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Rear Separation in a Rotating Fluid at Moderate Taylor Numbers

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***Abstract*—**The motion of a sphere moving along the axis of a rotating viscous fluid is studied at high Reynolds numbers and moderate values of Taylor number. The Higher Order Compact Scheme is used to solve the governing Navier-Stokes equations. The equations are written in the form of Stream function, Vorticity function and angular velocity which are highly non-linear, coupled and elliptic partial differential equations. The flow is governed by two parameters Reynolds number (Re) and Taylor number (*T*). For very low values of Re and *T*, the results agree with the available experimental and theoretical results in the literature. The results are obtained at higher values of Re and moderate values of *T* and compared with the experimental results. The results are fourth order accurate.

***Keywords*—**Navier_Stokes equations, Taylor number, Reynolds number, Higher order compact scheme, Rotating Fluid.

I. INTRODUCTION

IN a fluid of constant density ρ and kinematic viscosity ν, which is in solid body rotation with angular velocity Ω, a sphere of radius 'a' moves with the speed U along the axis of rotation which is free to rotate about the same axis. The motion created by a sphere is of great interest since in this flow two kinds of basic motion rotation and translation, interact and modify each other. In a viscous flow the flow depends upon two dimensionless parameters namely the Ekman number E and the Rossby number R_0 where

$$E = \frac{\nu}{\omega_0 a^2} \quad , \quad R_0 = \frac{U}{2\omega_0 a}$$

With ω_0 is the constant angular velocity at large enough distances from the sphere. The Taylor number (*T*) and the Reynolds number (Re) are defined as

$$T = \frac{1}{E} \quad , \quad \mathrm{Re} = \frac{2R_0}{E} \cdot$$

Proudman (1916) theoretically predicted that in such a kind of flow, a column of fluid is pushed ahead of the sphere like a solid mass having zero axial velocity relative to the moving body. This prediction was confirmed by [7] experimentally. This phenomenon is now referred as the Taylor column. [2] studied the flow theoretically by the method of singular perturbation technique, for small values of Re and T.

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These results were experimentally verified by [4]. Maxworthy also found that for a fixed Reynolds number, when the rotation parameter T increases, the drag also increases. [5] experimentally observed forward separation of the axi-symmetric flow for large values of Re and *T*. Dennis et al (1982) analyzed the problem by the method of second order finite differences for Reynolds numbers from 0 to 0.5 and values of T ranging from 0 to 0.5. Using series expansion method and monopole approximation method, Weisenborn (1984) studied the problem for zero Reynolds number and values of *T* ranging from zero to infinity. In this paper the problem is studied using Higher order compact scheme for higher values of Re and moderate values of *T*.

II. FORMULATION OF THE PROBLEM.

We have considered a sphere of radius 'a' whose centre is fixed at origin with the coordinate axes are fixed in direction. The sphere is moving with constant velocity U in the negative Z direction in an incompressible viscous fluid. The Navier-Stokes equations governing the steady axi-symmetric flow can be written as three coupled, nonlinear, elliptic partial differential equations. The governing equations in spherical polar co-ordinates (r, θ, φ) with the transformation $r = e^{\xi}$ are

$$\frac{\partial^2 \psi}{\partial \xi^2} + \frac{\partial^2 \psi}{\partial \theta^2} - \frac{\partial \psi}{\partial \xi} - \cot\theta \frac{\partial \psi}{\partial \theta} = -e^{3\xi} \varsigma \sin\theta \tag{1}$$

$$\frac{\partial^2 \varsigma}{\partial \xi^2} + \frac{\partial^2 \varsigma}{\partial \theta^2} - \frac{\partial \varsigma}{\partial \xi} - \cot\theta \frac{\partial \varsigma}{\partial \theta} = \frac{R}{e^{\xi} \sin\theta} \left\{ \begin{array}{l} \left(\frac{\partial \psi}{\partial \theta} \cdot \frac{\partial \varsigma}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \cdot \frac{\partial \varsigma}{\partial \theta} \right) + \\ 2\left(\cot\theta \frac{\partial \psi}{\partial \xi} - \frac{\partial \psi}{\partial \theta} \right) \varsigma - \frac{2T^2}{R} \left(\cot\theta \frac{\partial \Omega}{\partial \xi} - \frac{\partial \Omega}{\partial \theta} \right) \Omega \end{array} \right\} \tag{2}$$

$$\frac{\partial^2 \Omega}{\partial \xi^2} + \frac{\partial^2 \Omega}{\partial \theta^2} - \frac{\partial \Omega}{\partial \xi} - \cot\theta \frac{\partial \Omega}{\partial \theta} = \frac{R}{e^{\xi} \sin\theta} \left\{ \left(\frac{\partial \psi}{\partial \theta} \cdot \frac{\partial \Omega}{\partial \xi} - \frac{\partial \psi}{\partial \xi} \cdot \frac{\partial \Omega}{\partial \theta} \right) \right\} \tag{3}$$

where ψ is the dimensionless Stream function, $\frac{\varsigma}{e^{\xi} \sin\theta}$ is the vorticity and $\frac{\Omega}{e^{\xi} \sin\theta}$ is the angular velocity. The dimensionless velocity components are related to ψ and Ω by the equations

$$v_r = \frac{e^{-2\xi}}{\sin\theta} \frac{\partial \psi}{\partial \theta}, \; v_\theta = -\frac{e^{-\xi}}{\sin\theta} \frac{\partial \psi}{\partial \xi}, \; v_\varphi = \frac{e^{-\xi}}{\sin\theta} \Omega.$$

III. BOUNDARY CONDITIONS

The equations (1)-(3) are solved subject to the following boundary conditions

On the surface of the sphere (ξ=0)

$$\psi = \frac{\partial \psi}{\partial \xi} = 0,\ \Omega = \tilde{\omega} \sin^2 \theta$$

At large distances from the sphere $(\xi \to \infty)$

$$\psi = \frac{1}{2} e^{2\xi} \sin^2 \theta,\ \Omega = e^{2\xi} \sin^2 \theta, \varsigma = 0$$

Along the axis of symmetry $(\theta = 0^0, \theta = 180^0)$

$$\psi = \varsigma = \frac{\partial \Omega}{\partial \theta} = 0.$$

IV. NUMERICAL METHOD

The above governing equations are discretized using higher order compact scheme (HOCS) on nine point compact stencil. The first and second order derivatives are approximated by the central differences along with their leading errors as follows.

$$\frac{\partial U}{\partial \xi} = \delta_\xi U - \frac{h^2}{6} \cdot \frac{\partial^3 U}{\partial \xi^3} + O(h^4)$$

$$\frac{\partial^2 U}{\partial \xi^2} = \delta_\xi^2 U - \frac{h^2}{12} \cdot \frac{\partial^4 U}{\partial \xi^4} + O(h^4)$$

$$\frac{\partial U}{\partial \theta} = \delta_\theta U - \frac{k^2}{6} \cdot \frac{\partial^3 U}{\partial \theta^3} + O(k^4)$$

$$\frac{\partial^2 U}{\partial \theta^2} = \delta_\theta^2 U - \frac{k^2}{12} \cdot \frac{\partial^4 U}{\partial \theta^4} + O(k^4)$$

where

$$\delta_\xi U_{i,j} = \frac{U_{i+1,j} - U_{i-1,j}}{2h}, \qquad \delta_\xi^2 U_{i,j} = \frac{U_{i+1,j} - 2U_{i,j} + U_{i-1,j}}{h^2}$$

$$\delta_\theta U_{i,j} = \frac{U_{i,j+1} - U_{i,j-1}}{2k}, \qquad \delta_\theta^2 U_{i,j} = \frac{U_{i,j+1} - 2U_{i,j} + U_{i,j-1}}{k^2}$$

The third and fourth derivatives occurring above are calculated by differentiating the given partial differential equation which introduces cross derivatives which are approximated by second order central differences. The matrices associated with the finite difference equations are diagonally dominant. The algebraic equations are solved using line Gauss-Seidel iteration method. Among the variables, we first solved for Ω, and then for ς, and then for ψ. Convergence is said to have been achieved when the norm of the dynamic residuals is less than 10^{-6}. The vorticity on the surface of the sphere is obtained using Briley (1971) formula

$$,\ \omega_{1,j} = -\frac{108\ \psi_{2,j} - 27\ \psi_{3,j} + 4\psi_{4,j}}{18\ h^2 \sin\ \theta}.$$

The value of Ω on the surface is depending on the value of $\tilde{\omega}$. Childress (1964) had given the expression as

$$\tilde{\omega} = \chi(\alpha) \cdot \mathrm{Re}$$

with

$$\alpha = \frac{2T}{\mathrm{Re}^2}$$

where

$$\chi(\alpha) = \frac{3i}{8\alpha} \int_0^1 \left\{ (t^2 + 4i\alpha t)^{\frac{1}{2}} - (t^2 - 4i\alpha t)^{\frac{1}{2}} \right\} (3t^3 - t) dt \cdot$$

For small values of α ranging from 0 to 1, the values of χ(α) found by Childress (1964) are provided in Table. II. For large values of α, the corresponding value of χ(α) can be obtained from the polynomial expression (Childress 1964)

$$\chi(\alpha) = -\frac{\sqrt{2}}{5\sqrt{\alpha}} \left(1 - \frac{75}{616} \cdot \frac{1}{\alpha} + \frac{35}{4992} \cdot \frac{1}{\alpha^2} - \frac{45}{28160} \cdot \frac{1}{\alpha^3} + \cdots \right)$$

V. RESULTS AND DISCUSSION

A far field of 21.58 times the radius of the sphere is considered in all the numerical simulations which are performed in the finest grid of 240 × 128. Numerical investigations were carried out for Reynolds numbers of 25, 40 and 100 and rotating parameter (T) from 0 to 4.0. The initial solution for $T = 0$ is taken as $\psi = 0$, $\varsigma = 0$ and $\Omega = 0$ at all the grid points inside the boundary. In finding the solution for higher values of Re and T, the solutions obtained for lower values of Re and T are used as the starting solution. Using the solution Drag co-efficient C_D and $D/D_s = (C_{D*}\mathrm{Re})/12$ were calculated.

The results are computed for Re = 0.12 at T = 0.025, 0.05, 0.075 and 0.1 in different grids and is presented in Table.I to show the grid independence. D/D_s values for Re = 0.12 at different values of T are presented in Table. III along with experimental values of [4] and numerical results of Childress (1964), Weisenborn (1984) and Dennis et.al (1982). The obtained results are in agreement with experimental results of Maxworthy (1965) and numerical results of Weisenborn (1984) and Dennis et.al (1982).

The results are also obtained at higher values of Re = 25, 40 and 100 and for values of T ranging from 0 to 4.0 and the drag coefficient values are presented in Fig 1. From this figure it is clear that the drag coefficient increases with increase of T which agree with experimental results of Maxworthy (1965). The streamlines for Re = 25,40 and 100 for different values of T are presented in Fig. 2 - Fig.4 respectively. It is observed that as T increases the separation length and angle increases.

TABLE I
GRID INDEPENDENCE OF **D/D_s** VALUES

T	48X48	64X64	80X80	96X96
0.025	1.102923596	1.102716017	1.102575950	1.102523755
0.05	1.148402303	1.147745547	1.147226625	1.147042066
0.075	1.185303233	1.184232138	1.183373917	1.183061173
0.1	1.216450312	1.216110115	1.214975468	1.214502026

TABLE II
VALUES OF FOR SMALL VALUES OF α

α	0	0.1	0.23	0.37	0.5	0.67	0.83
χ(α)	0.375	0.375	0.367	0.337	0.31	0.287	0.266

TABLE III
COMPARISON OF **(D/D$_s$) -1** VALUES WITH NUMERICAL AND EXPERIMENTAL VALUES

T	(1)	(2)	(3)	(4)	(5)	(6)
0.025	0.09	0.1	0.1	0.11	0.1	0.09±0.01
0.05	0.13	0.14	0.14	-----	0.15	0.13±0.02
0.075	0.16	0.18	0.18	-----	0.18	0.18±0.02
0.1	0.18	0.21	0.21	-----	0.22	0.22±0.03
0.2	0.26	0.32	0.32	-----	0.33	0.30±0.04
0.25	0.29	0.37	0.37	0.31	0.37	0.37±0.04
0.5	0.4	0.56	0.57	0.63	0.56	0.57±0.05
0.75	0.49	0.72	0.74	-----	0.71	0.75±0.05
1	0.57	0.83	0.9	-----	0.86	-----

Numerical results of (1) Childress(1964), (2) Weisenborn Series Expansion (1984), (3) Weisenborn Monopole approximation(1984), (4) Dennis et al(1982), (5) Present results and (6) Experimental results of Maxworthy(1965).

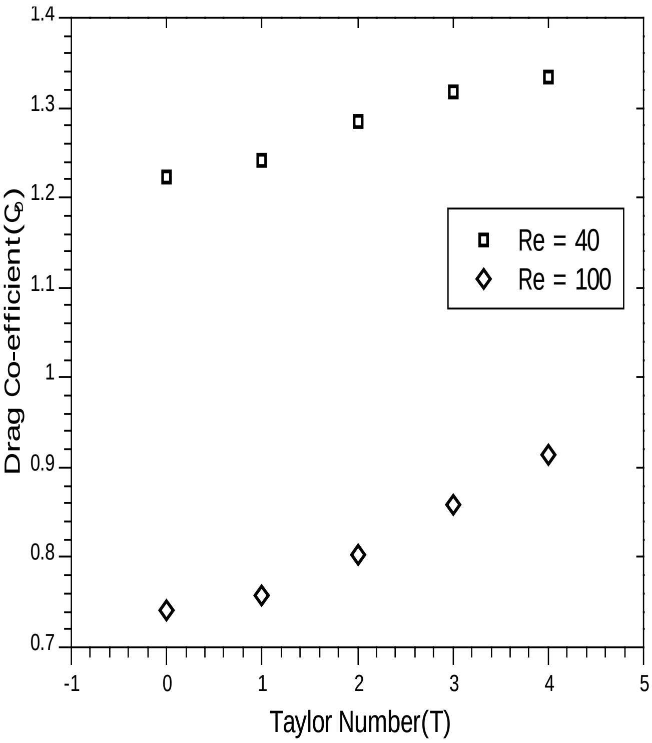

Fig. 1 Drag co-efficients corresponding to different Taylor numbers for Re=40 and 100

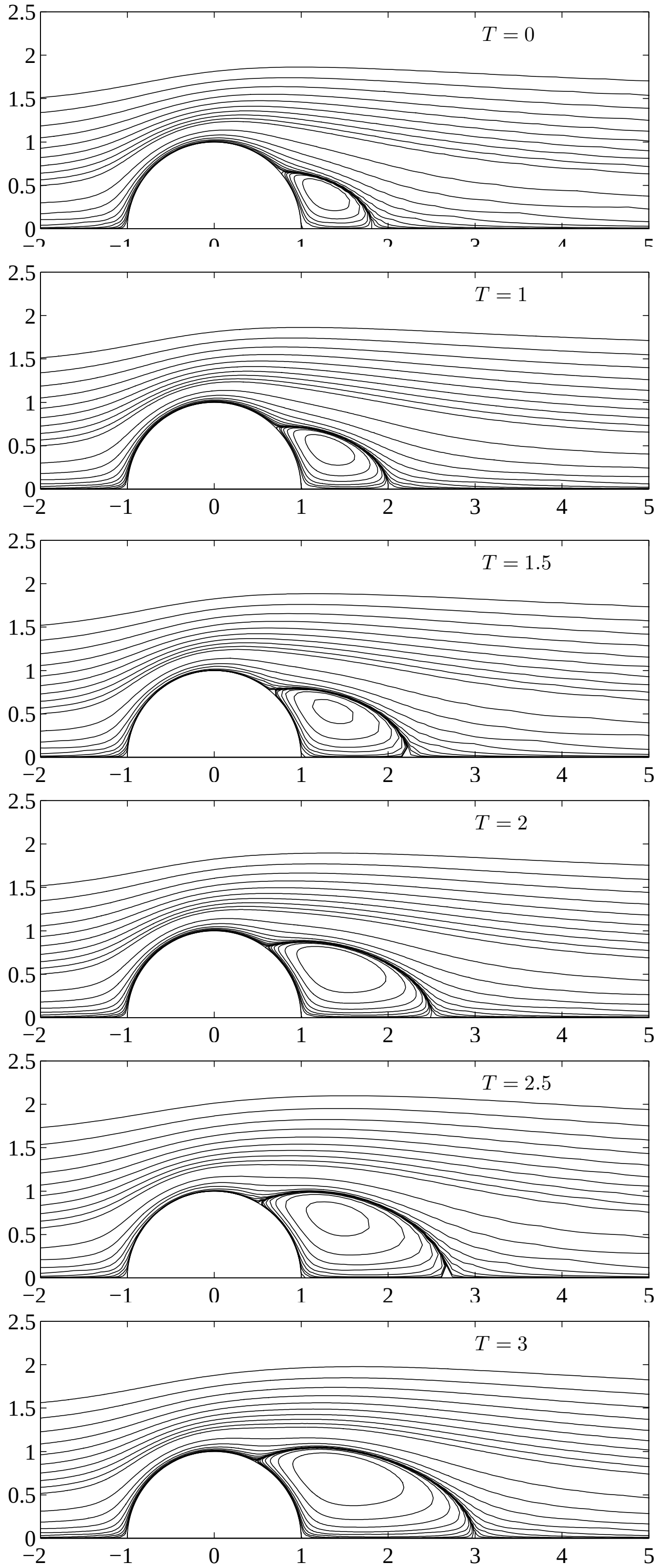

Fig. 2 Streamlines for Re=25 at T=0,1,1.5,2,2.5 and 3

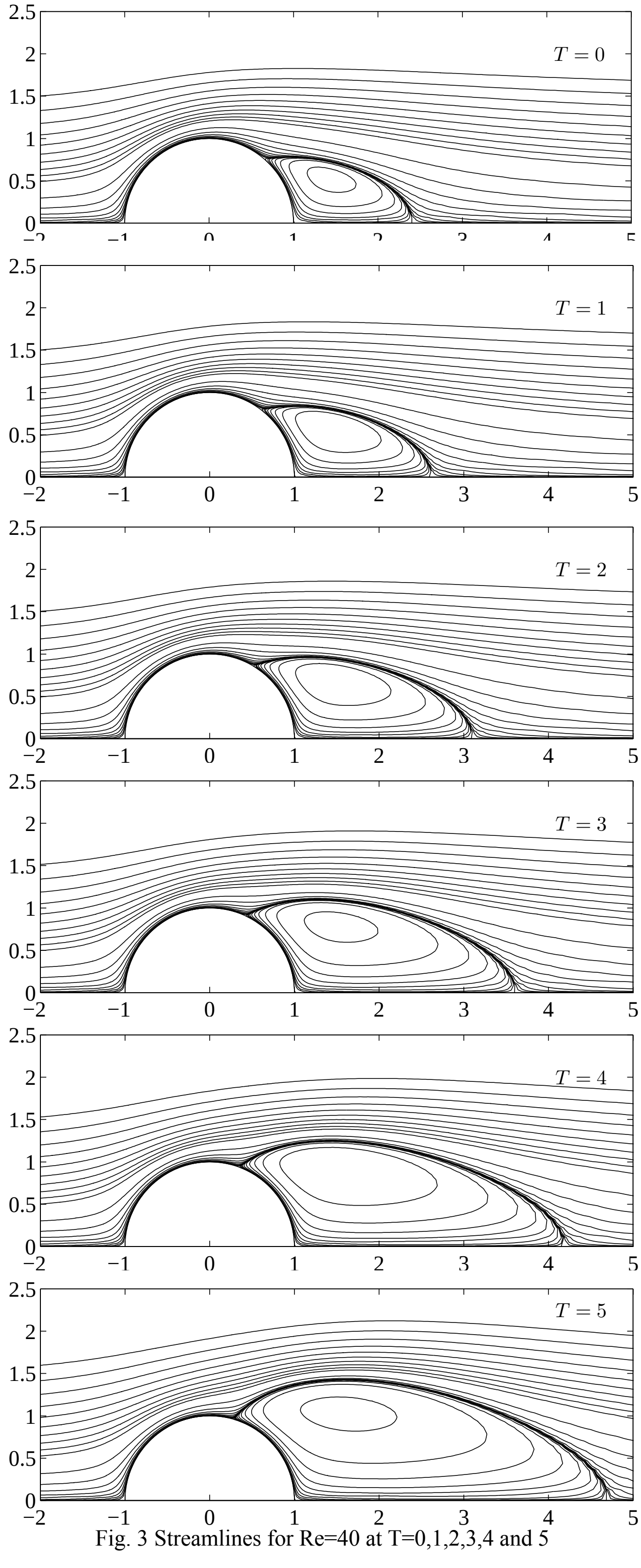

Fig. 3 Streamlines for Re=40 at T=0,1,2,3,4 and 5

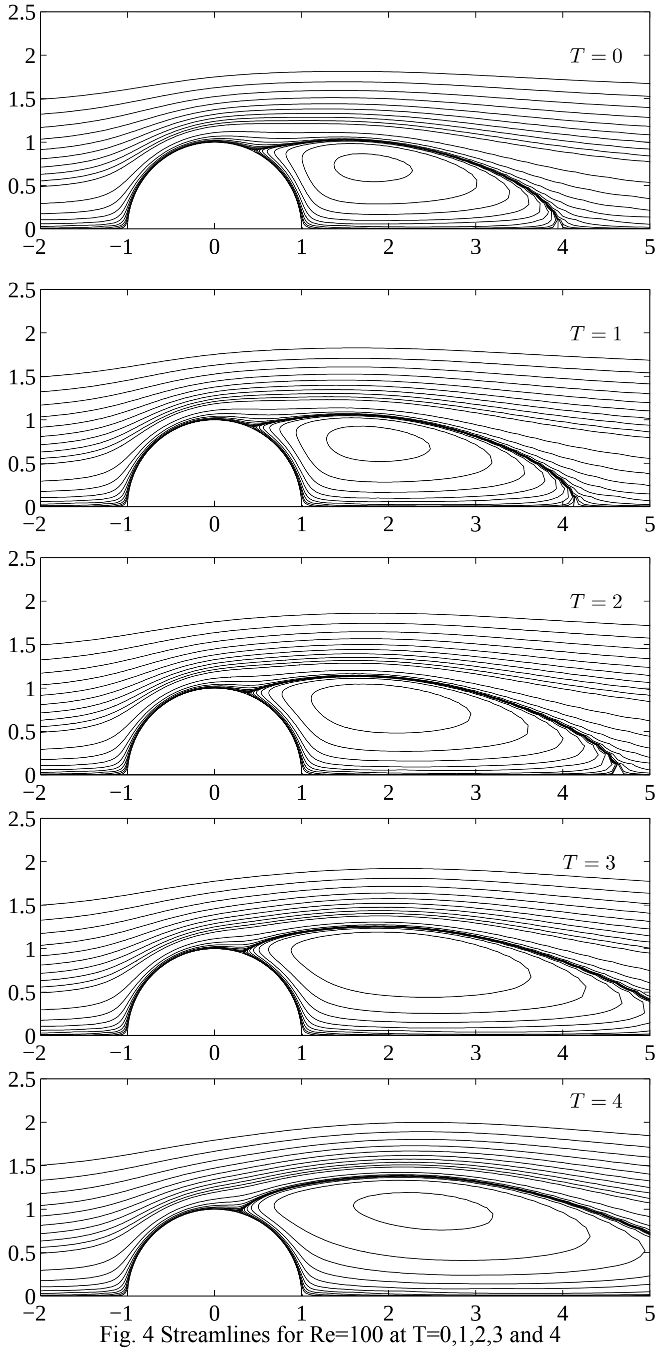

Fig. 4 Streamlines for Re=100 at T=0,1,2,3 and 4

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