Tracking Control of a Linear Parabolic PDE with In-domain Point Actuators

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Abstract—This paper addresses the problem of asymptotic tracking control of a linear parabolic partial differential equation with in-domain point actuation. As the considered model is a non-standard partial differential equation, we firstly developed a map that allows transforming this problem into a standard boundary control problem to which existing infinite-dimensional system control methods can be applied. Then, a combination of energy multiplier and differential flatness methods is used to design an asymptotic tracking controller. This control scheme consists of stabilizing state-feedback derived from the energy multiplier method and feed-forward control based on the flatness property of the system. This approach represents a systematic procedure to design tracking control laws for a class of partial differential equations with in-domain point actuation. The applicability and system performance are assessed by simulation studies.

Keywords—Tracking Control, In-domain point actuation, Partial Differential Equations.

I. INTRODUCTION

Linear parabolic Partial Differential Equations (PDE) arise in the mathematical modeling of many physical, chemical and biological phenomena as well as many other areas such as heat process, astrophysics, economy, financial modeling, etc. In most of these systems, control inputs are applied on the boundary of the system, which is known as the boundary control problem [1]. However, for efficiency and application related concerns, the control inputs may be placed in the domain of the system. Integrating a number of control inputs acting in the domain will lead to a non-standard inhomogeneous PDE [2], [3]. The main topic of this article is the tracking control design of a special class of linear parabolic PDE using in-domain actuators.

One of the viable methods for tackling non standard PDE systems control is to transform the original form into a standard boundary value model, so that it may be represented as a standard Cauchy problem to which functional analytic setting based on semigroup and other related tools are applicable (see, e.g., [3], [4], [5], [6]). There are therefore at least four categories of tools that can be used to assess the stability of a strongly continuous semigroup: 1) time domain criteria; 2) frequency domain criteria; 3) spectral analysis method; and 4) energy multiplier method. The method of energy multiplier, in a very similar way to the Lyapunov method for finite

dimensional system, guarantees the stability of the system if one can show that the time derivative of an energy-like function is dissipative along the solutions of the system. We applied this method in the present context to find stabilizing feedback control laws.

To further improve system performance and efficiency regarding the specifications on, for instance, start-up and transition behaviors, one may consider the tracking problem, in which the control objective is to make system outputs to follow prescribed and desired reference trajectories [7], [8], [9], [10], [11]. In this paper, particular attention will be paid on the method of differential flatness which is a powerful tool for tracking control problems.

The concept of flat systems is originally developed for the control of finite-dimensional nonlinear systems [12], [13], [14], [15]. Roughly speaking, differential flatness is a system property describing the ability to explicitly express all states and inputs by the so-called flat output and its time-derivatives up to a certain problem-dependent order. This tool is particularly advantageous for solving motion or trajectory planning problems, as well as set-point tracking control. The tracking control can be carried out easily in this framework, in which the desired behavior can be specified through reference trajectories. Therefore, a high system performance can be expected. The flat system approach has further been successfully extended to a variety of infinite-dimensional systems (see, e.g., [7], [10], [16], [17], [18], [19]). A notable work is presented in [7] in which flatness method is applied to a boundary tracking control of a parabolic PDE. The proposed control algorithm consists in applying the control signal to one end in order to steer the other end of the linear PDE, which is stabilized by back-stepping, to a prescribed trajectory. Inspired by this idea, this paper will demonstrate a new application of differential flatness to the control of PDEs with in-domain point actuators.

The proposed approach combines the method of energy multiplier and differential flatness to achieve asymptotically tracking control problems. To make the issue under consideration more tangible, we describe the model on the basis of a benchmark model of heat equation. Toward this end, we consider the heat transfer in a one dimensional bar with a number of point actuators distributed along the domain of the system. The basic idea consists in mapping the original system dynamics into a standard boundary control form to which energy multiplier method may be applied to find a state-feedback boundary control that can guarantee the exponential stability of the target system. It is shown then that the reformulation of the target system allows the introduction of additional degrees-of-freedom, which can be exploited for flatness-based trajectory planning and feed-forward control

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design. The obtained algorithm is a combination of feedback and feed-forward control, which enables the stabilization of the closed-loop system around prescribed trajectories for the outputs of the original system.

This paper aims at presenting a systematic approach for the tracking control problem of a class of linear parabolic PDEs by using N degrees-of-freedom (N-DOF) of the system to approximate infinite-DOF trajectories. Besides, the distributed nature of the system is kept and the control design is preformed using the infinite-dimensional dynamic model. The control synthesis includes then the development of : 1) a map that will transform the original system into standard boundary value form; 2) a state feedback law by applying the energy multiplier method; and 3) a feed-forward law using the property of system flatness. The combination of state feedback and flatness-based feed-forward control results in an asymptotic tracking control law, which allows the system to follow prescribed output trajectories.

The remainder of the paper is organized as follows. Section II describes the mathematical model of a representative in-domain actuated system. Section III deals with control synthesis. A simulation study is carried out in Section IV, and, finally, Section V provides some conclusions and discussions for future works.

II. PROBLEM STATEMENT AND MODELING

There are several industrial processes which tend to heat or cool a certain substance in accordance with a pre-specified profile, such as processes in the areas of glass industry [10], of polymerization reactors [18], and of fixed-bed tubular reactors for production or degradation, of activated sludge processes, or of catalytic converters for emission control and purification [20]. The final product is produced by passing the raw material along a feeder whose temperature follows a desired trajectory. The feeder, as illustrated in Fig. 1, typically consists of a continuous bar that is internally equipped with actuators. Since cooling or heating occurs through the continuous bar, there is a high amount of cross-talk among the actuators. As a result, the change of temperature of any actuator alters the condition experienced by the neighbor actuators. Due to this thermal coupling between actuators, closed-loop control is generally used to achieve accurate feeder corrective temperature profiles. A prerequisite toward this goal is the development of a dynamical model of the system.

Assuming that the size of actuators is much smaller than the length of the continuous feeder bar, the effect of the actuators can be mathematically modeled as the point actuation in the domain of the system. Therefore, the dynamics of the temperature of the bar, $w(x,t)$ at position x and time t, can be modeled by a heat equation with point actuators [1]

$$\frac{\partial w(x,t)}{\partial t} - \alpha \frac{\partial^2 w(x,t)}{\partial x^2} = \sum_{i=1}^{N} \alpha_i(t)\delta(x-x_i), \ x \in \Omega, \ t \in \mathbb{R}^+,$$

$$\tag{1a}$$

$$\frac{\partial w(x,t)}{\partial n} = 0 \text{ on } \partial\Omega,$$

$$\tag{1b}$$

$$w(x,0) = w^0(x), \ x \in \Omega,$$

$$\tag{1c}$$

where $\alpha_i(t)$ is the control input at each actuation spot $x = x_i$, described by the Delta Dirac function, $\delta(x-x_i)$, Ω and $\partial\Omega$ represent the domain and its boundary on which the system is defined, and α is the Péclet number which is defined, for diffusion of heat, as

$$\alpha \triangleq \frac{C\rho}{\lambda l} \tag{2}$$

where C, ρ, λ, and l are specific glass heat, glass density, glass heat conductivity, and feeder length, respectively. Note that (1a) takes into account heat diffusion and generation in the domain. (1b) implies a problem of homogenous Neumann boundary value where n stands for the conormal vectors of $\partial\Omega$. The initial condition is defined via (1c). Figure 1 illustrates the schematic of in-domain control of a feeder.
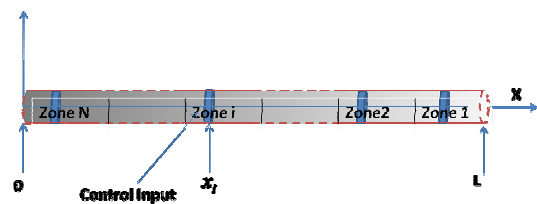

Fig. 1. Schematic of the in-domain actuator feeder

Following the terminology of [3], this type of PDE is called "point actuator PDEs." Our case study is limited to one dimensional bar so that $\Omega \in \mathbb{R}$, and $\partial\Omega$ inclds $\Gamma_1 : x = 0$ and $\Gamma_2 : x = L$. However, for the development of our approach, we are always using the general notation, Ω, Γ_1, Γ_2, etc; when there is no confusion. Besides, for notation convenience we use w instead of $w(x,t)$ and Laplace operator, Δ, for $\partial^2 w(x,t)/\partial x^2$.

III. CLOSED-LOOP CONTROL DESIGN

A. Mapping Point Actuation to Neumann Boundary Form

Note that the model given in (1) is a non-standard PDE due to unbounded operator of Delta Dirac function in the domain of the system (see the right-hand-side of (1a)). The control of such systems has not received sufficient attention in the existing literature [21], [3]. To deal with this non-standard model, we introduce a map which allows transforming of the system into a standard form to which the existing stabilizing methods may be applied. To this end, we consider the following mixed boundary value PDE as a target system:

$$\frac{\partial w(x,t)}{\partial t} - \alpha \frac{\partial^2 w(x,t)}{\partial x^2} = 0, \ x \in \Omega, \ t \in \mathbb{R}^+, \tag{3a}$$

$$w(x,t) = 0 \text{ on } \Gamma_1, \tag{3b}$$

$$\frac{\partial w}{\partial n} = g(t) \text{ on } \Gamma_2, \tag{3c}$$

$$w(x,0) = w^0(x), \ x \in \Omega, \tag{3d}$$

where Γ_2 is a non-empty relatively open subset of $\partial\Omega$ and $\Gamma_1 \bigcup \Gamma_2 = \partial\Omega$.

Our aim is to find a relationship to map the effect of the control signals appearing on the right-hand-side of (1a) to a boundary value, $g(t)$, in (3c).

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Note that (1a) does not have classic solutions for $w \in C^2(\Omega) \bigcap C(\bar{\Omega})$. Otherwise $\partial^2 w(x,t)/\partial x^2$ would be a continuous function on Ω, which is not possible because $\sum_{i=1}^{N} \alpha_i(t) \delta(x - x_i)$ is not continuous on Ω [22].

Therefore, our goal is to ultimately ensure that systems (1) and (3) possess an identical weak solution. Toward this end, we consider the special Sobolev space:

$$H_{0,\Gamma_1}^1(\Omega) = \{v \in H_1(\Omega) : v = 0 \text{ on } \Gamma_1\}, \tag{4}$$

where $H^1(\Omega)$ is the Hilbert space associated with the inner product and the inner norm. The reasoning behind the consideration of this special Sobolev space is that we seek to preserve the information at Γ_2. As it will become apparent in the following, this choice enables us to map the inputs on the right-hand side of the original system to Γ_2 of the target system.

We multiply the differential equation (3a) by an arbitrary test function $v(x) \in H_{0,\Gamma_1}^1(\Omega)$, i.e., a smooth enough test function and not necessarily zero at Γ_2. By preforming an integration by parts over Ω, the following weak form will be obtained:

$$\int_\Omega w_t(x,t) \cdot v(x) dx + \int_\Omega \nabla w(x,t) \cdot \nabla v(x) dx$$
$$= \int_{\Gamma_2} \frac{\partial w(x,t)}{\partial n} v(x) d\gamma, \quad \forall v, w \in H_{0,\Gamma_1}^1(\Omega). \tag{5}$$

Note that the difference between this derived weak form and the usual weak form is the boundary integral term on Γ_2. This term is not vanished because the test function is not identically zero at Γ_2.

When applying the Lax-Milgram theorem with $V = H_{0,\Gamma_1}^1$, the existence and the uniqueness of a weak solution to this mixed boundary value PDE easily follows. For a proof of this result, we refer to, e.g., [23], [24].

Since in H_{0,Γ_1}^1, either zero or non-zero values can be applied to Γ_2, the weak form of (1a) can be defined based on this space. Similarly for (1a) we obtain:

$$\int_\Omega w_t(x,t) \cdot v(x) dx + \int_\Omega \nabla w(x,t) \cdot \nabla v(x) dx$$
$$= \sum_{i=1}^{N} \int_\Omega \alpha_i(t) \delta(x - x_i) v(x) dx, \quad \forall v, w \in H^1(\Omega). \tag{6}$$

In a similar way, the existence and the uniqueness of a weak solution to this problem can be shown.

Therefore, the sufficient condition for (6) and (5) to have an identical weak solution is:

$$\sum_{i=1}^{N} \int_\Omega \alpha_i(t) \delta(x - x_i) v(x) dx - \int_{\Gamma_2} \frac{\partial w(x,t)}{\partial n} v(x) d\gamma = 0. \tag{7}$$

This is readily obtained by subtracting (6) from (5). We choose a test function as follows:

$$v(x) = \xi + \|(x - \xi)^2\|, \quad x \in \Omega, \; \xi \in \{x_1, x_2, \cdots, x_N\} . \tag{8}$$

This function is indeed the static Green's function or influenced function which is the fundamental static solution of the equation for each unit source point at position ξ. In this case, choosing Green's function as a test function is the key point that guarantees the solvability of the map between the boundary value and the in-domain points (as it will be shown in the following section). For simplicity of notation, we denote this function by $G(x, \xi)$.

Accordingly, the corresponding boundary value may be expressed in terms of in-domain actuation signals as follows:

$$g(t) = \frac{\partial w(x,t)}{\partial n} = \frac{\sum_{i=1}^{N} \alpha_i(t) G(x_i, \xi)}{\int_{\Gamma_2} G(x, \xi) dx}, \; \xi \in \{x_1, x_2, \cdots, x_N\} \tag{9}$$

Equation (9) bridges the non-standard equation (1a) with the standard mixed boundary value form (3a). g is the function of time and free parameter of ξ. We will show it in the rest by $g(t)|_{x_i}$ for each $\xi = x_i$. The boundary condition of g may be compared to oblique derivative boundary condition [25] which is in general of the following form:

$$\sum_i a_i \frac{\partial w}{\partial n} \cos(\alpha_i) + \sigma(x) w = g \text{ on } \partial \Omega, \tag{10}$$

where α_i is the angle between the unit outward normal vector n to $\partial \Omega$ and the x axis.

In (9) the term $\frac{G(x_i, \xi)}{\int_{\Gamma_2} G(x, \xi) dx}$ denotes indeed the obliqueness of each boundary value associated to ξ to Γ_2. The value measured at Γ_2 is the resultant of each oblique vector associated to each actuator.

Subsequently, we can use the existing methods to deal with PDEs control through boundary condition and then apply the results back to the original in-domain actuation control using (9).

B. Exponential Stabilization by Energy Multiplier Method

In the previous subsection, we developed a map to transform the non-standard original system to a standard boundary system. In this section, we employ the method of energy multiplier to stabilize the boundary value PDE. The approach presented in this section is mainly adopted from [4]. The stabilizing controller, that we will shortly examine, will be applied back to the original system using (9) to obtain the actual control signals.

Henceforth, we use the method of energy multiplier to find a boundary feedback control law to make the system dissipative. Toward this end, we consider the following H_1-norm

$$E = \frac{1}{2} \int_\Omega w^2(x,t) dx + \frac{\alpha}{2} \int_\Omega w_x^2(x,t) dx, \tag{11}$$

where α is the Péclet number defined in (2).

Then, the time derivative of $E(t)$ along the solution of (3a) is given by:

$$\dot{E}(t) = \alpha \int_\Omega w w_{xx} dx + \alpha \int_\Omega w_x w_{tx} dx$$
$$= \alpha w(x) w_x(x)|_{\Gamma_2} - \alpha \int_\Omega w_x^2 dx + \alpha w_t(x) w_x(x)|_{\Gamma_2}$$
$$- \alpha \int_\Omega w_{xx}(x) w_t(x) dx. \tag{12}$$

If we choose

$$w_x(x,t) (w(x,t) + w_t(x,t))|_{\Gamma_2} \leq 0, \tag{13}$$

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we have

$$\dot{E}(t) \leq -\int_\Omega w_x^2 dx - \int_\Omega w_{xx}^2 dx.$$

By applying Poincaré inequality [1], we obtain

$$\dot{E}(t) \leq -\frac{1}{4}\int_\Omega w^2 dx - \frac{1}{4}\int_\Omega w_x^2 dx$$
$$\leq -\frac{1}{2}E.$$

Consequently, the energy stored in the system is dissipative and the system is exponentially stable. To satisfy the stability condition (13), we can choose the boundary control as follow:

$$g(t) = k_g w_x(x,t)|_{\Gamma_2} = -k_g \left(w_t(x,t)|_{\Gamma_2} + w(x,t)|_{\Gamma_2}\right), \quad (14)$$

where $k_g > 0$ is a constant controller gain. Substituting $g(t)$ from (9) leads to:

$$g(t) = \frac{\sum_{i=1}^N \alpha_i(t) G(x_i, \xi)}{\int_{\Gamma_2} G(x, \xi) d\gamma_2} = -k_g \left(w_t(x,t)|_{\Gamma_2} + w(x,t)|_{\Gamma_2}\right). \quad (15)$$

It is worth mentioning that we can suppose that the measurement of the temperature and its time derivative on the boundary, $w(x,t)|_{\Gamma_2}$ and $w_t(x,t)|_{\Gamma_2}$, is available. In fact, this is quite feasible and practical and there is no need for intrusion inside the system for the measurement. Therefore, $g(t)$ can be implemented by closed-loop state feedback.

Stabilizing feedback control law (15) is constrained by N unknown variables in the original system. Therefore, for a given $g(t)$, we can freely choose $N-1$ actuation signals and deduce one of them, e.g. $\alpha_j(t)$. Thus the control signal associated to the actuator at $\xi = x_j$ can be computed by:

$$\alpha_j(t) = -\frac{1}{G(x_j, \xi_j)} \sum_{i=1, i\neq j}^N \alpha_i(t) G(x_i, \xi_j)$$
$$- k_g \left(w(x,t)|_{\Gamma_2} + w_t(x,t)|_{\Gamma_2}\right) \int_{\Gamma_2} G(x, \xi_j) d\gamma. \quad (16)$$

So far, we have obtained the condition to compute one actuation signal for the original system based on the state feedback control. However, the remaining $N-1$ degrees-of-freedom of the system has not yet been utilized. In the next section, we make use of this free DOF potential to develop a feed-forward controller based on the well-known technique of differential flatness, which is broadly used for tracking control in finite dimensional systems.

C. Flatness-based Tracking Control

In the previous section, we developed a condition on input signals that must be satisfied in order to make the system exponentially stable. In fact, (13) limits the system trajectory to a stable region, yet the $N-1$ degrees-of-freedom of the system are left free. Given this available capacity, we can use the technique of differential flatness to generate reference trajectories.

Our objective now is to approximate a spatially continuous trajectory at some specific points, i.e. at x_k, which are specified by a reference profile, $w_d(x,t)$. Therefore, we want to find $N-$

1 control signals such that the outputs of the system, $w(x_k,t)$, follow the reference profile $w_d(x_k,t)$. We claim that

$$w_k(t) = w(x_k,t), \text{ for } k = 1,\cdots,N; \ k\neq j \quad (17)$$

are "flat outputs" of the system. This means that we have chosen the $N-1$ free inputs of the system as flat outputs. To find the reference input we need to find the full state trajectory of the system. Toward this end, (3a) for the case of thin bar of length l in the Laplace variable s reads:

$$s\hat{w}(x,s) = \alpha\hat{w}_{xx}(x,s), \ x \in (0,l), \quad (18a)$$
$$\hat{w}_x(l,s) = g(s), \quad (18b)$$
$$\hat{w}(0,s) = 0, \quad (18c)$$

where $\hat{w}$ stands for the Laplace transform of w. It represents a second-order boundary-value ordinary differential equation (ODE) with respect to x. Thus, we obtain the following solution:

$$\hat{w}(x,s) = \frac{\sinh\left(\sqrt{\frac{s}{\alpha}}x\right)}{\sqrt{\frac{s}{\alpha}}\cosh\left(\sqrt{\frac{s}{\alpha}}\right)} g(s). \quad (19)$$

(19) stands for the full-state trajectory of the system in Laplace space which simultaneously satisfies (3a-3d). We want the flat outputs of the system, as claimed in (17), to follow the reference output trajectory. Therefore, we assign the prescribed values to flat outputs and rearrange (19) in terms of control signals:

$$\hat{g}(s)|_{x_k} = \frac{\sqrt{\frac{s}{\alpha}}\cosh\left(\sqrt{\frac{s}{\alpha}}\right)}{\sinh\left(\sqrt{\frac{s}{\alpha}}x_k\right)} \hat{w}_k(s). \quad (20)$$

$g(t)|_{x_k}$ can be computed by taking the inverse Laplace transform of (20). To numerically compute the inverse Laplace transform of $\hat{g}(s)|_{x_k}$, $\cosh(\cdot)$ and $(\sinh(\cdot))^{-1}$ may be expanded by Taylor expansion around $x = x_k \neq 0$. This expansion, consisting of different powers of s, corresponds to the different order of time derivative of $w_i(t)$. In other words, whenever $t \mapsto w(t)$ is in a desired trajectory, $t \mapsto (w(t), g(t))$ defined by the inverse Laplace transform of (19) and (20) is a trajectory of (18a-18c). This shows the flatness of the considered system [26].

Note that $x_k = 0$ is a singular point of $(\sinh(\cdot))^{-1}$. This implies that we should not place actuators at $x = 0$. In fact, $x = 0$ cannot be considered as a flat output of the heat equation in the presence of Neumann boundary values.

To ensure the convergence of (20), $t \mapsto w(t)$ must be a smooth function. However, in general it can not be an arbitrary analytic function. Based on the recommendation of [27], to steer the system from an initial temperature profile at t_0 to a final one at t_1, we select $w_d(t)$ as a Gevrey-Roumieu function on $[t_0, t_1]$ of order greater than 1 but smaller or equal to 2 with initial and final Taylor expansions imposed by the initial and

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final temperature profiles:

$$w_d(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \dfrac{\displaystyle\int_0^t \exp(-1/(\tau(1-\tau)))d\tau}{\displaystyle\int_0^1 \exp(-1/(\tau(1-\tau)))d\tau} & \text{if } t \in (0,1) \\ 1 & \text{if } t \geq 1 \end{cases}$$

With such a function, we may be able to compute the feed-forward control signals by using (20), which steer the system from one profile to another.

To find the admissible control signal of each actuator, $\alpha_i(t)$, corresponding to each output $\xi = x_k$, $k = 1, 2, \cdots, N$, $k \neq j$, we substitute (9) into the inverse Laplace transform of (20) at each output point. This leads to $N-1$ equations of $N-1$ unknown variables α_i as follows:

$$\sum_{i=1, i\neq j}^{N} G(x_i, x_k)\alpha_i(t) = -\left(G(x_j, \xi)\alpha_j(t) - \beta(t, \xi)\right)\big|_{\xi=x_k}$$

$$k = 1, 2, \cdots, N, k \neq j, \qquad (21)$$

with

$$\beta(t, \xi) = \mathscr{L}^{-1}\left(g(s)|_{x_k}\right) \int_\Gamma G(x, \xi) d\gamma \qquad (22)$$

where $\mathscr{L}^{-1}\left(g(s)|_{x_k}\right)$ is the inverse Laplace transform of (20) at each flat output, x_k. The $(N-1)$ feed-forward control signals, $\alpha_i(t)$, are determined by solving the $N-1$ equations of (21). We remind that the remaining $\alpha_j(t)$, which defines the stabilizing feedback control, is determined by (16).

IV. SIMULATION STUDY

In simulations, we consider a feeder of unit length. The feeder is equipped with three actuators placed at $\xi = 0.25, 0.5, 0.75$. The closed-loop state feedback control is given in (14) and the control signals for the actuation of the original system can be computed by (16) and (21).

To simplify the implementation, the control signals $\alpha_1(t)$, $\alpha_2(t)$, and $\alpha_3(t)$ are computed recursively. More specifically, given the solution of (20) and for the $\alpha_1(t)$ obtained in the previous step, $\alpha_2(t)$ and $\alpha_3(t)$ can be computed by:

$$\begin{pmatrix} \alpha_2(t) \\ \alpha_3(t) \end{pmatrix} = \begin{pmatrix} G(0.5, 0.5) & G(0.75, 0.5) \\ G(0.5, 0.75) & G(0.75, 0.75) \end{pmatrix}^{-1} \\ \times \begin{pmatrix} -G(0.25, 0.5)\alpha_1(t) + \beta(t, 0.5) \\ -G(0.25, 0.75)\alpha_1(t) + \beta(t, 0.75) \end{pmatrix}.$$

For computing $\mathscr{L}^{-1}\left(g(s)|_{\xi_k}\right)$, we consider the coefficients of the first three terms of Taylor expansion around ξ_k of $\sqrt{\frac{s}{\alpha}}\cosh(\sqrt{\frac{s}{\alpha}})/\sinh(\sqrt{\frac{s}{\alpha}}\xi_k)$, and multiply by $w_d(t)$, $\dot{w}_d(t)$, and $\ddot{w}_d(t)$[1].

From the stability condition of (16) and taking the feedback control gain $k_g = 1$, α_1 is obtained by:

$$\alpha_1(t) = -\frac{G(0.5, 0.25)\alpha_2(t) + G(0.75, 0.25)\alpha_3(t)}{G(0.25, 0.25)} \\ - (w(1,t) + w_t(1,t)) \int_0^1 G(x, 0.25)dx.$$

[1] The Matlab M-file for carrying out the computation is available upon request at `amir.badkoubeh@polymtl.ca`

Figure 2 illustrates the evolution of temperature in the system by applying the developed algorithm with feed-forward and feedback control. It can be seen that the closed-loop system is stabilized around the profile generated by the feed-forward control. The reference trajectories, $w_{d1}(t)$ at $x = 0.5$ and $w_{d2}(t)$ at $x = 0.75$ are illustrated in Fig. 3. The stabilizing feedback signal, $\alpha_1(t)$ and the feed-forward controls, $\alpha_2(t)$ and $\alpha_3(t)$ are depicted in Fig. 4. The results clearly illustrate the performance of the proposed approach to steer the system from an initial profile $w(x, t_0)$ to a final profile $w(x, t_1)$.
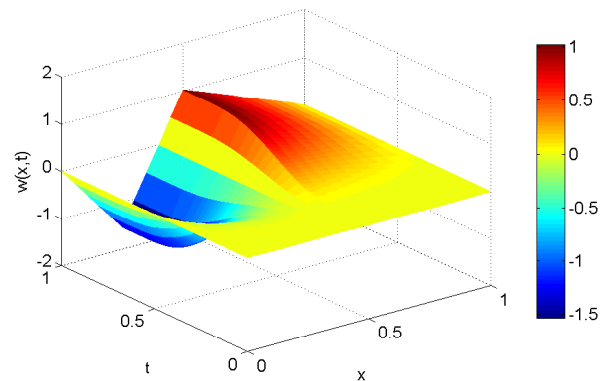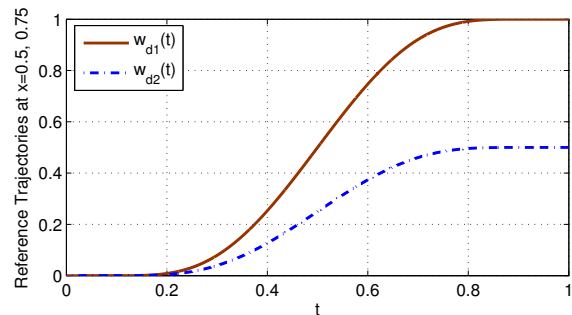

Fig. 2. Temperature evolution for $t \in [0, 1]$.

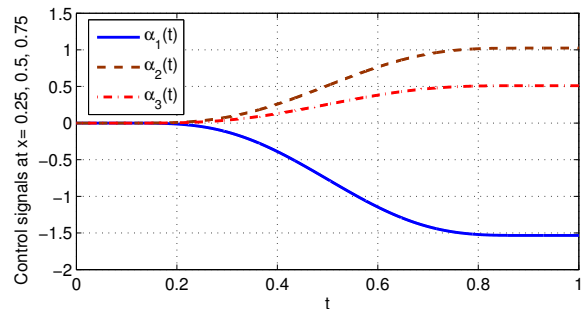Fig. 3. Reference trajectories, $w_{d1}(t)$ at $x = 0.5$ and $w_{d2}(t)$ at $x = 0.75$.

Fig. 4. Temperature evolution at $x = 0.25, 0.5, 0.75$ where the feedback feedforward signals, $\alpha_1(t), \alpha_2(t)$ and $\alpha_3(t)$, are applied.

V. Conclusion

This paper presented a systematic approach to deal with the problem of asymptotic tracking control of a linear heat equation with point actuation. This approach consists in firstly mapping the non-standard model of the process into a standard boundary value PDE. Then, a combination of energy multiplier and differential flatness is used in the design of tracking controller. Finally, the control signal of each actuator is deduced by assigning the stabilizing feedback to one actuator and by applying flatness-based feed-forward control to the other ones. System performance with the designed control is evaluated by numerical simulations, which demonstrates the viability and efficiency of the proposed approach.

It is wroth noting that the only data required for the implementation are the temperature evolution and its time derivative on the boundary. In fact, both sets of data on the boundary are practically accessible without requiring any intrusion through the domain for collocating a new set of sensors.

Furthermore, the property of differential flatness used for trajectory planning can be extended to linear heat equations with spatially and temporally varying parameters, where the Laplace transform seems not applicable, by using the formal power series as proposed in [7] to attain the full state trajectory of the system.

Finally, to gain a better tracking performance, it is suggested to consider a control on the boundary condition (1b) and to assign the stabilizing feedback signal to this actuator. Consequently, all the interior actuators can be utilized to generate spatial approximation of reference trajectories.

Acknowledgement

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