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Fuzzy Estimation of Parameters in Statistical Models

A. Falsafain, S. M. Taheri, and M. Mashinchi

***Abstract*—Using a set of confidence intervals, we develop a common approach, to construct a fuzzy set as an estimator for unknown parameters in statistical models. We investigate a method to derive the explicit and unique membership function of such fuzzy estimators. The proposed method has been used to derive the fuzzy estimators of the parameters of a Normal distribution and some functions of parameters of two Normal distributions, as well as the parameters of the Exponential and Poisson distributions.**

***Keywords*—Confidence interval. Fuzzy number. Fuzzy estimation.**

I. Introduction and Preliminaries

RECENTLY Buckley [2] introduced a method of estimation for parameters in statistical models. He used a set of confidence intervals producing a triangular shaped fuzzy number for the estimator, without giving any explicit membership functions of such fuzzy estimators. In this paper, by developing Buckley's approach, we introduce a method to find the explicit formula for membership functions of the fuzzy estimations in statistical models not known before. We apply this method to derive the explicit and unique membership functions for fuzzy estimation of the parameters of a Normal population. We also derive such membership functions for fuzzy estimations of $\mu_1 - \mu_2$ and σ_1^2/σ_2^2 of two Normal populations, and also to estimation of parameters in Exponential and Poisson distributions.

First, let us introduce the notations we will use in this paper. Following [2], we use the notations $\overline{A}, \overline{B}, \overline{C}, \ldots$ representing fuzzy sets. If $\overline{A}$ is a fuzzy set, then $\overline{A}(x) \in [0,1]$is the membership function for $\overline{A}$. An $\alpha-$ cut of $\overline{A}$, written $\overline{A}[\alpha]$, is defined as $\left\{x|\alpha \leqslant \overline{A}(x)\right\}$, for $0 \leqslant \alpha \leqslant 1$. A fuzzy number $\overline{N}$ is a fuzzy subset of the real numbers satisfying: (1) $\overline{N}(x) = 1$ for some x (normalized), and (2) $\overline{N}(\alpha)$ is a closed, bounded interval for $0 < \alpha \leq 1$. A triangular fuzzy number $\overline{T}$ is defined by three numbers $a_1 < a_2 < a_3$ where the graph of $\overline{T}$ is triangular with base on the interval $[a_1, a_3]$ and vertex at $x = a_2$ ($\overline{T}(a_2) = 1$). We write $\overline{T} = (a_1, a_2, a_3)$ for triangular fuzzy numbers. A triangular shaped fuzzy number has curves, not straight line segments, for the sides of the triangular. For any fuzzy number $\overline{N}$ we have $\bar{N}[\alpha] = [\theta_1(\alpha), \theta_2(\alpha)]$ for all α, $0 < \alpha \leq 1$, which describes the closed, bounded intervals as functions of α. Finally, we recall some known facts of fuzzy sets in the following theorems that will be used in next sections.

A. Falsafain and M. Mashinchi are with the Department of Statistics, Faculty of Mathematics and Computer Sciences, Shahid Bahonar University of Kerman, Kerman, Iran (e-mail: ali-falsafain@yahoo.com, mashinchi@mail.uk.ac.ir).

S.M. Taheri is with the School of Mathematical Sciences, Isfahan University of Technology, Isfahan 84156-83111, Iran, (e-mail: Taheri@cc.iut.ac.ir).

Theorem *1.1:* (resolution identity) [6, 8] Let $\overline{A}$ be a fuzzy subset of the set U. For all $x \in U$, we have $\overline{A}(x) = \sup\left\{\alpha\, I_{\overline{A}[\alpha]}(x)\,|\alpha \in [0,1]\right\}$, where $I_{\overline{A}[\alpha]}$ is defined as

$$I_{\overline{A}[\alpha]}(x) = \begin{cases} 1 & x \in \overline{A}[\alpha] \\ 0 & x \notin \overline{A}[\alpha] \end{cases}.$$

Theorem *1.2 (1, 6, 8):* Let A and B be mappings from a set U into a partially ordered set C. If $\overline{A}[\alpha] = \overline{B}[\alpha]$ for all $\alpha \in C$, then $\overline{A} = \overline{B}$.

Throughout the paper we assume that $\Re$ and $\Re^+$ are the set of all real numbers and set of all non-negative real numbers, respectively.

In the sequel, after introducing some notations, we recall Buckley's approach in Section 2. In Section 3, we obtain the fuzzy estimators of a parameter of a statistical model in a general framework. In Section 4, we obtain the fuzzy estimators of the parameters of a Normal distribution in different cases. Section 5 provides the fuzzy estimator for the difference of means in two independent Normal distributions. In Section 6, we derive the membership function of fuzzy estimator of the ratio of variances for two Normal distributions. In Sections 7 and 8, we apply the introduced method to obtain the fuzzy estimations for parameters of Exponential and Poisson distributions, respectively. In Section 9, we provide a conclusion. It is worth mentioning that the proposed method is essentially based on the pivotal approach to construct confidence intervals in classical statistics. Therefore, for any estimation problem, in which there exists a pivotal quantity, one can apply the introduced method to derive a fuzzy estimator.

II. Fuzzy Estimation

First, we briefly recall the method of Buckley [2]. Let $X_1, X_2, ..., X_n$ be a random sample from a distribution with probability density function (or probability mass function) $f(x;\theta)$, with observed value $x_1, x_2, ..., x_n$. Assume that θ is a single unknown parameter and it must be estimated. Based on these observations, one can obtain $(1-\beta)\,100\%$ confidence intervals for $0.01 \leq \beta \leq 1$, using the usual statistical methods [3,4,9]. Denote these confidence intervals as $[\theta_1(\beta), \theta_2(\beta)]$. We are using β here since α, usually employed for confidence intervals, is reserved for $\alpha-$cuts of fuzzy sets. Notice that for $\beta = 1$ we have a point not an interval. Now place these confidence intervals one on top of the other, respectively from 0.01 to one, to produce a fuzzy number $\overline{\theta}$ whose $\alpha-$cuts are the confidence intervals. Starting at 0.01 is arbitrary and we can begin at 0.001 or 0.005 etc. All that is needed is to finish the "bottom" of $\overline{\theta}$ to make it a complete fuzzy number. We will simply drop the graph of $\overline{\theta}$ straight down

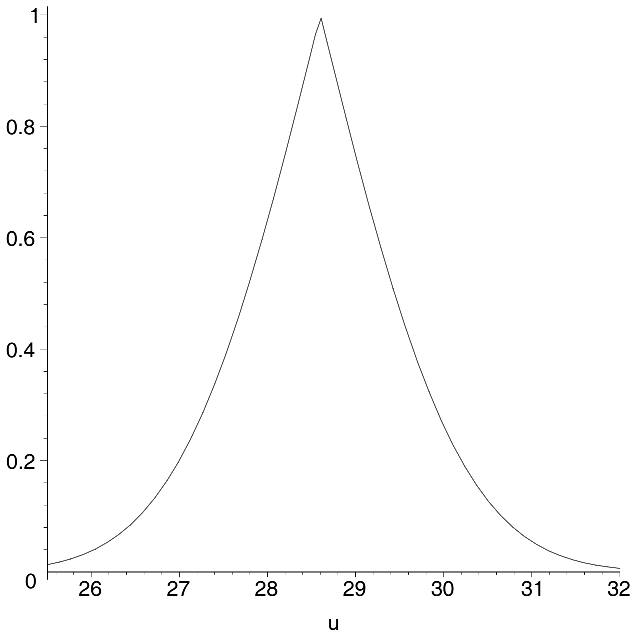

Fig. 1. Fuzzy mean $\overline{\mu}$ in Example 2.1

to complete its $\alpha-$cuts. So that $\overline{\theta}[\alpha] = [\theta_1(0.01), \theta_2(0.01)]$, for all $0 \leq \alpha \leq 0.01$. It is easy to generalize the method to the case where θ is a vector of parameters. We illustrate above method by the following example to estimate the mean of a Normal distribution [2].

Example *2.1:* [2] Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from a Normal population with unknown mean μ and known variance $\sigma^2 = 100$. Let $x_1, x_2, ..., x_n$ be the observations of the random sample. Suppose the mean of this random sample turns out to be 28.6. Then $(1-\beta)100\%$ confidence interval for μ is

$$\left[28.6 - z_{1-\beta/2}10/\sqrt{n}, 28.6 + z_{1-\beta/2}10/\sqrt{n}\right]$$

for all $0 < \beta \leq 1$, where $z_{1-\beta/2}$ is defined as

$$\int_{-\infty}^{z_{1-\beta/2}} N(0,1)dx = 1 - \beta/2,$$

and $N(0,1)$ denote the standard Normal density function. Assuming $n = 64$, we have shown the graph of $\overline{\mu}$ in Fig 1, using the software Maple [7]. Now if we drop the graph straight down from end points of interval for $\beta = 0.01$, then we will have a triangular shaped fuzzy number depicted in Fig.1.

In the next sections, we develop this method, where we use the $(1-\beta)100\%$ confidence intervals not only for $0.01 \leq \beta \leq 1$, but for all $0 \leq \beta \leq 1$. Especially, we consider $\Re$ as the confidence interval for the case $\beta = 0$.

III. Estimation the Parameter in a Statistical Model (General Case)

In this section, we derive the explicit and unique membership functions for fuzzy estimators of the parameters of a distribution in general.

Theorem *3.1:* Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n from a distribution with unknown parameter θ. If, based on observations $x_1, x_2, ..., x_n$, we consider $[\theta_1(\beta), \theta_2(\beta)]$, as a $(1-\beta)100\%$ confidence interval for θ, then the fuzzy estimation of θ is a fuzzy set with the following unique membership function

$$\bar{\theta}(u) = \min\left\{\theta_1^{-1}(u), (-\theta_2)^{-1}(-u), 1\right\}.$$

Proof: . $(1-\beta)100\%$ confidence intervals for θ are $[\theta_1(\beta), \theta_2(\beta)]$. Now change β to α and obtain intervals $[\theta_1(\alpha), \theta_2(\alpha)]$, the $\alpha-$cuts of a fuzzy set, say $\bar{\theta}$. Since these intervals are nested, we can obtain a unique fuzzy set based on them [6]. Furthermore, by Theorem 1.1, we have

$$\overline{\theta}(u) = \sup\left\{\alpha | u \in \overline{\theta}[\alpha]\right\}.$$

Now, if $u \in \overline{\theta}[\alpha]$, then $\theta_1(\alpha) \leq u \leq \theta_2(\alpha)$. Obviously, $\theta_1(\alpha)$ and $\theta_2(\alpha)$ are monotone functions of α, so that From $\theta_1(\alpha) \leq u$ we have $\alpha \leq \theta_1^{-1}(u)$. Also, $-\theta_2(\alpha) \leq -$ and therefore $\alpha \leq (-\theta_2)^{-1}(-u)$. $\theta_1^{-1}(u)$ or $(-\theta_2)^{-1}(-u)$ may be greater than one and then we have

$$\bar{\theta}(u) = \min\left\{\theta_1^{-1}(u), (-\theta_2)^{-1}(-u), 1\right\}.$$

The above theorem is basically in this paper and in the following sections we investigate fuzzy membership functions of parameter in some cases by using it.

IV. Estimation the Parameters of a Normal Distribution

In this section, we derive the explicit and unique membership functions for fuzzy estimators of the parameters of Normal distribution, in different cases, using Theorem 3.1.

Theorem *4.1:* Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n, from a Normal distribution with unknown mean μ and known variance σ^2. If based on observations $x_1, x_2, ..., x_n$, we consider

$$\left[\bar{x} - z_{1-\beta/2}\sigma/\sqrt{n}, \bar{x} + z_{1-\beta/2}\sigma/\sqrt{n}\right],$$

as a $(1-\beta)100\%$ confidence interval for μ, the fuzzy estimation of μ is a fuzzy set of $\Re$, with the following unique membership function

$$\overline{\mu}(u) = 2F\left(-\left|\frac{u-\bar{x}}{\sigma/\sqrt{n}}\right|\right), u \in \Re,$$

where F is the distribution function of the standard Normal distribution.

Proof: . Here $\theta_1(\alpha) = \bar{x} - z_{1-\beta/2}\sigma/\sqrt{n}$. Then $\theta_1^{-1}(u) = 2\left[1 - F\left(\frac{\bar{X}-u}{\sigma/\sqrt{n}}\right)\right]$. Also $\theta_2(\alpha) = \bar{x} + z_{1-\beta/2}\sigma/\sqrt{n}$ and therefore $[-\theta_2]^{-1}(-u) = 2\left[1 - F\left(\frac{u-\bar{X}}{\sigma/\sqrt{n}}\right)\right]$. Based on Theorem 3.1

$$\bar{\theta}(u) = \min\left\{2\left[1 - F\left(\frac{\bar{X}-u}{\sigma/\sqrt{n}}\right)\right], 2\left[1 - F\left(\frac{u-\bar{X}}{\sigma/\sqrt{n}}\right)\right], 1\right\}$$

that is equal to $\overline{\mu}(u) = 2F\left(-\left|\frac{u-\bar{x}}{\sigma/\sqrt{n}}\right|\right)$, $u \in \Re$.

Example *4.2:* Let $X_1, X_2, ..., X_{100}$ be a random sample of size 100 from a Normal distribution $N(\mu, \sigma^2)$, where $\sigma^2 = 4$,

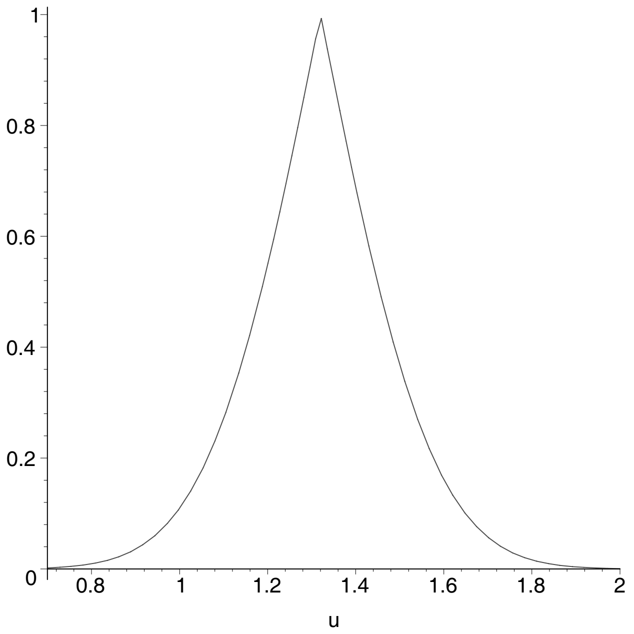

Fig. 2. Fuzzy estimation $\overline{\mu}$ in Example 4.1

with observed values $x_1, x_2, ..., x_{100}$, with $\bar{x} = 1.32$. Based on Theorem 4.1, the unique membership function of the fuzzy estimation $\overline{\mu}$ of μ is

$$\begin{aligned}\overline{\mu}(u) &= 2F\left(-\left|\frac{u-1.32}{2/\sqrt{100}}\right|\right) \\ &= 2F\left(-\left|5u-6.6\right|\right). \qquad (1)\end{aligned}$$

Therefore, according to our data, we estimate the parameter μ to be almost 1.32. Almost 1.32 is represented by the above membership function. The graph of its membership function is shown in Fig 2.

In the following theorem, we consider the case in which the variance of population is unknown.

Theorem *4.3:* Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n, from a Normal distribution with unknown mean and unknown variance σ^2. If based on observations, we consider

$$\left[\bar{x} - t_{1-\beta/2} s/\sqrt{n}, \bar{x} + t_{1-\beta/2} s/\sqrt{n}\right] ,$$

as a $(1-\beta)\,100\%$ confidence interval for μ, the fuzzy estimation of μ is a fuzzy set of $\Re$, with the following unique membership function

$$\overline{\mu}(u) = 2F\left(-\left|\frac{u-\bar{x}}{s/\sqrt{n}}\right|\right), u \in \Re,$$

where F is the distribution function of t-student distribution with n-1 degrees of freedom and s^2 is the sample variance.

Proof: . The proof is similar to the proof of Theorem 4.1.

The next theorem deals with the estimation of the variance of a Normal population.

Theorem *4.4:* Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n, from a Normal distribution with unknown mean μ and unknown variance σ^2. If based on observations $x_1, x_2, ..., x_n$, we consider

$$\left[\frac{(n-1)\,s^2}{\chi^2_{1-\beta/2,n-1}}, \frac{(n-1)\,s^2}{\chi^2_{\beta/2,n-1}}\right],$$

as a $(1-\beta)\,100\%$ confidence interval for σ^2, the fuzzy estimation of σ^2 is a fuzzy set of $\Re^+$, with the following unique membership function

$$\overline{\sigma^2}(u) = \begin{cases} 2F\left[\frac{(n-1)s^2}{u}\right] & 0 < \frac{(n-1)s^2}{u} \le M, \\ 2\left\{1 - F\left[\frac{(n-1)s^2}{u}\right]\right\}, & M \le \frac{(n-1)s^2}{u}, \end{cases} \quad u \in \Re^+,$$

where F is the distribution function of chi-squared with n-1 degrees of freedom, M is median and $\chi^2_{1-\beta/2,n-1}$ and $\chi^2_{\beta/2,n-1}$ are the percentiles of this distribution, respectively.

Proof: . Here $\theta_1(\alpha) = \frac{(n-1)s^2}{\chi^2_{1-\alpha/2}}$. Then $\theta_1^{-1}(u) = 2\left[1 - F\left(\frac{(n-1)s^2}{u}\right)\right]$. Also $\theta_2(\alpha) = \frac{(n-1)s^2}{\chi^2_{\alpha/2}}$ and therefore $[-\theta_2]^{-1}(-u) = 2\left[F\left(\frac{(n-1)s^2}{u}\right)\right]$. Based on theorem 3.1 $\bar{\theta}(u) = \min\left\{2\left[1 - F\left(\frac{(n-1)s^2}{u}\right)\right], 2\left[F\left(\frac{(n-1)s^2}{u}\right)\right], 1\right\}$ that is equal to

$$\overline{\sigma^2}(u) = \begin{cases} 2F\left[\frac{(n-1)s^2}{u}\right], & 0 < \frac{(n-1)s^2}{u} \le M, \\ 2\left\{1 - F\left[\frac{(n-1)s^2}{u}\right]\right\}, & M \le \frac{(n-1)s^2}{u}, \end{cases} , u \in \Re^+.$$

Example *4.5:* Let $X_1, X_2, ..., X_{10}$ be a random sample of size 10 from a Normal distribution with unknown mean and unknown variance. Suppose that $s = 0.4$ (data has been gotten from [5]). Based on Theorem 4.3, the unique membership function of the fuzzy estimator $\overline{\sigma^2}$ of σ^2 is

$$\begin{aligned}\overline{\sigma^2}(u) &= \begin{cases} 2F\left[\frac{9(0.16)}{u}\right], & 0 < \frac{9(0.16)}{u} \le 8.343 \\ 2\left\{1 - F\left[\frac{9(0.16)}{u}\right]\right\}, & 8.343 \le \frac{9(0.16)}{u} \end{cases} \\ &= \begin{cases} 2F\left[\frac{1.44}{u}\right], & 0.173 \le u \\ 2\left\{1 - F\left[\frac{1.44}{u}\right]\right\}, & 0 < u \le 0.173 \end{cases}\end{aligned}$$

Therefore, with respect to our data, we estimate σ^2 to be almost 0.173. Almost 0.173 is a fuzzy set with above membership function. The graph of its membership function is shown in Fig 3.

V. Fuzzy Estimation of $\mu_1 - \mu_2$ in two Normal Distributions

In this section we provide the unique fuzzy estimator for the difference of means in two independent Normal distributions.

A. Variances are known

Theorem *5.1:* Suppose that $X_1, X_2, ..., X_m$ is random sample of size m from a Normal distribution with unknown mean μ_1 and known variance σ_1^2, and $Y_1, Y_2, ..., Y_n$ is random

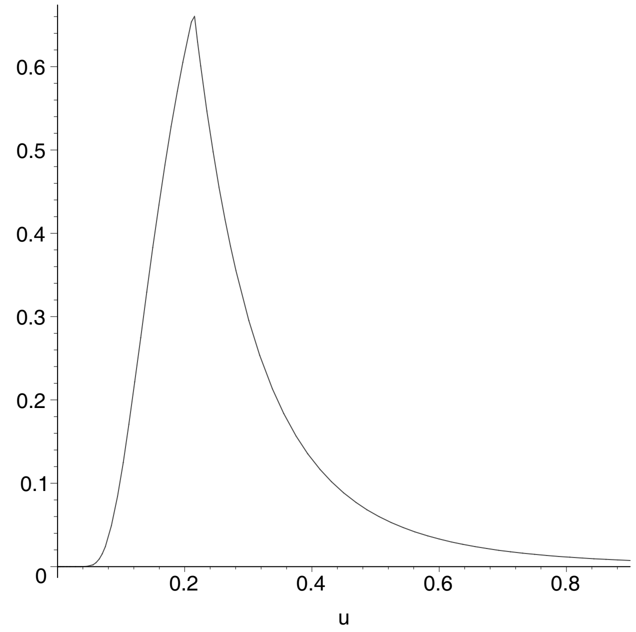

Fig. 3. Fuzzy estimation $\overline{\sigma^2}$ in Example 3.2

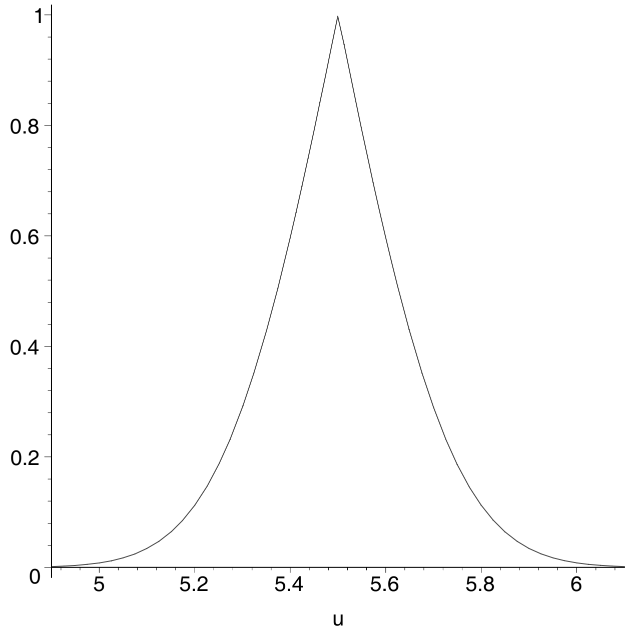

Fig. 4. Fuzzy estimation $\overline{(\mu_1 - \mu_2)}$ in Example 4.1.1

sample of size n from a Normal distribution with unknown mean μ_2 and known variance σ_2^2 (independent of the first sample). If, based on observations $x_1, x_2, ..., x_m$ and $y_1, y_2, ...y_n$, we consider

$$\left[\bar{x} - \bar{y} - z_{1-\beta/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}, \bar{x} - \bar{y} + z_{1-\beta/2}\sqrt{\frac{\sigma_1^2}{m} + \frac{\sigma_2^2}{n}}\right],$$

as a $(1-\beta)\,100\%$ confidence interval for $\mu_1 - \mu_2$, the fuzzy estimation of $\mu_1 - \mu_2$ is a fuzzy set of $\Re$, with the following unique membership function

$$\overline{(\mu_1 - \mu_2)}\,(u) = 2\left\{F\left[-\left|\frac{u - (\bar{x} - \bar{y})}{\sqrt{\sigma_1^2/m + \sigma_2^2/n}}\right|\right]\right\}, u \in \Re.$$

Proof: . The proof can be done in a similar way to previous theorems.

Example *5.2:* Let $X_1, X_2, ..., X_{100}$ be a random sample of size 100 from normal distribution with unknown mean μ_1 and variance $\sigma_1^2 = 2$ and $Y_1, Y_2, ...Y_{64}$ be a random sample of size 64 from Normal distribution with unknown mean μ_2 and variance $\sigma_2^2 = 1$ with observed values $x_1, x_2, ..., x_{100}$ and $y_1, y_2, ..., y_{64}$ with $\bar{x} = 10.6$ and $\bar{y} = 5.1$. Based on Theorem 5.1.1 the unique membership function of the fuzzy estimator $\overline{(\mu_1 - \mu_2)}$ of $\mu_1 - \mu_2$ is

$$\begin{aligned}\overline{(\mu_1 - \mu_2)}\,(u) &= 2\left[F\left(-\left|\frac{u - (10.6 - 5.1)}{\sqrt{2/100 + 1/64}}\right|\right)\right] \\ &= 2\left[F\left(-\left|\frac{u - 5.5}{0.189}\right|\right)\right].\end{aligned}$$

Therefore, based on our sample, we estimate $\mu_1 - \mu_2$ to be almost 5.5. Almost 5.5 is represented by a fuzzy set with above membership function. The graph of its membership function is shown in Fig 4.

B. Variances are unknown but equal

Theorem *5.3:* Suppose that $X_1, X_2, ..., X_m$ is a random sample of size m from a Normal distribution with unknown mean μ_1, and, independently, $Y_1, Y_2, ..., Y_n$ is a random sample of size n from a Normal distribution with unknown mean μ_2. Suppose that both variances are unknown but equal to σ^2. If, based on observations $x_1, x_2, ..., x_m$ and $y_1, y_2, ...y_n$, we consider

$$\left[\bar{x} - \bar{y} - t_{1-\beta/2}\sqrt{\frac{1}{m} + \frac{1}{n}}\sqrt{s_P^2}, \bar{x} - \bar{y} + t_{1-\beta/2}\sqrt{\frac{1}{m} + \frac{1}{n}}\sqrt{s_P^2}\right]$$

as a $(1-\beta)\,100\%$ confidence interval for $\mu_1 - \mu_2$, the fuzzy estimation of $\mu_1 - \mu_2$ is a fuzzy set of $\Re$ with the following unique membership function

$$\overline{(\mu_1 - \mu_2)}\,(u) = 2F\left[-\left|\frac{u - (\bar{x} - \bar{y})}{\sqrt{1/m + 1/n}\sqrt{s_P^2}}\right|\right]$$

where F is the distribution function of t-student with m+n-2 degrees of freedom, $t_{\beta/2}$ and $t_{1-\beta/2}$ are percentiles of this distribution respectively, s_1^2 and s_2^2 are sample variances and s_P^2 is equal to $\frac{(m-1)s_1^2 + (n-1)s_2^2}{m+n-2}$.

Proof: . The proof can be done in a similar way to previous theorems.

Example *5.4:* Let $X_1, X_2, ..., X_{10}$ be a random sample of size 10 from a Normal distribution with unknown mean μ_1, and $Y_1, Y_2, ..., Y_{15}$ be a random sample of size 15 from a Normal distribution with unknown mean μ_2 (independent of the first sample) and both variances are unknown and equal to σ^2. Suppose that the observed values of samples are $x_1, x_2, ..., x_{10}$ and $y_1, y_2, ...y_{15}$ with $\bar{x} = 2.1$ and $\bar{y} = 4.4$, $s_1^2 = 1.64$, and $s_2^2 = 1.71$. Based on Theorem 4.2.1 the unique membership function of the fuzzy estimator $\overline{(\mu_1 - \mu_2)}$

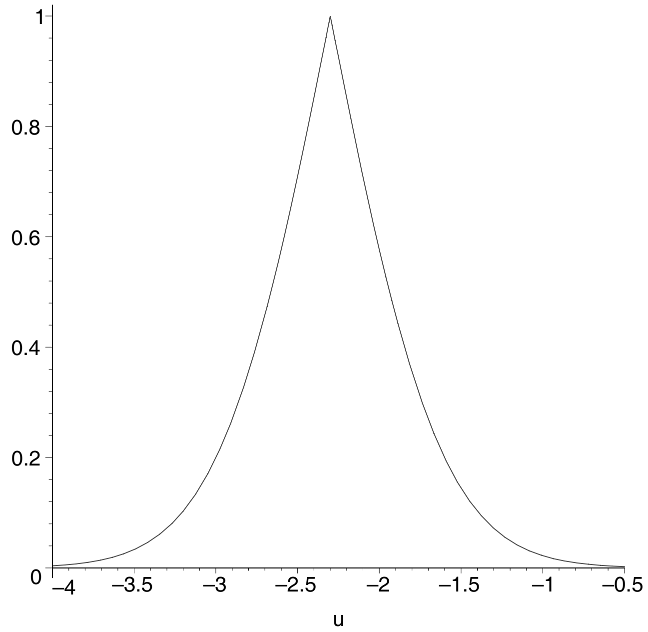

Fig. 5. Fuzzy estimation $\overline{(\mu_1 - \mu_2)}$ in Example 4.2.1

of $\mu_1 - \mu_2$ is

$$\begin{aligned} \overline{(\mu_1 - \mu_2)}(u) &= 2F\left[-\left|\frac{u-(2.1-4.4)}{\sqrt{1/10+1/15}\sqrt{1.68}}\right|\right] \\ &= 2\left[F\left(-\left|\frac{u+2.3}{0.53}\right|\right)\right] \end{aligned}$$

Therefore, we estimate the difference of means of two populations, $\mu_1 - \mu_2$, almost -2.3. Here, almost -2.3 is represented by a fuzzy set with above membership function. The graph of its membership function is shown in Fig 5.

VI. Fuzzy Estimation of σ_1^2/σ_2^2 in two Normal Distributions

In this section we provide the unique membership function of fuzzy estimator of the ratio of variances for two Normal distributions.

Theorem *6.1:* Suppose that $X_1, X_2, ..., X_m$ is a random sample of size m from a Normal distribution with unknown mean μ_1 and unknown variance σ_1^2, and $Y_1, Y_2, ..., Y_n$ is a random sample of size n from a Normal distribution with unknown mean μ_2 and unknown variance σ_2^2 (independent of the first distribution). If, based on observations $x_1, x_2, ..., x_m$ and $y_1, y_2, ..., y_n$, we consider

$$\left[\frac{s_1^2}{s_2^2} f_{\beta/2,n-1,m-1}, \frac{s_1^2}{s_2^2} f_{1-\beta/2,n-1,m-1}\right],$$

as a $(1-\beta)\,100\%$ confidence interval for $\frac{\sigma_1^2}{\sigma_2^2}$, then the fuzzy estimation of $\frac{\sigma_1^2}{\sigma_2^2}$ is a fuzzy set of $\Re$ with the following unique membership function

$$\overline{\left(\frac{\sigma_1^2}{\sigma_2^2}\right)}(u) = \begin{cases} 2F\left(u\frac{s_2^2}{s_1^2}\right), & 0 < u\frac{s_2^2}{s_1^2} \le M \\ 2\left[1 - F\left(u\frac{s_2^2}{s_1^2}\right)\right], & M \le u\frac{s_2^2}{s_1^2} \end{cases}$$

where F is the distribution function of Fisher distribution with n-1 and m-1 degrees of freedom, M is the median of this distribution, $f_{1-\beta/2}$ and $f_{\beta/2}$ are respectively the percentiles of distribution of $F_{m-1,n-1}$, and s_1^2 and s_2^2 are the sample variances.

Proof: . The proof can be done, in a similar way to previous theorems.

Example *6.2:* Let $X_1, X_2, ..., X_{12}$ be a random sample of size 12 from a Normal distribution with unknown mean and unknown variance, and (independently) $Y_1, Y_2, ..., Y_{10}$ be a random sample of size 10 from a Normal distribution with unknown mean and unknown variance. Suppose that the observed values of samples are $x_1, x_2, ..., x_{12}$ and $y_1, y_2, ..., y_{10}$ with $s_1^2 = 5.29$ and $s_2^2 = 2.25$ (data has been gotten from [5]). Based on Theorem 6.1 the unique membership function of the fuzzy estimator $\left(\frac{\sigma_1^2}{\sigma_2^2}\right)$ of $\frac{\sigma_1^2}{\sigma_2^2}$ is

$$\begin{aligned} \overline{\left(\frac{\sigma_1^2}{\sigma_2^2}\right)}(u) &= \begin{cases} 2F\left(u\frac{2.25}{5.29}\right) & 0 < u\frac{2.25}{5.29} \le 0.986 \\ 2\left[1 - F\left(u\frac{2.25}{5.29}\right)\right] & 0.986 \le u\frac{2.25}{5.29} \end{cases} \\ &= \begin{cases} 2F(0.425u) & 0 < u \le 2.318 \\ 2\left[1 - F(0.425u)\right] & 2.318 \le u \end{cases} \end{aligned}$$

Therefore, we estimate the ratio of variances of two populations, σ_1^2/σ_2^2, almost 2.318. Here, almost 2.318 is represented by a fuzzy set with above membership function. The graph of its membership function is shown in Fig 6.

VII. Fuzzy Estimation of λ in Exponential Distribution

Theorem *7.1:* Suppose that $X_1, X_2, ..., X_n$ be a random sample of size n, from an exponential distribution. If, based on observations $x_1, x_2, ..., x_n$, we consider

$$\left[\frac{\chi^2_{\beta/2,2n}}{2\sum_{i=1}^{n} x_i}, \frac{\chi^2_{1-\beta/2,2n}}{2\sum_{i=1}^{n} x_i}\right],$$

as $(1-\beta)100\%$ confidence interval for λ, then the fuzzy estimation of λ is a fuzzy set of $\Re^+$, with the following unique membership function

$$\overline{\lambda}(u) = \begin{cases} 2F\left[2u\sum_{i=1}^{n} x_i\right], & 0 < 2u\sum_{i=1}^{n} x_i \le M, \\ 2\left\{1 - F\left[2u\sum_{i=1}^{n} x_i\right]\right\}, & M \le 2u\sum_{i=1}^{n} x_i, \end{cases}$$

where F is the distribution function of chi-squared with 2n degrees of freedom, M is median and $\chi^2_{1-\beta/2,2n}$ and $\chi^2_{\beta/2,2n}$ are the percentiles of this distribution, respectively.

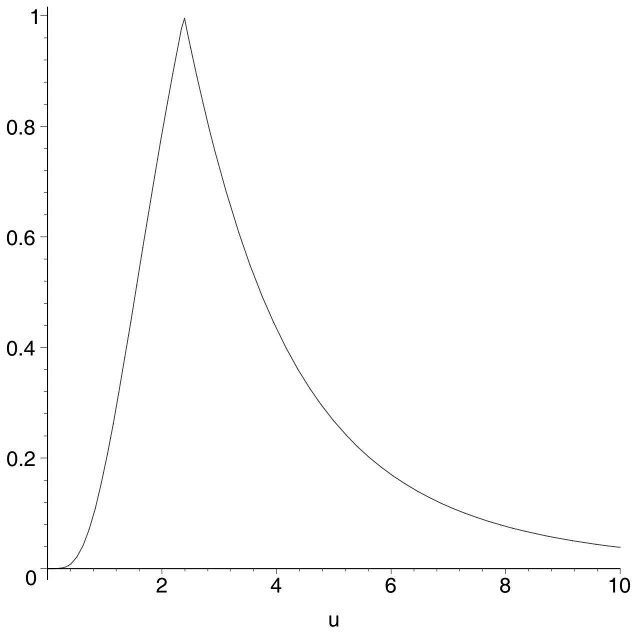

Fig. 6. Fuzzy estimation $\overline{\left(\frac{\sigma_1^2}{\sigma_2^2}\right)}$ in Example 5.1

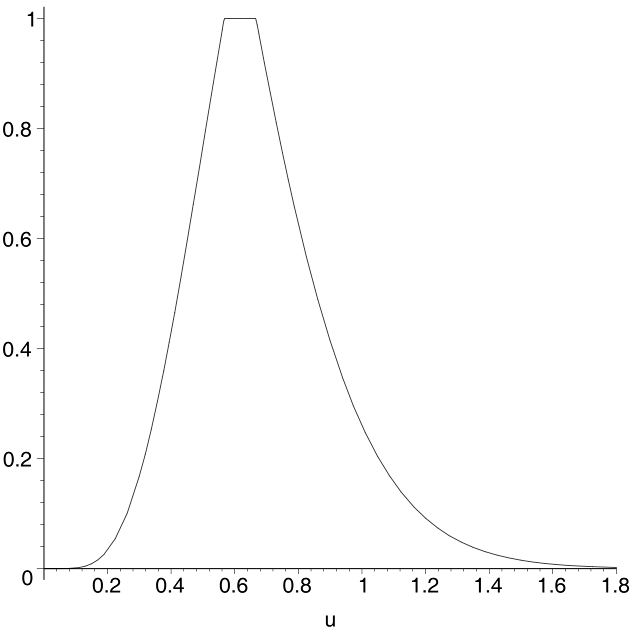

Fig. 7. Fuzzy estimation $\bar{\lambda}$ in Example 7.1

Proof: . The proof can be done in a similar way to previous Theorems.

VIII. Fuzzy Estimation of λ in Poisson Distribution

Theorem *8.1:* Suppose that $X_1, X_2, ..., X_n$ is a random sample of size n, from a Poisson distribution. If, based on observations $x_1, x_2, ..., x_n$, we consider

$$\left[\frac{\chi^2_{\beta/2,2y}}{2n}, \frac{\chi^2_{1-\beta/2,2y+2}}{2n}\right]$$

when $y = \sum_{i=1}^{n} x_i \neq 0$, and

$$\left[0, \frac{\chi^2_{1-\beta/2,2}}{2n}\right]$$

when $y = \sum_{i=1}^{n} x_i = 0$, as a $(1-\beta)\,100\%$ confidence interval for λ [3], then the fuzzy estimation of λ is a fuzzy set of $\Re^+$, with the following unique membership function

$$\bar{\lambda}(u) = \begin{cases} \min\left\{2F_1(2nu), 2\left[1 - F_2(2nu)\right], 1\right\}, & y \neq 0 \\ 2\left[1 - F_3(2nu)\right], & y = 0 \end{cases}$$

or

$$\bar{\lambda}(u) = \begin{cases} 2F_1(2nu), & 0 < 2nu \leq M_1 \\ 1, & M_1 \leq 2nu \leq M_2 \\ 2\left[1 - F_2(2nu)\right], & M_2 \leq 2nu \end{cases}$$

when $y \neq 0$ and

$$\bar{\lambda}(u) = 2\left[1 - F_3(2nu)\right]$$

when $y = 0$, where F_1, F_2, F_3 are the distribution functions of chi-squared with 2y, 2y+2 and 2 degrees of freedom, and M_1 and M_2 are the medians of two first distributions, respectively.

Proof: . Here we have $\theta_1(\alpha) = \frac{\chi^2_{\alpha/2,2y}}{2n}$ and $\theta_2(\alpha) = \frac{\chi^2_{1-\alpha/2,2y+2}}{2n}$ for $y \neq 0$. Therefore $\theta_1^{-1}(u) = 2F_1(2nu)$ and $\left(-\theta_2^{-1}\right)(-u) = 2\left(1 - F_2(2nu)\right)$. For $y = 0$ we have $\theta_1(\alpha) = 0$ that do not contain α and we are ignorant of it. Also $\theta_2(\alpha)$ is equal $\frac{\chi^2_{1-\alpha/2,2}}{2n}$ and $\left(-\theta_2^{-1}\right)(-u) = 2\left(1 - F_3(2nu)\right)$. The proof is obtained by minimizing the suitable phrases.

Example *8.2:* Let $X_1, X_2, ..., X_{10}$ be a random sample of size 10 from a Poisson distribution with unknown parameter λ, with observed value $y = \sum_{i=1}^{10} x_i = 6$. Based on Theorem 8.1, the membership function of the fuzzy estimator $\bar{\lambda}$ of λ is

$$\begin{aligned} \bar{\lambda}(u) &= \begin{cases} 2F_1(20u), & 0 < 20u \leq 11.34 \\ 1, & 11.34 \leq 20u \leq 13.339 \\ 2\left[1 - F_2(20u)\right], & 13.339 \leq 20u \end{cases} \\ &= \begin{cases} 2F_1(20u), & 0 < u \leq 0.567 \\ 1, & 0.567 \leq u \leq 0.667 \\ 2\left[1 - F_2(20u)\right], & 0.667 \leq u \end{cases} \end{aligned}$$

where F_1 and F_2 are distribution functions of chi-squared distribution with 12 and 14 degrees of freedom respectively. Therefore, we estimate λ almost $(0.567, 0.667)$. Here, almost $(0.567, 0.667)$ is represented by a fuzzy set with above membership function. The graph of its membership function is shown in Fig 7.

IX. Conclusion

We developed a method to derive the explicit and unique membership functions of so-called fuzzy estimations. Fuzzy estimations of the parameters of Normal distribution, and some functions of the parameters of two Normal distributions, have been derived for different cases. The method has been also applied to estimate the parameters of the Exponential and Poisson distributions. Thanks to resolution identity (Theorem 1.1) and characterization of fuzzy sets in terms of their $\alpha-$cuts (Theorem 1.2) that made this unique and explicit fuzzy estimations available. By these tools, the proposed method can be used for estimating parameters of other statistical models as well. In particular, further research is concerned with extending the proposed method to fuzzy estimation and prediction in fuzzy regression models. This makes the proposed method promising having interesting scientific potential impact on the future research in the field of fuzzy statistics.

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