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Non approximately inner tensor product of $C^\star$–algebras

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Abstract—In this paper, we show that $C^\star$– tensor product of an arbitrary $C^\star$– algebra A, (not unital necessary) and a $C^\star$– algebra B without ground state, have no approximately inner strongly continuous one–parameter group of $\star$–automorphisms.

Keywords—One–parameter group, $C^\star$– tensor product, Approximately inner, Ground state.

I. Introduction

Suppose $\{\alpha_t; -\infty < t < \infty\}$ is strongly continuous one–parameter group of $\star$– automorphisms of a $C^\star$– algebra A, where by strongly continuous we mean $\|\alpha_t(a) - a\| \to 0$, as $t \to 0$, for each $a \in A$. We say the group $\{\alpha_t\}$ is approximately inner if there exist a sequence $\{h_n\}$ of hermitian elements of A such that

$$\|e^{ith_n} a e^{ith_n} - \alpha_t(a)\| \to 0,$$

as $n \to \infty$, for each $a \in A$, where for fixed a the convergence is uniform for t in compact set.

In quantum field theory and statistical mechanics, one of the describes a physical system in the terms of a $C^\star$–algebra A.

In quantum lattice systems the dynamics is given by approximately inner one–parameter groups of $\star$–automorphism (see the references [5], [6]). It follows that quantum lattice systems have ground state. Recall, has shown the existence of ground state for quantum lattice system in [[4], theorems 2(c) and 4].

If α_t and β_t are strongly continuous one–parameters group of $\star$–automorphism with infinitesimal quantum δ_1 and δ_2 for $C^\star$– algebras A and B respectively, then $\{\alpha_t \otimes \beta_t\}$ is strongly continuous one–parameter group for $A \otimes B$ with infinitesimal quantum $\Big((\delta_1 \otimes I) + (I \otimes \delta_2)\Big)$.

In this paper we shoe that tensor product of an arbitrary $C^\star$– algebras A (not unital necessary) and a $C^\star$– algebras B without ground state, have no approximately inner strongly continues one–parameter group of $\star$–automorphisms.

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II. Preliminaries

in working with a strongly continuous one –parameter group of $\star$– automorphisms α_t it is often useful to introduce the unbounded derivation δ which generates the group. suppose α_t is a strongly continuous group of $\star$–automorphisms of a $C^\star$– algebra A. the generator of the group α_t is a derivation δ given by

$$\delta(a) = \lim_{t \to 0} (\alpha_t(a) - a)/t$$

where the domain $D(\delta)$ of δ is the linear manifolds of all $a \in A$ such that the above limit exists in the sense of norm convergence. It follows from semigroup theory (see [7]) and the fact that α_t are $\star$–automorphisms that δ has the properties,

i) $D(\delta)$ is a norm dense linear subset of A and δ is linear mapping of $D(\delta)$ into A.
ii) $D(\delta)$ is an algebra and if $a, b \in D(\delta)$ then $ab \in D(\delta)$ and $\delta(ab) = \delta(a)b + a\delta(b)$
iii) $D(\delta)$ is a $\star$–algebra and if $a \in D(\delta)$ then $a^\star \in D(\delta)$ and $\delta(a^\star) = \delta(a)^\star$
iv) δ is closed i.e, if $a_n \in D(\delta)$, $\|a_n - a\| \to 0$ and $\|\delta(a_n) - b\| \to 0$ as $n \to \infty$ then $a \in D(\delta)$ and $\delta(a) = b$.

We present the definitive of a ground state on a $C^\star$–algebra with respect to a one–parameter group of $\star$–automorphism this definitive is essentially the spectral condition of quantum field theory.

Definition 2.1: Suppose $\{\alpha_t\}$ is a one–parameter group of $\star$–automorphism of a $C^\star$–algebra A, we say ω is a ground state of A for the group $\{\alpha_t\}$, if ω is a state of A with the property, if $a, b \in A$ then $\omega(a\alpha_t(b))$ is a continuous function of t and

$$\int h(t)\omega(a\alpha_t(b))dt = 0,$$

for all continuous L^1–functions h whose Fourier transform

$$\tilde{h}(\lambda) = \frac{1}{\sqrt{2\pi}} \int e^{-it\lambda} h(t)dt,$$

vanishes on the negative real axis $(-\infty, 0]$.

Theorem 2.1: Suppose $\{\alpha_t\}$ is a one–parameter group of $\star$–automorphism of a $C^\star$–algebra A, suppose δ is the generator of $\{\alpha_t\}$ and D is a core for δ , then a state ω is a ground state for $\{\alpha_t\}$ if and only if

$$-i\omega(a^\star\delta(a)) \geq 0$$

for all $a \in D$.
Proof. See [1]

Theorem 2.2: Suppose $\{\alpha_t\}$ is a strongly continuous one–parameter group of $\star$–automorphisms of a $C^\star$–algebra A, suppose $\{\alpha_t\}$ is approximately inner, then there exists a ground state ω for $\{\alpha_t\}$. This ground state need not be unique.
Proof. See [1]

Let A, B be $C^\star$–algebras and $A \otimes B$ be theirs algebraic tensor product. Let π_1, π_2 be faithful representation of A, B on Hilbert spaces H_1, H_2 respectively, and define

$$\| \sum_j a_j \otimes b_j \|_s = \| \sum_j \pi_1(a_j) \otimes \pi_2(b_j) \|$$

where $a_j \in A, b_j \in B$ and the norm on the right hand side is the operator norm on the Hilbert space $H_1 \oplus H_2$. This norm is the Spatial $C^\star$–norm on $A \otimes B$ and refer to the $C^\star$–algebra $A \otimes_s B$ as the spatial tensor product of A and B.

Let δ_1, δ_2 be generators of strongly continuous one-parameter groups of automorphisms on A, B respectively. We define $\delta_1 \otimes I + I \otimes \delta_2$ on $A \otimes B$ by

$$(\delta_1 \otimes I + I \otimes \delta_2)(a \otimes b) = \delta_1(a) \otimes b + a \otimes \delta_2(b)$$

where $(a \in D(\delta_1), b \in D(\delta_2))$

In this paper we denote $\delta_1 \otimes I + I \otimes \delta_2$ by $\delta_1 \otimes \delta_2$. The $\delta_1 \otimes \delta_2$ is closable $\star$–derivation and its closure is an infinitesimal generator on $A \otimes_s B$. [1]

Let G be a locally compact group and let μ be a left invariant Haar measure on G and let $L^1(G)$ be the Banach space of all complex valued μ–integrable functions on G. For $f, g \in L^1(G)$ define a multiplication $\star$ and $\star$–operation as follows:

$$\begin{aligned} f \star g(x) &= \int_G f(xy) g(y^{-1}) dy \\ &= \int_G f(y) g(y^{-1}x) dy, \end{aligned}$$

and $f^\star(x)\Delta(x^{-1})\bar{f}(x^{-1})$, where Δ is the modular function on G. Let $f \in L^1(G)$ and define the operator L_f on $L^2(G)$ by $L_f(g) = f \star g, (g \in L^2(G))$, then the mapping $f \to L_f$ is a bounded representation of the algebra $L^1(G)$. For instance, $4L_{f_1}L_{f_2}(g) = f_1 \star (f_2 \star g) = (f_1 \star f_2) \star g = L_{f_1 \star f_2}(g).4$ Hence $L_{f_1 \star f_2} = L_{f_1} L_{f_2}$ the inequality

$$\|f \star g\|_2 \leq \|f\|_1 \|g\|_2,$$

implies that $\|L_f\| \leq \|f\|_1$, where $f \in L^1(G), g \in L^2(G)$.

Suppose $K(G)$ is the set of all complex–valued square summable functions on G with compact support. $K(G)$ is$\star$–subalgebra of $L^1(G) \cap L^2(G)$. Define

$$T(G) = \{L_f : \ f \in K(G)\},$$

and $C_r^\star(G)$ to be the $C^\star$–algebra generated by $T(G)$. $C_r^\star(G)$ is the reduced $C^\star$–algebra of G.

If $f \in L^1(G)$, there exist a sequence $\{f_n\}$ in $K(G)$ such that $\|f_n - f\|_1 \to 0$ thus

$$\|L_{f_n} - L_f\| \leq \|f_n - f\|_1 \to 0,$$

therefore $C_r^\star(G)$ is the $C^\star$–algebra generated by the set

$$\{L_f \ : \ f \in L^1(G)\}.$$

Define $K'(G)$ by

$$K'(G) = \{f \in L^1(G) : f \text{ has a compact support}\}$$

then $K(G) \subseteq K'(G)$ and $\star$–subalgebra

$$D = \{L_f : \ f \in K'(G)\}$$

is dense in $C_r^\star(G)$

Let θ be a complex–valued measurable function on G , such that θ is bounded on any compact subset of G.

if $f \in K'(G)$, then $\theta f \in K'(G)$. Since

$$\begin{aligned} \int_G |\theta(x) f(x)| dx &= \int_G |\theta(x)||f(x)| dx \\ &\leq \sup_{x \in C} |\theta(x)\|f\|_1 \end{aligned}$$

where C is the support of f.

Suppose $\mathrm{Hom}(G, R)$ is the set of all real–valued homomorphisms from G to R and θ is a continuous homomorphism in $\mathrm{Hom}(G, R)$.

We define δ_θ from D into D by $\delta_\theta(L_f) = iL_{\theta f}$. Niknam in [] has shown that δ_θ is closable $\star$–derivation and its closure is an infinitesimal generator of $C_r^\star(G)$.

Theorem 2.3: Let G be a locally compact group and $\theta \in \mathrm{Hom}(G, R)$ be measurable function, then δ_θ is closable $\star$–derivation from D to D and its closure $\bar{\delta_\theta}$ is an infinitesimal generator of a strongly continuous one–parameter group of $\star$–automorphisms.
Proof. See [1]

In the proof of the above theorem, if we define $\alpha_t \ : D \to D$ be $\alpha_t(L_f) = L_{e^{it\theta_f}}$, where $f \in K'(G)$, then $\{\alpha_t\}$ is a strongly continuous one–parameter group of $\star$–automorphisms by infinitesimal generator δ_θ, for $\theta \in \mathrm{Hom}(G, R)$.

III. The result

In this section, the main result of this work mentioned as a following theorem.

Theorem 3.1: Let A and B be $C^\star$–algebra and B is not unital necessary. Suppose that $\{\alpha_t\}$ and $\{\beta_t\}$ are strongly continuous one–parameter group of $\star$–automorphisms on A and B with infinitesimal generators δ_1 and δ_2 respectively. If $\{\alpha_t\}$ has not ground state and if there exist an element $x \in B$ such that $\delta_2(x) = 0$, then, the one–parameter automorphism group $\{\alpha_t \otimes \beta_t\}$ of $A \otimes_s B$ is not approximately inner.

Proof: If $\{\alpha_t \otimes \beta_t\}$ were approximately inner, then, by using Theorem 2.2, would be a ground state ω for $\{\alpha_t \otimes \beta_t\}$ an $A \otimes_s B$. Let Φ be the state on A defined by

$$\Phi(a) = \omega(a \otimes x^\star x)$$

where $\delta_2(x) = 0$.

Since

$$\begin{aligned}
&(a \otimes x)^\star(\delta_1 \otimes \delta_2)(a \otimes x)\\
&\quad = (a \otimes x)^\star(\delta_1 \otimes I + I \otimes \delta_2)(a \otimes x)\\
&\quad = (a^\star \otimes x^\star)\big[(\delta_1 \otimes I)(a \otimes x) + (I \otimes \delta_2)(a \otimes x)\big]\\
&\quad = (a^\star \otimes x^\star)\big[(\delta_1(a) \otimes x) + (a \otimes \delta_2(x))\big]\\
&\quad = (a^\star \otimes x^\star)(\delta_1(a) \otimes x)\\
&\quad = (a^\star \delta_1(a) \otimes x^\star x),
\end{aligned}$$

Hence

$$\begin{aligned}
-i\Phi(a^\star \delta_1(a)) &= -i\omega(a^\star \delta_1(a) \otimes x^\star x)\\
&= -i\omega\Big((a \otimes x)^\star(\delta_1 \otimes \delta_2)(a \otimes x)\Big) \geq 0,
\end{aligned}$$

it follows by theorem 2.1 that Φ would be a ground state for $\{\alpha_t\}$, the contradiction shows that $\{\alpha_t \otimes \beta_t\}$ is not approximately inner.

Following example clear above theorem:

Example 3.1: If $G = R$ be is a locally compact group, then by Theorem 2.3, there exist a strongly continuous one–parameter group of $\star$–automorphisms $\{\alpha_t\}$ with infinitesimal generator δ_θ for $\theta \in \mathrm{Hom}(R, R)$ of reduced $C^\star$–algebra $C_r^\star(R)$, for function $f \in L^1(R)$, by

$$f(x) = \begin{cases} 0 & x \in Q \\ 1 & x \in R - Q \end{cases}$$

we have $\delta_\theta(L_f) = 0$, hence if $\{\beta_t\}$ be a one–parameter group of $\star$–automorphisms on $C^\star$–algebra B without ground state, then by theorem 3.1 $\{\alpha_t \otimes \beta_t\}$ is a strongly continuous one–parameter group of $\star$–automorphisms on $C_r^\star(R) \otimes A$ that is not approximately inner. In particular if G be a discrete group, then by [2], $C_r^\star(G)$ has a one parameter group without ground state. Hence, we can apply it instead A in above example.

IV. Conclusion

In this paper, we had shown that tensor product of an arbitrary $C^\star$– algebra A, (not unital necessary) and a $C^\star$–algebra B without ground state, have no approximately inner strongly continuous one–parameter group of $\star$–automorphisms.

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