomenon, possibility portfolio models integrate the past equities and experts' judgments to catch variations to equity markets more plausibly [8], those researchers' effort has been developed by others, such as: Zhang [4]; Lacagnina [5]; Takashi, Ishii [6],[7]; Chen, G., [8],[9], and others [6],[10], and [11].

On the other hand, a Mixed Integer Multi Objective Non Linear Programming (MIMONLP) is an efficient technique to model and solve decision problems in which several conflicting and incommensurable objectives are to be optimized simultaneously subject to specified constraints [12].

MONLP model with fuzzy parameters in its objective functions and/or constraints is called a Fuzzy MONLP problem. For the reason that the values of the parameters in a MONLP model are often imprecisely or ambiguously understood to the experts it may be more appropriate to interpret the experts' understanding of these parameters as fuzzy values. Moreover, this could be more appropriate than modeling the MONLP problem where there are random values in the model parameters [5].

In [2],[13] the authors supposed a few models considering the future scenario with fuzzy returns and multi-objective programming problem or even portfolio multi-objective optimization, while according to our comprehensive survey there are no one develops a fuzzy model regarding portfolio considering several equity markets' diversification. Through a survey, to a considerable extend, we could not conceive any research in multi objective models considering multi markets.

However, considering the psychology of DMs to diversify capital in several markets, and the uncertainty of given information, since it is difficult to predict important factors either decision parameters [e.g. Return on Equities, Maximum Portfolio Tolerance, Return associated to particular Scenario, number of shares per equity (Quantities or Volume), and others] or decision variables (i.e. The proportion of the total investment devoted to equity bought/sold by DM onto a particular scenario) Hence, the future return, and other parameters of future scenario can be adopted. Here, the problem is to maximize the fuzzy returns in the future

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[1] Duration between an equity's purchase and its sale

scenarios which are often in conflict with each other, where the total risk is considered fuzzy, considering several equities from several equity markets. The portfolio problem has to be converted into a Mixed Integer Nonlinear Multi-objective Programming Problem Model.

In the rest of this paper, a fuzzy multi-objective portfolio optimization model is established, and a solution method shows the utilization of fuzzy mathematics into the multi-objective fuzzy nonlinear portfolio program; finally, we gave a numerical example with comparative analysis.

III. The Framework of the Portfolio Problem

The Dimensional Space for the Portfolio Equity Market $(DSPEM)$, consists of $\{(x_{1111}, \dots, x_{1mce}, \dots, x_{2mce}, \dots, x_{Smce}), m = 1, \dots, NMRK, c = 1, \dots, NCAT_m, e = 1, \dots, NEQT_{cm}, s = 1, \dots, S\} \subseteq \mathbb{R}^{s \times m \times c \times e}$, and x_{smce} is the e^{th} equity in the c^{th} industrial category in the m^{th} market in scenario (s), Where (S) is the maximum number of scenarios, $NMRK$, is the maximum number of markets, $NCAT_m$, is the maximum number of industrial categories per market (m), and the maximum number of equities per category (c) at market (m) is $NEQT_{cm}$. As in Fig. 1, for example, we have $DSPEM$ for two scenarios, two markets, with different structure of industrial categories.

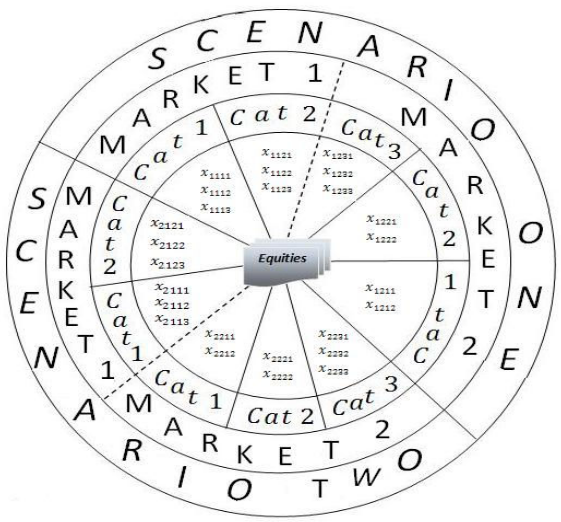

Fig. 1 A diagram shows $DSPEM$ for two scenarios, two markets, with different structure of industrial categories

In the rest of this section we define the list of decision variables, parameters and spotting on conditions that shall be satisfied.

A. List of Decision Variables:

We describe four types of decision variables as follows:

x^+_{smce}: The proportion of the total investment in future scenario devoted to the e^{th} equity bought in the c^{th} industrial category in the m^{th} market in scenario (s), $x^+ = (x^+_{1111}, \dots x^+_{smce})$ is the decision vector consist of $s \cdot m \cdot c \cdot e$ decision variables corresponding to equities bought.

x^-_{smce}: The proportion of the total investment in future scenario devoted to the e^{th} equity sold in the c^{th} industrial category in the m^{th} market in scenario (s), $x^- = (x^-_{1111}, \dots x^-_{smce})$, is the decision vector consist of $s \cdot m \cdot c \cdot e$ decision variables corresponding to equities sold.

τ_s: The fraction between budget and total amount paid for investments, regarding the number of shares for the equities traded in scenario (s), that has to be minimized, and $\tau_s \leq \varepsilon_s, \ \forall \ s = 1,2, \dots, S.$

$\tilde{Q}^+_{smce}$: A decision variable representing the number of shares devoted to e^{th} equity bought in the c^{th} industrial category in the m^{th} market at scenario (s), suppose that the *volume of shares* traded is the total number of a listed equity's shares that the DM is expected to be traded on, expressed as $(\tilde{Q}^+_{smce})$, and they are non-negative Integers.

$$\tilde{Q}^+_{smce} = \frac{\tilde{B}_s \cdot x^+_{smce}}{\tilde{a}^+_{smce}} \ \forall \ \tilde{Q}^+_{smce} \geq 0, \quad \& \ \ Integer,$$

$\tilde{Q}^-_{smce}$: A decision variable representing the number of shares devoted to e^{th} equity sold in the c^{th} industrial category in the m^{th} market at scenario (s), suppose that the *volume of shares* traded is the total number of a listed equity's shares that the DM is expected to be traded on, expressed as $(\tilde{Q}^-_{smce})$, and they are non-negative Integers.

$$\tilde{Q}^-_{smce} = \frac{\tilde{B}_s \cdot x^-_{smce}}{\tilde{a}^-_{smce}} \ \forall \ \tilde{Q}^-_{smce} \geq 0, \quad \& \ \ Integer,$$

B. The List of calculated –Decision Variables:

x^+_{mce}, x^-_{mce}: The proportion of investment devoted to the e^{th} equity bought/sold respectively in the c^{th} industrial category in the m^{th} market in the whole portfolio, where $x^+_{mce} = \sum_{s=1}^{S} x^+_{smce}$, and $x^-_{mce} = \sum_{s=1}^{S} x^-_{smce}$,

C. List of Decision Parameters:

α_s: The Weights expressing the probability of each scenario (s) to be occurred in the portfolio, $0 \leq \alpha_s \leq 1$, Where $\sum_{s=1}^{S} \alpha_s = 1 \ \forall \ s = 1,2, \dots, S,$

k_{smce} : The *Transaction Cost* [2]per the e^{th} equity (bought /sold) in the c^{th} industrial category in the m^{th} market in scenario (s), and *Transaction Costs* has to be paid for both bought/sold Transactions on the equity. However, transaction costs for any market are non-fuzzy numbers.

k_{mce}: The *Transaction* Cost per *the* e^{th} equity (bought/sold) in the c^{th} industrial category in the m^{th} market,

$\tilde{p}^+_{smce}$: The price of proportion of total *investment* devoted to the e^{th} equity bought in the c^{th} industrial category in the m^{th} market in scenario (s),

$\tilde{p}^-_{smce}$: The price of *proportion* of total investment devoted to the e^{th} equity sold in the c^{th} industrial category in the m^{th} market in scenario (s),

$\tilde{B}_s$: The *Capital Budget* for investments devoted to scenario (s) including *Trans. Costs*,

$\tilde{B}$: The *Capital Budget including* transaction costs for the

[2] Transaction costs are fixed for the long term per market, according to the *Markets' Capital Market Association* rules.

whole portfolio investments, and should satisfy the following condition $\sum_{s=1}^{S} \tilde{B}_s \leq \tilde{B}$,

ε_s: The *minimum* price can be used to buy a one share from the equities have been traded on in the scenario (s).

$\tilde{\omega}$: The Maximum Available Risk accepted by the DM,

$\widetilde{NR}_{sm}$: The *minimum* Net Return Ratio for the scenario (s), that is calculated with respect to the budget assigned to the market (m) in scenario (s),

$\widetilde{NR}_s$: The minimum Net Return Ratio for scenario (s), and should satisfy the following condition: $\sum_{m=1}^{NCAT_m} \alpha_s \cdot \widetilde{NR}_{sm} \leq \widetilde{NR}_s$, $\forall\ 1 \leq s \leq S$.

ρ: The minimal total revenue for portfolio that is satisfied by the DM, it is a proportion from invested budget, and should satisfy the following condition $\sum_{s=1}^{S} \alpha_s \cdot \widetilde{NR}_s + B \leq \rho$,

$C_s(x^+, x^-)$: The number of total *Transaction Costs* carried by DM for all Equities included in scenario (s),

u^+_{smce}, u^-_{smce}: The upper pounds of x^+_{smce}, x^-_{smce} corresponding to proportion of *Investment* devoted to the e^{th} equity bought / sold respectively, in the c^{th} industrial category in the m^{th} market in scenario (s),

l^+_{smce}, l^-_{smce}: The lower pounds of x^+_{smce}, x^-_{smce} corresponding to proportion of *Investment* devoted to the e^{th} equity bought or sold respectively, in the c^{th} industrial category in the m^{th} market in scenario (s),

$\tilde{\theta}_{smce}$: The maximum available proportion of the total investment devoted to x^+_{smce}, equity bought by DM through scenario (s) to be sold by DM,

D.Pre-Preparational Calculations:

k_{mce}: The weighted sum of all the transaction costs for the e^{th} equity (bought/sold) in the c^{th} industrial category in the m^{th} market at scenario (s), $k_{mce} = \sum_{s=1}^{S} \alpha_s \cdot k_{smce}$,

k_{sm}: The Transaction Cost for market m at scenario (s), $k_{sm} = \alpha_s \cdot \sum_{c=1}^{NCAT_m} \sum_{e=1}^{NEQT_{cm}} k_{smce}$, $1 \leq s \leq S$ & $m = 1, \dots, NMRK$,

k_s : The Transaction Cost associated to scenario (s), $k_s = \sum_{m=1}^{NMRK} k_{sm}$, *and* $1 \leq s \leq S$.

$\tilde{a}^+_{smce}$: The total fund for investments devoted to the e^{th} equity bought in the c^{th} industrial category in the m^{th} market in scenario (s), and can be expressed as the sum of the price devoted to the e^{th} equity bought in the c^{th} industrial category in the m^{th} market in scenario (s) plus the transaction cost per scenario (s), that is $\tilde{a}^+_{smce} = (1 + k_{sm}) \cdot \tilde{p}^+_{smce}$.

$\tilde{a}^-_{smce}$: The Total Fund has to be returned from investments devoted to the e^{th} equity sold in the c^{th} industrial category in the m^{th} market in scenario (s), it can be expressed in the same manner as $\tilde{a}^+_{smce}$ has been expressed, that is $\tilde{a}^-_{smce} = (1 + k_{sm}) \cdot \tilde{p}^-_{smce}$,

$\tilde{r}_{smce}$: The Rate of Return for the e^{th} equity in the c^{th} industrial category in the m^{th} market in scenario (s), Where $\tilde{r}_{smce} = \frac{\tilde{a}^-_{smce} - \tilde{a}^+_{smce}}{\tilde{a}^+_{smce}}$, assuming no Dividend Yield.

$\tilde{r}_{mce}$: The *rate of return* for the e^{th} equity in the c^{th} industrial category in the m^{th} market for the equity that x_{mce} has been calculated on, and it can be expressed as the weighted sum of all the fuzzy rate of returns for the e^{th} equity in the c^{th} industrial category in the m^{th} market for scenario(s),$\tilde{r}_{mce} = \sum_{s=1}^{S} \alpha_s \cdot \tilde{r}_{smce}$ $\forall\ m = 1,2, \dots, NMRK$, $c = 1,2, \dots, NCAT_m$, $e = 1,2, \dots, NEQT_{cm}$,

$\tilde{r}_{sm}$: The Return associated to scenario (s) at market m, $\tilde{r}_{sm} = \alpha_s \cdot \sum_{c=1}^{NCAT_m} \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{smce}$, $1 \leq s \leq S$ & $m = 1, \dots, NMRK$.

$\tilde{r}_s$: Return associated to scenario (s), where $\tilde{r}_s = \sum_{m=1}^{NMRK} \tilde{r}_{sm}$, *and* $1 \leq s \leq S$

IV. The Proposed Portfolio Selection Model

A. Objective Functions:

A several holding periods are the number of scenarios devoted to the portfolio maximum expected return for any scenario (s) that is expressed as follows:

$$Maximize\quad f(x_s) = \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{smce} \cdot (x^+_{smce} - x^-_{smce}) \qquad \forall \quad 1 \leq s \leq S,$$

Where $m = 1, \dots,\ NMRK,\ c = 1, \dots, NCAT_m,\ \&\ e = 1, \dots, NEQT_{cm}$,

B. Feasible set:

The set of constraints can be divided into two large blocks; global constraints on the portfolio (related to the whole portfolio), and possible temporal scenario constraints (effected by holding periods). Thus Constraints (C01, C04, C08, and C09) are global constraints on the portfolio, whereas Constraints (C02, C03, C05, C06, and C07) are classified as temporal scenario constraints. However constraints (C10, C11, and C12) are boundary restrictions constraints. Next each constraint are clarified.

(C01)- As in *Takashi and Ishii* [9], Lets redefine a constraint that represents the DM's satisfied *Risk in the whole portfolio*, this is as follows:

$$\sum_{s=1}^{S} \alpha_s \cdot \left(\sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NUMC_m} \cdot \sum_{e=1}^{NUME_{cm}} (\tilde{r}_{smce} - \tilde{r}_{mce}) \cdot (x^+_{smce} - x^-_{smce}) \right)^2 \leq \tilde{\omega}$$

Where $m = 1,2, \dots, NMRK$, $c = 1,2, \dots, NCAT_m$, *and* $e = 1,2, \dots, NEQT_{cm}$

(C02)- Also, it is more realistic expressing the minimum Net Return Ratio $\widetilde{NR}_{sm}$ regarding the scenario (s), that is calculated with respect to the budget assigned to the market (m), after ignoring transaction costs $C_s(x^+, x^-)$, in scenario (s), that can be constrained as follows:

$$\sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{sm} \cdot (x_{smce}^{+} - x_{smce}^{-}) - \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} k_{sm} \cdot (x_{smce}^{+} + x_{smce}^{-}) \geq \widetilde{NR}_{sm}$$

$\forall \quad s = 1,2, \dots S, \quad m = 1,2, \dots NMRK,$

(C03)- *The Minimum Net Return Ratio* for *the scenario* in the portfolio is $\widetilde{NR}_s$. Since it has been assumed that the *Minimum Net Return Ratio* for the scenario in the portfolio is calculated after paying the transaction costs for each scenario, that can be constrained as follows:

$$\sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_s \cdot (x_{smce}^{+} - x_{smce}^{-}) - \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} k_s \cdot (x_{smce}^{+} - x_{smce}^{-}) \geq \widetilde{NR}_s ,$$

Where $\tilde{r}_s = \alpha_s \cdot \sum_{m=1}^{NMRK} \tilde{r}_{sm},$

$k_s = \alpha_s \cdot \sum_{m=1}^{NMRK} k_{sm}, \qquad \forall \quad s = 1,2, \dots, S$

(C04)- Let us assume that the DM does not invest in a *Portfolio Capital* or a Budget that exceeds *The Budget Parameter* $(\tilde{B})$ in the portfolio then, we constrain the budget requirements as follows:

$$\sum_{s=1}^{S} \cdot \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} [(a_{smce}^{+} \cdot Q_{smce}^{+}) - (a_{smce}^{-} \cdot Q_{smce}^{-})] \leq \tilde{B}$$

(C05)- We express *The Budget Requirements* devoted to scenario (s) as follows:

$$\sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} [(\tilde{a}_{smce}^{+} \cdot Q_{smce}^{+}) - (\tilde{a}_{smce}^{-} \cdot Q_{smce}^{-})] \leq \tilde{B}_s \quad \forall \quad 1 \leq s \leq S$$

(C06)- It has been supposed that (τ_s) the fraction between budget, and total amount paid for investments regarding the number of shares for the equities bought traded in scenario (s), that has to be minimized, expressed as:

$$\tilde{B}_s - \left[\sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} (\tilde{a}_{smce}^{+} \cdot Q_{smce}^{+}) \right] \leq \tau_s$$

(C07)- It has been defined (ε_s) that (τ_s) cannot exceed in scenario (s) that has to be minimized, expressed as: $\tau_s \leq \varepsilon_s \quad \forall \quad 1 \leq s \leq S$

(C08)- We express (ρ) *the total Revenue for Portfolio*, that is represent a percentage of the total invested Capital Budget; then

$$\sum_{s=1}^{S} \cdot \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{smce} \cdot (x_{smce}^{+} - x_{smce}^{-}) \geq \rho,$$

Where (ρ) the minimal total revenue for portfolio that is satisfied by the DM, it is a proportion from total invested budget, and $\sum_{s=1}^{S} \alpha_s \cdot \widetilde{NR}_s + B \leq \rho$, should be satisfied, and $\tilde{r}_{smce}$ is the Rate of Return for the e^{th} equity in the c^{th} industrial category in the m^{th} market in scenario (s),

(C09)- We suppose *all funds must be invested* in equities that are available, that is expressed as:

$$\sum_{s=1}^{S} \cdot \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} x_{smce}^{+} = 1,$$

(C10)- It has been supposed that $\tilde{\theta}_{smce}$ *is the maximum available proportion of equity bought* (x_{smce}^{+}) *to be sold* (x_{smce}^{-}) by DM, it has been expressed as:
$x_{smce}^{-} \leq \tilde{\theta}_{smce} \cdot x_{smce}^{+}, \ 0 \leq \tilde{\theta}_{smce} \leq 1, \ m = 1, \dots, NMRK, \ c = 1, \dots, NCAT_m, \ \& \ e = 1, \dots, NEQT_{cm}$

(C11)- It has been supposed that $(u_{smce}^{+}, l_{smce}^{+})$ are the upper and lower bounds expressing the proportion of total investment devoted to *equities bought* expressed as follows:
$l_{smce}^{+} \leq x_{smce}^{+} \leq u_{smce}^{+}$
$\forall \ 1 \leq s \leq S, \ m = 1, \dots, NMRK, \ c = 1, \dots, NCAT_m, \ \& \ e = 1, \dots, NEQT_{cm},$

(C12)- It has been supposed that $(u_{smce}^{-}, l_{smce}^{-})$ are the upper and lower bounds expressing the proportion of total investment devoted to *equities sold* expressed as follows:
$l_{smce}^{-} \leq x_{smce}^{-} \leq u_{smce}^{-}, \forall \ 1 \leq s \leq S, \ m = 1, \dots, NMRK, \ c = 1, \dots, NCAT_m, \ \& \ e = 1, \dots, NEQT_{cm}$

Then, the Fuzzy Vector Optimization Portfolio Problem FVOP can take the form of Mixed Integer Non-Linear Fuzzy Multi-Objective, Mathematically that can be expressed as follows:

Maximize

$$f(s) = \sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{smce} \cdot (x_{smce}^{+} - x_{smce}^{-}), \quad \forall \ 1 \leq s \leq S$$

Subject To:

$$\sum_{s=1}^{S} \alpha_s \cdot \left(\sum_{m=1}^{NMRK} \cdot \sum_{c=1}^{NUMC_m} \cdot \sum_{e=1}^{NUME_{cm}} (\tilde{r}_{smce} - \tilde{r}_{mce}) \cdot (x_{smce}^{+} - x_{smce}^{-}) \right)^2 \leq \widetilde{\omega},$$

$$\sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} \tilde{r}_{sm} \cdot (x_{smce}^{+} - x_{smce}^{-}) - \sum_{c=1}^{NCAT_m} \cdot \sum_{e=1}^{NEQT_{cm}} k_{sm} \cdot (x_{smce}^{+} + x_{smce}^{-}) \geq \widetilde{NR}_{sm},$$

$$\forall\ s=1,2,\ldots S,\quad m=1,2,\ldots NMRK$$

$$\sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}\tilde{r}_s\cdot(x^+_{smce}-x^-_{smce}) - \sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}k_s\cdot(x^+_{smce}-x^-_{smce}) \geq \widetilde{NR}_s$$

$$\forall\ s=1,2,\ldots S,$$

$$\sum_{s=1}^{S}\cdot\sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}[(a^+_{smce}\cdot Q^+_{smce})-(a^-_{smce}\cdot Q^-_{smce})] \leq \tilde{B}$$

$$\alpha_s\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}[(\tilde{a}^+_{smce}\cdot Q^+_{smce})-(\tilde{a}^-_{smce}\cdot Q^-_{smce})] \leq \tilde{B}_s$$

$$\forall\ s=1,2,\ldots S,$$

$$\tilde{B}_s-\left[\sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}(\tilde{a}^+_{smce}\cdot Q^+_{smce})\right] \leq \tau_s$$

$$\forall\ s=1,2,\ldots S,$$

$$\tau_s \leq \varepsilon_s\ ,\quad s=1,2,\ldots S,$$

$$\sum_{s=1}^{S}\cdot\sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}\tilde{r}_{smce}\cdot(x^+_{smce}-x^-_{smce}) \geq \rho$$

$$\sum_{s=1}^{S}\cdot\sum_{m=1}^{NMRK}\cdot\sum_{c=1}^{NCAT_m}\cdot\sum_{e=1}^{NEQT_{cm}}x^+_{smce}=1,$$

$x^-_{smce} \leq \tilde{\theta}_{smce}\cdot x^+_{smce}$, Where $0 \leq \tilde{\theta}_{smce} \leq 1$,
$l^+_{smce}, \leq x^+_{smce} \leq u^+_{smce}$,
$l^-_{smce} \leq x^-_{smce} \leq u^-_{smce}$,
$k_{sm}>0, k_s>0, \tilde{r}_{sm}>0, \tilde{r}_s>0$
$\tilde{Q}^+_{smce}, \tilde{Q}^-_{smce} \in \mathbb{Z}^{\geq 0}$ & *Integers*

V. Solution Algorithm

The Solution algorithm for the proposed model consist of five main steps:

A. Initialization:

1- Set maximal number of $s=S$ where S is a number of objectives, (by the Expert & Analyst),

2- Set the required Weights $w_s=[0,1]$ $\forall$ $1 \leq s \leq S$, are the set of pre-defined weight required for each scenarios. (by the Expert 'Portfolio Manager'),

3- Set $\alpha_s \in [0,1]$ the probability of each scenario s to be occurred in the portfolio, $\sum_{s=1}^{S}\alpha_s=1$ (by Expert),

4- Set the number of industrial categories (c^{th} industrial categories), and the m^{th} market up to maximum number of markets to be traded on, and the e^{th} equity in the c^{th} industrial category in the m^{th} market in Scenario (s), shall be a model input parameters. (by the Expert & Analyst),

5- Set the equities expected returns ($\tilde{r}_{smce}$), and its transaction costs (k_{smce}), (by Expert),

6- set approximated *Risk Tolerance* ($\tilde{\omega}$), Budget B, B_s, Revenue (ρ). (by Expert),

7- Set the equities prices for every one of the buy/sell deals, p^-_{smce} p^+_{smce} & $\tilde{\theta}_{smce}$ (by Expert),

8- Set the upper pounds (u^+_{smce}), (u^-_{smce}), and lower pounds (l^+_{smce}), (l^-_{smce}) for investments, set (l^+_{smce}), (l^-_{smce}) equal to zero, and ε_s for all $1 \leq s \leq S$, (by Expert).

B. Pre-Preparational Calculations:

9- Set return for each scenario, and return for the scenario in each market, as well as the transaction costs for each scenario. *(by Expert),*

10- Set $k_{mce}=\sum_{s=1}^{S}\alpha_s\cdot k_{smce}$, and $\tilde{r}_{mce}=\sum_{s=1}^{S}\alpha_s\cdot\tilde{r}_{smce}$, *(*by Analyst*),*

11- Set $\tilde{r}_{sm}$, k_{sm} for all $1 \leq s \leq S$ & $m=1,\ldots,NMRK$ (By Expert),

12- Set $\tilde{r}_s=\sum_{m=1}^{NMRK}\tilde{r}_{sm}$, and $k_s=\sum_{m=1}^{NMRK}k_{sm}$, $\forall s=1,2,\ldots,S$, *(by Analyst),*

13- Set $\tilde{a}^+_{smce}=(1+k_{sm})\cdot\tilde{p}^+_{smce}$, $\tilde{a}^-_{smce}=(1+k_{sm})\cdot\tilde{p}^-_{smce}$

C. Determining the Fuzzy Membership Functions:

14- Set α-Cut $=\alpha^*$, Apply the increasing half-trapezoidal membership function for returns, and a decreasing function for risk; (by Analyst);

15- For representing the expected return on each equity existed in the scenarios of the portfolio, this can be written by next equation of increasing half-trapezoidal membership function. Fig. 2 shows the membership function for each equity return;

$$\mu_r(x)=\begin{cases}1 & if\ r(x)\leq r_i^0\ ,\\ 1+\dfrac{[r_i^0-r_i]}{\Delta r_i} & if\ r_i^0-\Delta r_i\leq r_i\leq r_i^0\ ,\\ 0 & otherwise.\end{cases}$$

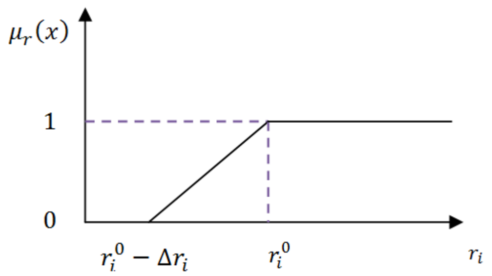

Fig. 2 Membership function for each equity return

16- For representing the maximum tolerance risk for the portfolio, this can be written by next equation of decreasing membership function. Fig. 3 shows the membership function for the portfolio risk;

$$\mu_\omega(p) = \begin{cases} 1 & if\ \omega(p) \le \omega_p^0 \quad , \\ 1 - \dfrac{[\omega_p - \omega_p^0]}{\Delta\omega_p} & if\ \omega_p^0 \le \omega_i \le \omega_p^0 + \Delta\omega_p \ , \\ 0 & otherwise. \end{cases}$$

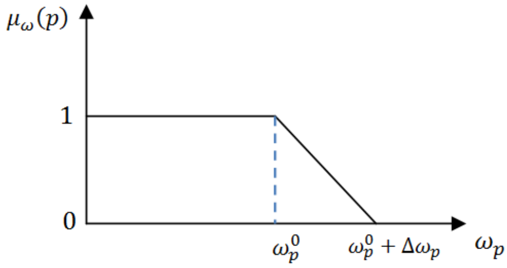

Fig. 3 The Membership function for the portfolio risk

D. De-fuzzyffication for the Model:

17- Solve the fuzzy MINL-VOP problem using the weighting method of VOP, and determine the sensitivity analysis, for comparative analysis. If satisfied solution, stop.

E. Solving the Model:

18- Ask the DM if the solution is satisfied, if yes Stop, and view results. If solution were not satisfied, set new Weights... go to (step 2).

19- End.

A Flowchart for the proposed model solution is illustrated in Fig. 4

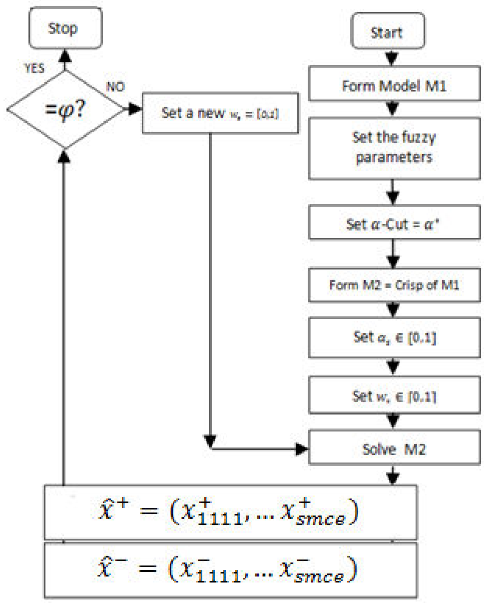

Fig. 4 Flowchart for the proposed model solution

VI. Numerical Example

In this section we give an example to illustrate the model for portfolio selection proposed in this paper. We suppose that one investor chooses eight different types of equities related to different number of industrial categories in two Stock Exchanges for his/her investments, assuming there are two scenarios, the first devoted to the one day settlement, and the second scenario devoted to the two days settlement given that the probabilities (α_s) of scenario one and two to occur are 40% and 60%, respectively; the budget for each scenario is 1,000,000 EGP; Whereas investor's required revenue at least 100,000 EGP, and for risk tolerance intervals its estimated to be between 0.035, and 0.07.

We present the given prices, expected rate of returns, and Theta's intervals (see Table I), whereas the expected rate of returns intervals for equities in the whole portfolio are shown in Table II.

TABLE I
THE PRICES, THE EXPECTED RATE OF RETURNS, AND THETA'S INTERVALS

Scenario #	Market #	Ind. Cat. #	Equity #	Parameters	Det. Val.	Min Val.	Max Val.	Parameters	Det Val.	Min Val.	Max Val.
1	1	1	1	Prices intervals for equities bought	221	212	223	The equities expected returns	0.040	0.066	0.066
			2		51	48	53		0.080	0.083	0.083
		2	1		111	107	116		0.070	0.037	0.038
			2		46	45	47		0.040	0.056	0.056
	2	1	1		121	118	124		0.040	0.034	0.041
			2		89	85	88		0.010	0.047	0.047
		2	1		13	9.8	12.5		0.170	0.224	0.224
			2		5	4.99	6		0.300	0.287	0.288
2	1	1	1		221	211	223		0.070	0.104	0.104
			2		50	48	53		0.100	0.135	0.135
		2	1		110	107	116		0.010	0.028	0.028
			2		45	45	47		0.020	0.045	0.045
	2	1	1		121	118	124		0.010	0.020	0.032
			2		88	85	88		0.030	0.060	0.068
		2	1		12	9.8	12.5		0.080	0.146	0.146
			2		5	5	6		0.050	0.360	0.366
1	1	1	1	Prices intervals for equities sold	230	225	236	Theta's intervals for equities	0.90	0.17	0.25
			2		54	52.5	55		0.90	0.66	0.72
		2	1		115	112	119		0.90	0.45	0.60
			2		48	48	49		0.90	0.70	0.88
	2	1	1		125	122	129		0.90	0.32	0.44
			2		89	89	91		0.90	0.80	0.92
		2	1		15	12	14		0.90	0.80	0.93
			2		6	7	8		0.90	0.30	0.44
2	1	1	1		238	233	239		0.90	0.55	0.80
			2		55	54.5	57		0.90	0.70	0.85
		2	1		112	110	119		0.90	0.50	0.59
			2		49	46.5	50		0.90	0.30	0.50
	2	1	1		122	121	128		0.90	0.60	0.92
			2		91	91	94		0.90	0.30	0.50
		2	1		14	11.8	13.99		0.90	0.42	0.69
			2		8	6.5	9		0.10	0.6	0.73

TABLE II
THE EXPECTED RATE OF RETURNS INTERVALS FOR EQUITIES
IN THE WHOLE PORTFOLIO

Market #	Ind. Cat. #	Equity #	Actual Val.	Min Val.	Max Val.
1	1	1	0.06	0.089	0.089
		2	0.09	0.056	0.056
	2	1	0.02	0.032	0.032
		2	0.03	0.042	0.050
2	1	1	0.02	0.025	0.036
		2	0.02	0.055	0.056
	2	1	0.12	0.120	0.177
		2	0.42	0.331	0.331

The summation of all decision variables related to the proportions of total equities are having the summation of one, whereas the summation of proportions of total equities sold is not exceeding One. After we run the proposed model deterministically once and fuzzed once again we found that output described in table III.

TABLE III
THE COMPARISON BETWEEN THE DETERMINISTIC AND FUZZY SOLUTIONS

Scenario #	Market #	Ind. Cat. #	Equity #	D. Variables	Type of solution: Determini-stic	Type of solution: Fuzzy	D. Variables	Type of solution: Determini-stic	Type of solution: Fuzzy
1	1	1	1	The proportions of total equities bought	0.011	0.120	The proportions of total equities sold	0.010	0.000
			2		0.287	0.210		0.020	0.000
		2	1		0.011	0.030		0.010	0.000
			2		0.011	0.050		0.010	0.000
	2	1	1		0.022	0.000		0.020	0.000
			2		0.011	0.000		0.010	0.000
		2	1		0.022	0.000		0.020	0.000
			2		0.011	0.053		0.010	0.000
2	1	1	1		0.011	0.050		0.010	0.000
			2		0.446	0.150		0.020	0.000
		2	1		0.011	0.060		0.010	0.000
			2		0.011	0.040		0.010	0.000
	2	1	1		0.010	0.027		0.000	0.000
			2		0.022	0.070		0.020	0.000
		2	1		0.011	0.090		0.010	0.000
			2		0.090	0.050		0.010	0.000
Summation					1.000	1.000		0.20	0.000

Tables IV, V show the number of shares to be invested in the e^{th} equity bought/sold respectively in the c^{th} industrial category in the m^{th} market in each scenario in the portfolio. Budgeting shows that the DM should deal with in his\her portfolio.

The model results obviously show that the proportions of total equities sold in the fuzzy solution are 0's, which indicates recommendation to DM to hold if he/she decides. However, budgeting shows that the total utilization of the budget in the deterministic solution is 999993l.e.

TABLE IV
THE DETERMINISTIC SOLUTION FOR THE NUMBER OF SHARES

Scenario No	Equity	Vol. for bought (1)	Lower of total amount paid for bought (2)	Total budget utilized (1 X 2)	Upper of total amount paid for bought (3)	Total budget utilized (1 X 3)
s=1	1111	50	221	11050	22	230
	1112	5626	51	286926	185	54
	1121	100	111	11100	43	115
	1122	242	46	11132	104	48
	1211	184	121	22264	80	125
	1212	125	89	11125	56	89
	1221	1709	13	22217	667	15
	1222	2222	5	11110	833	6
s=2	2111	50	221	11050	21	238
	2112	8918	50	445900	182	55
	2121	101	110	11110	45	112
	2122	247	45	11115	102	49
	2211	83	121	10043	0	122
	2212	253	88	22264	110	91
	2221	926	12	11112	357	14
	2222	18095	5	90475	625	8
	SUM=			**999993**		
	Tuo=			**7**		

TABLE V
THE FUZZY SOLUTIONS FOR THE NUMBER OF SHARES

Scenario No	equity	Volume for bought (1)	Total lower amount paid for bought (2)	Total upper amount paid for bought (3)	Average total amount paid for bought (4)=(2+3)/2	Fuzzy budget in Avg. (5)=4*1
s=1	1111	430	212	223	217.5	93525
	1112	3170	48	53	50.5	160085
	1121	207	107	116	111.5	23080.5
	1122	851	45	47	46	39146
	1211	0	118	124	121	0
	1212	0	85	88	86.5	0
	1221	0	9.8	12.5	11.15	0
	1222	7084	4.99	6	5.495	38926.58
s=2	2111	269	211	223	217	58373
	2112	3396	48	53	50.5	171498
	2121	621	107	116	111.5	69241.5
	2122	1021	45	47	46	46966
	2211	260	118	124	121	31460
	2212	955	85	88	86.5	82607.5
	2221	8640	9.8	12.5	11.15	96336
	2222	10000	5	6	5.5	55000
	SUM=					**966245.08**

Table VI shows the proportion of investment devoted to the e^{th} equity bought/sold respectively in the c^{th} industrial category in the m^{th} market in the whole portfolio, with comparison between the deterministic and the fuzzy solutions.

TABLE VI
THE PROPORTIONS OF TOTAL EQUITIES BOUGHT AND SOLD IN THE PORTFOLIO

Market #	Ind. Cat. #	Equity #	D. Variables	Type of solution: Determin-istic	Type of solution: Fuzzy
1	1	1	The proportions of total equities bought in the portfolio	0.022	0.170
		2		0.733	0.360
	2	1		0.022	0.090
		2		0.022	0.090
2	1	1		0.032	0.027
		2		0.033	0.070
	2	1		0.033	0.090
		2		0.102	0.103
1	1	1	The proportions of total equities Sold in the Portfolio	0.010	0.000
		2		0.020	0.000
	2	1		0.010	0.000
		2		0.010	0.000
2	1	1		0.008	0.000
		2		0.016	0.000
	2	1		0.014	0.000
		2		0.010	0.000
Sum for proportions of total equities bought				1.000	1.000
Sum for proportions of total equities sold				0.020	0.000

Objective function values for the deterministic and fuzzy solutions are 0.053 and 0.060, respectively, whereas for both deterministic and fuzzy solutions τ_1 decision variable scenario 1= 6 l.e., and τ_2 for scenario 2= 7 l.e.. Fig 5. Describes the set of portfolios that has the maximum rate of return for every given level of risk, on other words the minimum risk for every potential rate of return, [1]. The fuzzy multi-objective portfolio optimization model performed well compared with its deterministic.

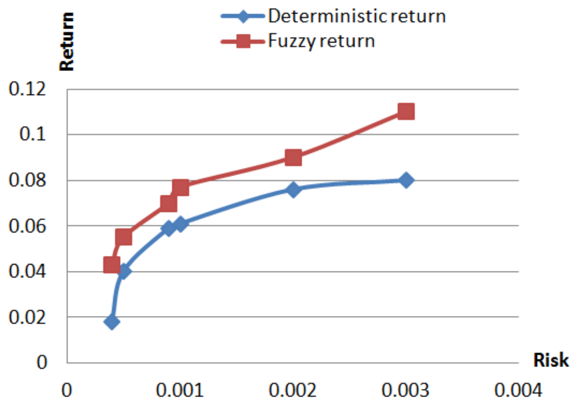

Fig. 5 Efficient Frontier for deterministic and fuzzy models

All computations were carried out on a windows PC using the General Algebraic Modeling System (GAMS) software, this is a high-level algebraic modeling system for large scale optimization. However, Basic Model Type: Mixed Integer Nonlinear Programs (MINLP).

VII. Conclusions and Future Research

This paper researches the portfolio selection theory using fuzzy mathematics theory. We have proposed a maximization model of fuzzy returns in future scenarios, the fuzzy extension of multi-objective mean-variance portfolio selection problem considering equity markets' future scenarios about net returns have been considered, and it has been proposed its solution method, with an example. The parameters of investment return and target risk are fuzzed. Then, these are described by linear half trapezoidal membership function. By comparative analyses, we get some following conclusions.

(1) In aspect of the model design, the portfolio selection model based on linear half trapezoidal membership function includes not only historical data, but also DMs' expectation. That's in accord with human psychology and fact gives more reliable solution when compare with the deterministic.

(2) The portfolio future return and risk aren't only one value, but several fuzzy values can be considered through the concept of future scenarios. However the fuzzy model is able to represent the expert knowledge as well DMs' subjective expectation.

(3) Comparing Markowitz's programming model [1], [14], and [15] the calculation process of our model is more practical.

Certainly, there are many other aspects which should be studied in the field of fuzzy multi-Scenario portfolio optimization with multi-markets. Some of these aspects are:

(a)-A parametric analysis on the solution for the proposed model on a life data from different Equity markets.

(b)-Developing the model in the context of short selling.

(c)-Adapting Heuristics' Search *Techniques*, either with increasing complexity as the number of markets becomes larger or adding non-smooth constraints to this model.

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