The Spanning Laceability of k-ary n-cubes when k is Even

Yuan-Kang Shih, Shu-Li Chang, and Shin-Shin Kao

Abstract—Q_n^k has been shown as an alternative to the hypercube family. For any even integer $k \geq 4$ and any integer $n \geq 2$, Q_n^k is a bipartite graph. In this paper, we will prove that given any pair of vertices, w and b, from different partite sets of Q_n^k, there exist $2n$ internally disjoint paths between w and b, denoted by $\{P_i \mid 0 \leq i \leq 2n-1\}$, such that $\bigcup_{i=0}^{2n-1} P_i$ covers all vertices of Q_n^k. The result is optimal since each vertex of Q_n^k has exactly $2n$ neighbors.

Keywords—container, Hamiltonian, k-ary n-cube, m^*-connected.

I. INTRODUCTION

The k-ary n-cube, denoted by Q_n^k, has been proposed as an alternative to the hypercube since it shares many nice properties of Q_n such as regular degrees, vertex symmetry, edge symmetry, recursive structure, etc.. The underlying topology of many machines is based on k-ary n-cubes, such as the Cray T3E, the iWARP, the Cray T3D and so on. Please see [1], [4], [11], [17]. Many reseachers have been working on k-ary n-cubes. For example, Stewart and Xiang [20] proved that the k-ary n-cube is edge-bipancyclic and bipanconnected for $k \geq 3$ and $n \geq 2$ and k being even. Namely, any edge of a k-ary n-cube Q_n^k lies on a cycle of any even length r for $4 \leq r \leq |Q_n^k|$, where $|Q_n^k|$ is the total number of vertices of Q_n^k. Besides, given two vertices u and v of Q_n^k, there exists a path of any even length r between u and v for $d(u,v) \leq r \leq |Q_n^k|$, where $d(u,v)$ is the distance between u and v. Other studies about fault tolerance on k-ary n-cubes can be found in [8], [23]. Recently, there are many studies about the spanning connectivity for interconnection networks and graphs [9]. A graph $H = (B \bigcup W, E)$ is *bipartite* if $V(H)$ is the union of two disjoint sets B and W such that every edge joins B with W. It is easy to see that any bipartite graph with at least three vertices is not hamiltonian connected except K_2. Note that any (nontrivial) bipartite graph except K_2 cannot be hamiltonian connected, whereas a bipartite graph is *hamiltonian laceable* if there exists a hamiltonian path between any two vertices u, v with $u \in B$ and $v \in W$ [22]. A graph $H = (B \bigcup W, E)$ is a *balanced* bipartite graph if $|V(B)| = |V(W)|$. Throughout this thesis, we only work on Q_n^k with $k \geq 4$ an even integer and $n \geq 2$, which are balanced bipartite graphs. A bipartite graph $H = (B \bigcup W, E)$ is m^*-*laceable* if given a white vertex $w \in W$ and a black vertex $b \in B$, there exist(s) m internal disjoint paths between w and b, denoted by P_i for

$0 \leq i \leq m-1$, such that $\bigcup_0^{m-1} P_i$ covers V. The *spanning laceability* of a graph H, $\kappa^*(H)$, is the largest integer k such that H is m^*-laceable for every m with $1 \leq m \leq k$. A higher spanning connectivity/laceability of the interconnection network implies a more efficient communication between processors. About the spanning connectivity and the spanning laceability, readers can refer to [6], [7], [12]–[15].

In this paper, we want to show the spanning laceability of k-ary n-cubes for any even integer $k \geq 4$. More precisely, we show that given a white vertex w and a black vertex b of a k-ary n-cube Q_n^k, there exist(s) m internally disjoint path(s) between w and b whose union covers all vertices of Q_n^k for $1 \leq m \leq 2n$. The result is optimal since any vertex in Q_n^k has exactly $2n$ neighbors. This paper is organized as follows. In Section 2, we introduce the graph terminologies and symbols that will be used in the paper and the definition of Q_n^k. In Section 3, we show our main results.

II. PRELIMINARIES

Throughout this paper, we follow [3] for the graph definitions and notations. The sets of vertices and edges of a graph G are denoted by $V(G)$ and $E(G)$, respectively. If u, v are vertices of a graph G such that there is an edge $e = (u,v) \in E(G)$ between u and v, then we say that the vertices u and v are *adjacent* in G. The *degree* of any vertex x is the number of distinct vertices adjacent to x. A *path* P between two vertices v_0 and v_k is represented by $P = \langle v_0, v_1, \ldots, v_k \rangle$, where each pair of consecutive vertices are connected by an edge. We use P^{-1} to denote the path $\langle v_k, v_{k-1}, v_{k-2}, \ldots, v_0 \rangle$. We also write the path $P = \langle v_0, v_1, \ldots, v_k \rangle$ as $\langle v_0, v_1, \ldots, v_i, Q, v_j, v_{j+1}, \ldots, v_k \rangle$, where Q denotes the path $\langle v_i, v_{i+1}, \ldots, v_j \rangle$. A *hamiltonian path* between u and v, where u and v are two distinct vertices of G, is a path joining u to v that visits every vertex of G exactly once. A *cycle* is a path of at least three vertices such that the first vertex is the same as the last vertex. A *hamiltonian cycle* of G is a cycle that traverses every vertex of G exactly once. A *hamiltonian graph* is a graph with a hamiltonian cycle. A graph G is *connected* if there is a path between any two distinct vertices in G and is *hamiltonian connected* if there is a hamiltonian path between any two distinct vertices in G [18]. A graph $H = (W \cup B, E)$ is *bipartite* if $V(H) = W \cup B$ and $E(H)$ is a subset of $\{(w, b) | w \in W, b \in B\}$. A bipartite graph H is *hamiltonian laceable* if there is a hamiltonian path between any two distinct vertices from different partite sets in H.

A graph G is *k-connected* if there exists $V' \subseteq V(G)$ with $|V'| = k$ such that $G - V'$ is disconnected and $G - V''$ is

Y.-K. Shih is with the Department of Computer Science, National Chiao Tung University, Hsinchu City, Taiwan 30010, R.O.C.

S.-L. Chang and S.-S. Kao are with the Department of Applied Mathematics, Chung-Yuan Christian University, Chungli, Taiwan 32023, R.O.C.

Correspondence to: Professor S.-S. Kao, e-mail: skao@math.cycu.edu.tw.

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connected for any $V'' \subseteq V(G)$ with $|V''| < k$. It follows from Menger's Theorem [16] that for every k-connected graph G, there exist k internally vertex-disjoint paths between any pair of distinct vertices of G. A *k-container* $C(u,v)$ in a graph G is a set of k internally vertex-disjoint paths between two distinct vertices u and v. We say that a graph G has a *spanning k-container* between u and v, denoted by $C(u,v)$, if $C(u,v)$ is a k-container that covers all vertices of G. A spanning k-container is also abbreviated as a k^*-*container* for simplicity. A graph G is k^*-*connected* if there is a k^*-container between any pair of vertices of G. Obviously, a graph G is hamiltonian connected if and only if G is 1^*-connected, and G is hamiltonian if and only if G is 2^*-connected. Lin et al. [13] defined the concept of *spanning connectivity*. The *spanning connectivity* of a graph G, $\kappa^*(G)$, is the largest integer k such that G is w^*-connected for all $1 \leq w \leq k$. Similarly, a bipartite graph H is k^*-*laceable* if there is a k^*-container between any pair of two vertices from different partite sets of H. Also, a bipartite graph H is hamiltonian laceable if and only if H is 1^*-laceable, and H is hamiltonian if and only if H is 2^*-laceable. So, the *spanning laceability* of a bipartite graph H, $\kappa^*(H)$, is the largest integer k such that H is m^*-laceable for all $1 \leq m \leq k$.

The k-ary n-cube, Q_n^k, is defined for all integers $k \geq 2$ and $n \geq 1$. The subclass Q_n^2 is the well-studied hypercube family. The subclass Q_1^k with $k \geq 3$ is defined as the cycle of length k. The k-ary n-cube, Q_n^k, for $k \geq 3$ and $n \geq 2$ is defined as follows. Let $u \in V(Q_n^k)$ be represented by $(u(0), u(1), ..., u(n-1))$, where $0 \leq u(i) \leq k-1$. Two vertices u and v are adjacent if and only if $|u(i) - v(i)| = 1$ or $k-1$ for some i and $u(j) = v(j)$ for any $0 \leq j \leq n-1$ with $j \neq i$. It is shown that Q_n^k is bipartite if k is even [10]. Here we mention some properties of Q_n^k that will be used in this paper.

Q_n^k is *vertex symmetric* (and *edge symmetric*) [10]. It means that given any two distinct vertices v and v' of Q_n^k, there is an automorphism of Q_n^k mapping v to v'. Note that each vertex of Q_n^k is represented by a n-bit tuple. We will call the dth-bit *the dth dimension*. We can partition Q_n^k over dimension d by fixing the dth element of any vertex tuple at some value a for every $a \in \{0, 1, ..., k-1\}$. This results in k copies of Q_{n-1}^k, denoted by $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, ..., Q_{n-1}^{k,k-1}$, with corresponding vertices in $Q_{n-1}^{k,0}, Q_{n-1}^{k,1}, ..., Q_{n-1}^{k,k-1}$ joined in a cycle of length k (in dimension d) [19].

In this article, we always partition Q_n^k over the 0-th dimension by letting $V(Q_{n-1}^{k,i}) = \{((i), v(1), v(2), \ldots, v(n-1)) \mid 0 \leq v(j) \leq k-1, \forall 1 \leq j \leq n-1\}$ for $0 \leq i \leq k-1$. Given a vertex $x = (x(0), x(1), \ldots, x(n-1)) \in V(Q_n^k)$, the symbol $x^j = ((j), x(1), x(2), \ldots, x(n-1))$, where $0 \leq j \leq k-1$, is defined to be the vertex corresponding to x in $Q_{n-1}^{k,j}$ for simplicity. So, if $P = \langle x_0, x_1, \ldots, x_{n-1} \rangle$, P^j is represented by $\langle x_0^j, x_1^j, \ldots, x_{n-1}^j \rangle$. Throughout this paper, let $n \geq 2$ be an integer and $k \geq 4$ an even integer.

Theorem 1. *[10] For any even integer $k \geq 4$, Q_n^k is hamiltonian laceable for $n \geq 2$. In other words, Q_n^k is 1^*-laceable.*

Theorem 2. *[5] The graph Q_n^k is hamiltonian. In other words,* Q_n^k *is 2^*-laceable.*

III. MAIN RESULTS

Lemma 1. *Given Q_n^k and its k subcubes, $Q_{n-1}^{k,i}$, where $0 \leq i \leq k-1$. Let j and j' be two integers satisfying $0 \leq j \leq j' \leq k-1$, $w \in V(Q_{n-1}^{k,j})$ an arbitrary white vertex, and $b \in V(Q_{n-1}^{k,j'})$ an arbitrary black vertex. Then there exists a path between w and b that visits each vertex in $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'}$ exactly once.*

Proof: There are three cases.
Case 1. $j = j'$. W.L.O.G., let $j = j' = 0$. By Theorem 1, $Q_{n-1}^{k,0}$ is hamiltonian laceable. Thus, there is a hamiltonian path between w and b that visits each vertex of $Q_{n-1}^{k,0}$ exactly once.
Case 2. $j - j' = 1$. W.L.O.G., we can let $j = 0$ and $j' = 1$. Let w be a white vertex in $Q_{n-1}^{k,0}$ and b a black vertex in $Q_{n-1}^{k,1}$. We can find a pair of adjacent vertices x^0 and x^1 where x^0 is a black vertex of $Q_{n-1}^{k,0}$ and x^1 is a white vertex of $Q_{n-1}^{k,1}$. By Theorem 1, there exists a hamiltonian path P_0 of $Q_{n-1}^{k,0}$ between w and x^0, and a hamiltonian path P_1 of $Q_{n-1}^{k,1}$ between x^1 and b. Let $P = \langle w, P_0, x^0, x^1, P_1, b \rangle$. Hence P is the path between w and b that visits every vertex of $Q_{n-1}^{k,0}$ and $Q_{n-1}^{k,1}$ exactly once.
Case 3. $j - j' \geq 2$. Let w be a white vertex in $Q_{n-1}^{k,j}$ and b be a black vertex in $Q_{n-1}^{k,j'}$. There are $j - j' + 1$ k-ary $n - 1$-cubes, $Q_{n-1}^{k,j}, Q_{n-1}^{j,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'-1}$ and $Q_{n-1}^{k,j'}$. There are $j' - j$ pairs of adjacent vertices $x^r \in Q_{n-1}^{k,r}$ and $y^{r+1} \in Q_{n-1}^{k,r+1}$ where x^r is a black vertex and y^{r+1} is a white vertex for $j \leq r \leq j' - 1$. By Theorem 1, there is a hamiltonian path R_r of $Q_{n-1}^{k,r}$ joining y^r to x^r, where $j + 1 \leq r \leq j' - 1$. Again, with Theorem 1, there exists a hamiltonian path T of $Q_{n-1}^{k,j}$ joining w to x^j, and a hamiltonian path U of $Q_{n-1}^{k,j'}$ joining $y^{j'}$ to b. Let $P = \langle w, T, x^j, y^{j+1}, R_{j+1}, x^{j+1}, y^{j+2}, R_{j+2}, x^{j+2}, \ldots, y^{j'-1}, R_{j'-1}, x^{j-1}, y^{j'}, U, b \rangle$. Therefore, P is a path covering all the vertices of $Q_{n-1}^{k,j}, Q_{n-1}^{k,j+1}, Q_{n-1}^{k,j+2}, \ldots, Q_{n-1}^{k,j'}$ for $0 \leq j \leq j' \leq k-1$ between w and b. Please see Figure 1 for an illustration.
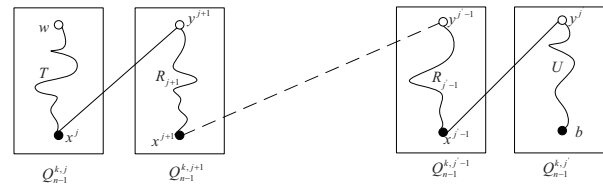

Fig. 1. The illustration for Case 3 of Lemma 1.

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Lemma 2. *Given Q_n^k and its k subcubes $Q_{n-1}^{k,i}$ for $0 \leq i \leq k-1$. Let w be a white vertex, b a black vertex in $Q_{n-1}^{k,i}$, and j an integer with $0 \leq i \leq j \leq k-1$. There exists a path between w and b that covers all the vertices of $Q_{n-1}^{k,i}, Q_{n-1}^{k,i+1}, \ldots,$ and $Q_{n-1}^{k,j}$.*

Proof: We consider the following two cases.

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Case 1. $j = i$. There is only one k-ary $(n-1)$-cube $Q_{n-1}^{k,i}$. By Theorem 1, the lemma holds in this case.

Case 2. $j \neq i$. There are $j - i + 1$ k-ary $(n-1)$-cubes. According to Theorem 1, there is hamiltonian path P_i that covers all the vertices of $Q_{n-1}^{k,i}$ between w and b of the form $\langle w, S_i, x^i, y^i, T_i, b \rangle$, where $\{x^i, y^i\}$ is an edge of $Q_{n-1}^{k,i}$ with $\{x^i, y^i\} \bigcap \{w, b\} = \emptyset$. Notice that by Theorem 1, $Q_{n-1}^{k,r}$ is hamiltonian laceable and hence there exists a hamiltonian path P_r between x^r and y^r of the form $\langle x^r, S_r, z^r, w^r, T_r, y^r \rangle$ for $i + 1 \leq r \leq j$. Let the required path between w and b be R, we have the following two subcubes.

Case 2.1. If $j - i + 1$ is even, then
$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3},$
$S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, x^j, S_j, z^j, w^j, T_j, y^j,$
$y^{j-1}, (T_{j-1})^{-1}, w^{j-1}, w^{j-2}, T_{j-2}, y^{j-2}, y^{j-3}, (T_{j-3})^{-1},$
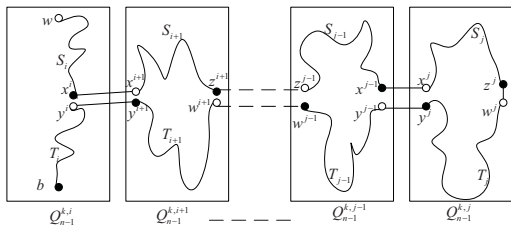$w^{j-3}, \ldots, y^{i+1}, y^i, T_i, b \rangle$. Please see Figure 2 for an illustration.

Fig. 2. The illustration for Lemma 2 when $j - i + 1$ is even.

Case 2.2. If $j - i + 1$ is odd, then
$R = \langle w, S_i, x^i, x^{i+1}, S_{i+1}, z^{i+1}, z^{i+2}, (S_{i+2})^{-1}, x^{i+2}, x^{i+3},$
$S_{i+3}, z^{i+3}, z^{i+4}, (S_{i+4})^{-1}, x^{i+4}, \ldots, z^j, (S_j)^{-1}, x^j, y^j,$
$(T_j)^{-1}, w^j, w^{j-1}, T_{j-1}, y^{j-1}, y^{j-2}, (T_{j-2})^{-1}, w^{j-2}, w^{j-3},$
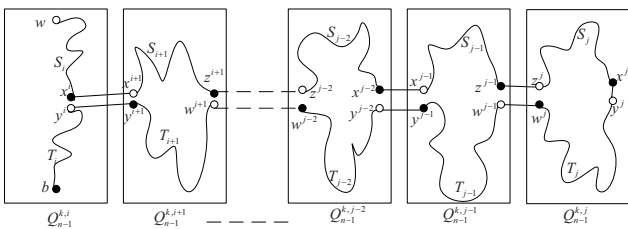$T_{j-3}, y^{j-3}, \ldots, y^{i+1}, y^i, T_i, b \rangle$. Please see Figure 2 for an illustration. ∎

Fig. 3. The illustration for Lemma 2 when $j - i + 1$ is odd.

Lemma 3. *The graph Q_2^4 is 3^*-laceable and 4^*-laceable.*

Proof: The proof is by brute force. Reader can refer to Appendix A. ∎

Lemma 4. *The graph Q_2^6 is 3^*-laceable and 4^*-laceable.*

Proof: By brute force, we constructed all spanning containers. Please see Appendix B. ∎

Lemma 5. *The graph Q_2^k is 3^*-laceable and 4^*-laceable for any even integer $k \geq 6$.*

Proof: With Lemma 4, we have shown that Q_2^6 is 3^*-laceable and 4^*-laceable. Now we will present a recursive

algorithm that uses a 3^*-container (resp. 4^*-container) of Q_2^k to construct a 3^*-container (resp. 4^*-container) of Q_2^{k+2}. Let R be a subset of $V(Q_2^k) \cup E(Q_2^k)$. We define a function, f, which maps R from Q_2^k into Q_2^{k+2} in the following way:
(1) If $(i, j) \in R \cap V(Q_2^k)$, where $0 \leq i, j \leq k - 1$, then

$$f((i,j)) = \begin{cases} (i, j) & \text{if } 0 \leq i, j \leq k-2; \\ (i+2, j) & \text{if } i = k-1, 0 \leq j \leq k-2; \\ (i, j+2) & \text{if } j = k-1, 0 \leq i \leq k-2; \\ (i+2, j+2) & \text{if } i = k-1 = j. \end{cases}$$

(2) If $((i,j), (i',j')) \in R \cap E(Q_2^k)$, where $i \leq i', j \leq j'$, then
$f(((i,j), (i',j')))$

$$= \begin{cases} ((i,j), (i',j')) & \text{if } 0 \leq i, j \leq k-3, \\ & 1 \leq i', j' \leq k-2; \\ ((i+2, j), (i'+2, j)) & \text{if } i = i' = k-1, \\ & 0 \leq j \leq k-3, \\ & 1 \leq j' \leq k-2; \\ ((i, j+2), (i', j'+2)) & \text{if } j = j' = k-1, \\ & 0 \leq i \leq k-3, \\ & 1 \leq i' \leq k-2; \\ ((i,j), (i', j'+2)) & \text{if } 0 \leq i = i' \leq k-2, \\ & j = 0, j' = k-1; \\ ((i,j), (i'+2, j')) & \text{if } 0 \leq j = j' \leq k-2, \\ & i = 0, i' = k-1; \\ ((i, j+2), (i'+2, j'+2)) & \text{if } i = 0, i' = k-1, \\ & j = j' = k-1; \\ ((i+2, j), (i'+2, j'+2)) & \text{if } j = 0, j' = k-1, \\ & i = i' = k-1. \end{cases}$$

Let w be a white vertex and b be a black vertex of Q_2^k. We say that a 3^*-container (resp. 4^*-container) $C(u, v)$ of Q_2^k is *regular* if $C(w, b)$ contains some edges in $\{((\alpha, k-2), (\alpha, k-1)) \mid 0 \leq \alpha \leq k-1\}$ and $\{((k-2, \beta), (k-1, \beta)) \mid 0 \leq \beta \leq k-1\}$. For example, all 3^*-containers and 4^*-containers of Q_2^6 constructed in Lemma 4 are regular. Let $C(w, b)$ be a regular 3^*-container (resp. 4^*-container) of Q_2^k with the endvertex set $P = \{w = (0,0), b = (x, y)\}$. We construct a regular 3^*-container (resp. 4^*-container) of Q_2^{k+2} with the endvertex set $f(P)$ using the following algorithm. Please see Figure 4 for an illustration.
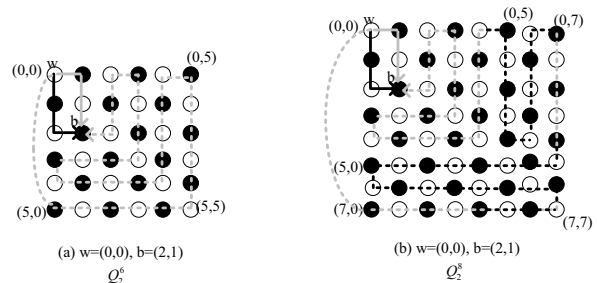

(a) w=(0,0), b=(2,1)
Q_2^6

(b) w=(0,0), b=(2,1)
Q_2^8

Fig. 4. Using the 4^*-container of Q_2^6 to construct the 4^*-container of Q_2^8.

Step 1. In Q_2^k, let $\{v_0, v_1, \ldots, v_{t-1}\}$ and $\{h_0, h_1, \ldots, h_{s-1}\}$ be finite sequences of indices satisfying the following requirements:
(1) $0 \leq v_0 < v_1 < \ldots < v_{t-1} \leq k - 1$ and $k - 1 \geq h_0 > h_1 > \ldots > h_{s-1} \geq 0$;

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(2) for $0 \le i \le k-1$, $((v_i, k-2), (v_i, k-1))$ is an edge of $C(w, b)$; for $0 \le j \le k-1$, $((k-2, h_j), (k-1, h_j))$ is an edge of $C(w, b)$.

Step 2. Let $\overline{C}(w, b)$ be the image in Q_2^{k+2} of $C(w, b) - (\{((v_i, k-2), (v_i, k-1)) \mid 0 \le i \le k-1\} \cup \{((k-2, h_j), (k-1, h_j)) \mid 0 \le j \le k-1\})$ under the function f. Please see Figure 5 for an illustration.
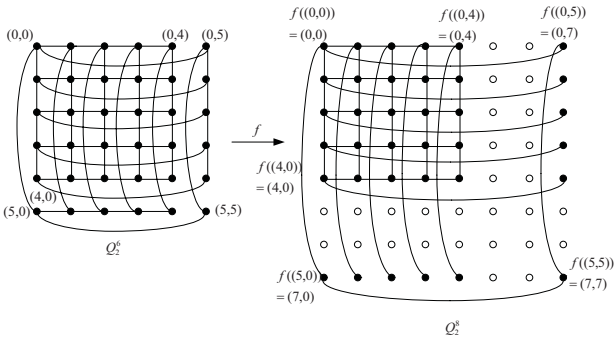

Fig. 5. Using function f to map a subset of edges and vertices of Q_2^6 into Q_2^8.

Step 3. For any two positive integers r and d, we use $[r]_d$ to denote $r(\textbf{mod } d)$. In Q_2^{k+2}, define the following path patterns, where r_1, r_2 are integers:

$$I_\alpha(r_1, r_2) = \langle (r_1, \alpha), ([r_1+1]_{k+2}, \alpha), \ldots, (r_2, \alpha)\rangle;$$
$$I_\alpha^{-1}(r_2, r_1) = \langle (r_2, \alpha), ([r_2-1]_{k+2}, \alpha), \ldots, (r_1, \alpha)\rangle;$$
$$H_\beta(r_1, r_2) = \langle (\beta, r_1), (\beta, [r_1+1]_{k+2}), \ldots, (\beta, r_2)\rangle;$$
$$H_\beta^{-1}(r_2, r_1) = \langle (\beta, r_2), (\beta, [r_2-1]_{k+2}), \ldots, (\beta, r_1)\rangle.$$

Let $\overline{v}_i = v_i + 2$ if $v_i = k-1$ and $\overline{v}_i = v_i$ if $0 \le v_i \le k-2$, and $\overline{h}_j = h_j + 2$ if $h_j = k-1$ and $\overline{h}_j = h_j$ if $0 \le h_j \le k-2$.

Case 1. $v_0 = k-1$.
Let $P_0 = \langle (k+1, k-2), (k+1, k-1), (0, k-1), I_{k-1}(0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, 0), (0, k), (k+1, k), (k+1, k+1)\rangle$.

<u>Case 1.1.</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 1.2.</u> $s \ge 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $0 \le i \le s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, [\overline{h}_0+1]_{k+2}), (k-1, [\overline{h}_0+1]_{k+2}), (k, [\overline{h}_0+1]_{k+2}), H_k([\overline{h}_0+1]_{k+2}, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 2. $v_{t-1} \le k-2$ and $((k-2, k-1), (k-1, k-1)) \in E(C(w, b))$ in Q_2^k.

<u>Case 2.1.</u> $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$.

<u>Case 2.1.1</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+$

$1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 2.1.2</u> $s = 2$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1+1), (k-1, \overline{h}_1+1), (k, \overline{h}_1+1), H_k(\overline{h}_1+1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$, and $\overline{P}_1 = \langle (k-2, \overline{h}_1), (k-1, \overline{h}_1), H_{k-1}^{-1}(\overline{h}_1, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_1), (k, \overline{h}_1), (k+1, \overline{h}_1)\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \overline{P}_0 \cup \overline{P}_1$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 2.1.3</u> $s \ge 3$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, \overline{h}_1+1), (k-1, \overline{h}_1+1), (k, \overline{h}_1+1), H_k(\overline{h}_1+1, k-1), (k, k-1), (k+1, k-1), I_{k-1}(k+1, [\overline{v}_0-1]_{k+2}), ([\overline{v}_0-1]_{k+2}, k-1), ([\overline{v}_0-1]_{k+2}, k), I_k^{-1}([\overline{v}_0-1]_{k+2}, k+1), (k+1, k), (k, k), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$, $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$ for $1 \le i \le s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1})\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 2.2.</u> $t \ge 2$.
Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1)\rangle$ for $0 \le i \le t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1)\rangle$.

<u>Case 2.2.1</u> $s = 1$.
Using the same $\overline{P}_0$ as in Case 2.1.1, then $\overline{C}(w, b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 2.2.2</u> $s = 2$.
Using the same $\overline{P}_0$ and $\overline{P}_1$ as in Case 2.1.2., then $\overline{C}(w, b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \overline{P}_0 \cup \overline{P}_1$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 2.2.3</u> $s \ge 3$.
Using the same $\{\overline{P}_i \mid 0 \le i \le s-1\}$ as in Case 2.1.3., then $\overline{C}(w, b) \cup \{P_i \mid 0 \le i \le t-1\} \cup \{\overline{P}_i \mid 0 \le i \le s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 3. $v_{t-1} \le k-2$ and $((k-2, k-1), (k-1, k-1)) \notin E(C(w, b))$ in Q_2^k.
<u>Case 3.1.</u> $t = 1$.
Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0+1), (k-1, \overline{h}_0+1), (k, \overline{h}_0+1), H_k(\overline{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0-1), (\overline{v}_0-1, k-1), (\overline{v}_0-1, k), I_k^{-1}(\overline{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1)\rangle$.

<u>Case 3.1.1</u> $s = 1$.
Let $\overline{P}_0 = \langle (k-2, \overline{h}_0), (k-1, \overline{h}_0), H_{k-1}^{-1}(\overline{h}_0, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_0), (k, \overline{h}_0), (k+1, \overline{h}_0)\rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

<u>Case 3.1.2</u> $s \ge 2$.
Let $\overline{P}_i = \langle (k-2, \overline{h}_i), (k-1, \overline{h}_i), H_{k-1}^{-1}(\overline{h}_i, \overline{h}_{i+1}+1), (k-1, \overline{h}_{i+1}+1), (k, \overline{h}_{i+1}+1), H_k(\overline{h}_{i+1}+1, \overline{h}_i), (k, \overline{h}_i), (k+1, \overline{h}_i)\rangle$

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for $0 \leq i \leq s-2$, and $\overline{P}_{s-1} = \langle (k-2, \overline{h}_{s-1}), (k-1, \overline{h}_{s-1}), H_{k-1}^{-1}(\overline{h}_{s-1}, 0), (k-1, 0), (k, 0), H_k(0, \overline{h}_{s-1}), (k, \overline{h}_{s-1}), (k+1, \overline{h}_{s-1}) \rangle$. Then $\overline{C}(w, b) \cup P_0 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 3.2. $t \geq 2$.

Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1) \rangle$ for $0 \leq i \leq t-2$, and $P_{t-1} = \langle (\overline{v}_{t-1}, k-2), (\overline{v}_{t-1}, k-1), I_{k-1}(\overline{v}_{t-1}, k-1), (k-1, k-1), H_{k-1}^{-1}(k-1, \overline{h}_0+1), (k-1, \overline{h}_0+1), (k, \overline{h}_0+1), H_k(\overline{h}_0+1, k-1), (k, k-1), (k+1, k-1), (0, k-1), I_{k-1}(0, \overline{v}_0-1), (\overline{v}_0-1, k-1), (\overline{v}_0-1, k), I_k^{-1}(\overline{v}_0-1, 0), (0, k), (k+1, k), (k, k), (k, k+1), (k-1, k+1), (k-1, k), I_k^{-1}(k-1, \overline{v}_{t-1}), (\overline{v}_{t-1}, k), (\overline{v}_{t-1}, k+1) \rangle$.

Case 3.2.1 $s = 1$.

Using the same $\overline{P}_0$ as in Case 3.1.1, then $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 3.2.2 $s \geq 2$.

Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 3.1.2., then $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 4. $v_{t-1} = k-1$ for some $t \geq 2$ and $v_0 = 0$.

Case 4.1. $t = 2$.

Let $P_0 = \langle (\overline{v}_0, k-2), (\overline{v}_0, k-1), I_{k-1}(\overline{v}_0, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_0), (\overline{v}_0, k), (\overline{v}_0, k+1) \rangle$, and $P_1 = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$.

Case 4.1.1. $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w, b) \cup P_0 \cup P_1 \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 4.1.2. $s \geq 2$.

Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(w, b) \cup P_0 \cup P_1 \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 4.2. $t \geq 3$.

Let $P_i = \langle (\overline{v}_i, k-2), (\overline{v}_i, k-1), I_{k-1}(\overline{v}_i, \overline{v}_{i+1}-1), (\overline{v}_{i+1}-1, k-1), (\overline{v}_{i+1}-1, k), I_k^{-1}(\overline{v}_{i+1}-1, \overline{v}_i), (\overline{v}_i, k), (\overline{v}_i, k+1) \rangle$ for $0 \leq i \leq t-3$, $P_{t-2} = \langle (\overline{v}_{t-2}, k-2), (\overline{v}_{t-2}, k-1), I_{k-1}(\overline{v}_{t-2}, k-2), (k-2, k-1), (k-2, k), I_k^{-1}(k-2, \overline{v}_{t-2}), (\overline{v}_{t-2}, k), (\overline{v}_{t-2}, k+1) \rangle$, and $P_{t-1} = \langle (k+1, k-2), (k+1, k-1), (k+1, k), (k+1, k+1) \rangle$.

Case 4.2.1. $s = 1$.

Using the same $\overline{P}_0$ as in Case 1.1., then $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \overline{P}_0$ is the 3*-container (or 4*-container) of Q_2^{k+2}.

Case 4.2.2. $s \geq 2$.

Using the same $\{\overline{P}_i \mid 0 \leq i \leq s-1\}$ as in Case 1.2., then $\overline{C}(w, b) \cup \{P_i \mid 0 \leq i \leq t-1\} \cup \{\overline{P}_i \mid 0 \leq i \leq s-1\}$ is the 3*-container (or 4*-container) of Q_2^{k+2}. ∎

Theorem 3. *For any integer $n \geq 2$ and any even integer $k \geq 4$, the graph Q_n^k is m^*-laceable where $1 \leq m \leq 2n$.*

Proof: According to Theorem 2-3 and Lemma 3-5, the theorem holds for any even integer $k \geq 4$ when $n = 2$. We will give the proof of the theorem by mathematical induction on n. By induction hypothesis, assume that $Q_{n-1}^{k,i}$ is m^*-laceable for $1 \leq m \leq 2n-2$, where $0 \leq i \leq k-1$. Given a white vertex $w \in V(Q_{n-1}^{k,j})$ and a black vertex $b \in V(Q_{n-1}^{k,j'})$. We will show that we can use the m^*-containers of $Q_{n-1}^{k,j}$ to construct a $(m+2)^*$-container of Q_n^k between w and b.

Case 1. For $j = j'$. Without loss of generality, we let $j = j' = 0$.

In this case, we have $\{w, b\} \in Q_{n-1}^{k,0}$. By induction hypothesis, there are m internal disjoint paths $\{P_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and b for $1 \leq m \leq 2n-2$. By Lemma 2, the exists a path S covering all vertices of $Q_{n-1}^{k,i}$ for $1 \leq i \leq k-2$ between w^1 and b^1. We can let $P_m = \langle w, w^1, S, b^1, b \rangle$. In $Q_{n-1}^{k,k-1}$, there exist a hamiltonian path R joining from w^{k-1} to b^{k-1} by Theorem 1. Also, we can let $P_{m+1} = \langle w, w^{k-1}, R, b^{k-1}, b \rangle$. Therefore, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 6 for an illustration.
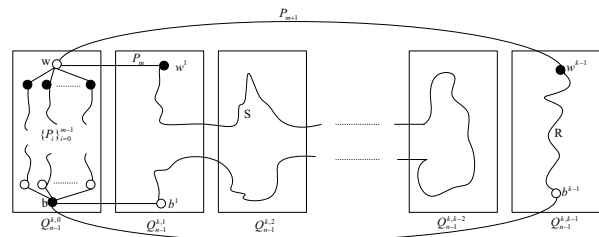

Fig. 6.   The illustration for Case 1 of Theorem 3.

Case 2. For $|j' - j| = 1$. Without loss of generality, we let $j = 0$ and $j' = 1$.

We have the following two cases.

Case 2.1. Suppose that $d(w, b) = 1$. It is easy to see that we can let $P_{m+1} = \langle w, b \rangle$.

Case 2.1.1. If $m = 1$.

Let z be any black vertex of $Q_{n-1}^{k,0}$. By Theorem 1, there exist a hamiltonian path S of $Q_{n-1}^{k,0}$ from w to z, and a hamiltonian path T of $Q_{n-1}^{k,1}$ from z^1 to b. So we set $P_0 = \langle w, S, z, z^1, T, b \rangle$. According to Lemma 1, a hamiltonian path R between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covers all vertices of $Q_{n-1}^{k,i}$ for $2 \leq i \leq k-1$. We can write P_1 as $\langle w, w^{k-1}, R, b^2, b \rangle$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 7 for an illustration.
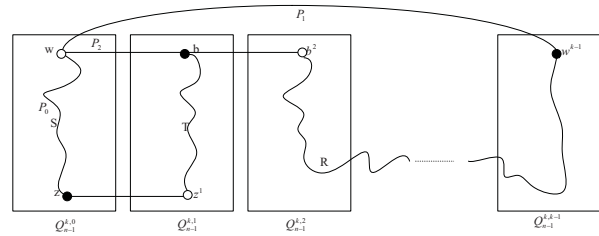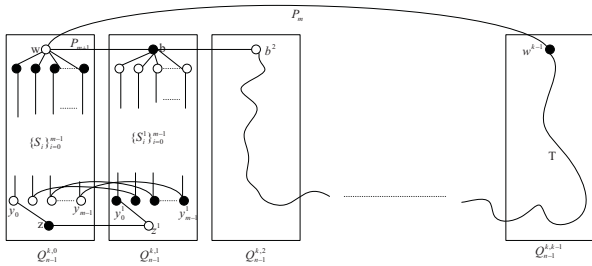

Fig. 7.   The illustration for Case 2.1.1 of Theorem 3.

Case 2.1.2. If $m \geq 2$.

According to the induction hypothesis, given any black vertex $z \in V(Q_{n-1}^{k,0} - N(w))$, there exist m internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and z for $2 \leq m \leq 2n-2$. Let $R_i = \langle w, S_i, y_i, z \rangle$ for $0 \leq i \leq m-1$. We set $P_0 = \langle w, S_0, y_0, z, z^1, y_0^1, (S_0^1)^{-1}, b \rangle$ and $P_i = \langle w, S_i, y_i, y_i^1, (S_i^1)^{-1}, b \rangle$ for $1 \leq i \leq m-1$. By

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Lemma 1, there is a hamiltonian path T between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. Set $P_m = \langle w, w^{k-1}, T, b^2, b \rangle$. Consequently, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 8 for an illustration.

Fig. 8. The illustration for Case 2.1.2 of Theorem 3.

Case 2.2. Suppose that $d(w,b) \ge 3$.
Case 2.2.1. If $m = 1$.
Given any black vertex z in $Q_{n-1}^{k,0}$, by Theorem 1, there is a hamiltonian path R of $Q_{n-1}^{k,0}$ joining from w to z. So there is also a hamiltonian path S of $Q_{n-1}^{k,1}$ between w^1 to z^1. We can set $S = \langle w^1, S_1', b, S_2', z^1 \rangle$. By Lemma 1, there exists a hamiltonian path T between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^2 \in Q_{n-1}^{k,2}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-1$. We let $P_0 = \langle w, R, z, z^1, (S_2')^{-1}, b \rangle$, $P_1 = \langle w, w^1, S_1', b \rangle$, and $P_2 = \langle w, w^{k-1}, T, b^2, b \rangle$. Therefore, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 9 for an illustration.
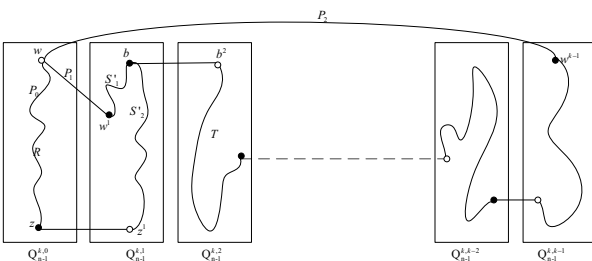

Fig. 9. The illustration for Case 2.2.1 of Theorem 3.

Case 2.2.2. If $m \ge 2$.
Let z be a black vertex of $V(Q_{n-1}^{k,0} - N(w))$. In $Q_{n-1}^{k,0}$, according to the induction hypothesis, there exist m internal disjoint paths $\{S_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,0}$ between w and z for $2 \le m \le 2n-2$. So as in $Q_{n-1}^{k,1}$, there exist m internal disjoint paths $\{T_i\}_{i=0}^{m-1}$ whose union covers all vertices of $Q_{n-1}^{k,1}$ between z^1 and b for $2 \le m \le 2n-2$. Let $T_0 = \langle z^1, y_0, T_0', x_0, w^1, T_0'', b \rangle$ and $T_i = \langle z^1, y_i, T_i', b \rangle$ for $1 \le i \le m-1$ in $Q_{n-1}^{k,1}$.
Case 2.2.2.1. If $b^0 \notin V(S_0)$.
Without loss of generality, let $b^0 \in V(S_{m-1})$. In $Q_{n-1}^{k,0}$, we also let $S_0 = \langle w, x_0^0, e, S_0', y_0^0, z \rangle$, $S_i = \langle w, S_i', y_0^0, z \rangle$ for $1 \le i \le m-2$, and $S_{m-1} = \langle w, S_{m-1}', b^0, f, S_{m-1}'', y_{m-1}^0, z \rangle$. A hamiltonian path R is embedded in $Q_{n-1}^{k,k-1}$ between w^{k-1} and f^{k-1} by Theorem 1. Write R as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1} \rangle$. Notice that g^{k-2} is a black vertex and b^2 is a white vertex.

According to Lemma 1, there is a hamiltonian path U between g^{k-2} and b^2 covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-2$. We can set $P_0 = \langle w, x_0^0, x_0, (T_0')^{-1}, y_0, z^1, y_{m-1}, T_{m-1}, b \rangle$, $P_1 = \langle w, w^1, T_0'', b \rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S_0', y_0^0, z, y_{m-1}^0, (S_{m-1}'')^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$, $P_3 = \langle w, S_{m-1}', b^0, b \rangle$, and $P_i = \langle w, S_{i-3}', y_{i-3}^0, y_{i-3}, T_{i-3}', b \rangle$ for $4 \le i \le m+1$. So, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 10 for an illustration.
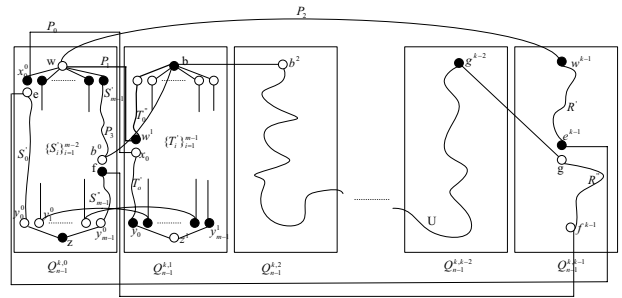

Fig. 10. The illustration for Case 2.2.2.1 of Theorem 3.

Case 2.2.2.2. If $b^0 \in V(S_0)$.
Let $S_0 = \langle w, x_0^0, e, S_0', b^0, f, S_0'', y_0^0, z \rangle$, and $S_i = \langle w, S_i', y_i^0, z \rangle$ for $1 \le i \le m-1$. A hamiltonian path R is embedded in $Q_{n-1}^{k,k-1}$ between w^{k-1} and f^{k-1} by Theorem 1. R is written as $\langle w^{k-1}, R', e^{k-1}, g, R'', f^{k-1} \rangle$. Notice that g^{k-2} is a black vertex and b^2 is a white vertex. According to Lemma 1, there is a hamiltonian path U between g^{k-2} and b^2 covering all vertices of $Q_{n-1}^{k,i}$ for $2 \le i \le k-2$. We let $P_0 = \langle w, x_0^0, x_0, (T_0')^{-1}, y_0, z^1, y_{m-1}, T_{m-1}', b \rangle$, $P_1 = \langle w, w^1, T_0'', b \rangle$, $P_2 = \langle w, w^{k-1}, R', e^{k-1}, e, S_0', b^0, b \rangle$, $P_3 = \langle w, S_{m-1}', y_{m-1}^0, z, y_0^0, (S_0'')^{-1}, f, f^{k-1}, (R'')^{-1}, g, g^{k-2}, U, b^2, b \rangle$, and $P_i = \langle w, S_{i-3}', y_{i-3}^0, y_{i-3}, T_{i-3}', b \rangle$ for $4 \le i \le m+1$. Hence, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 11 for an illustration.
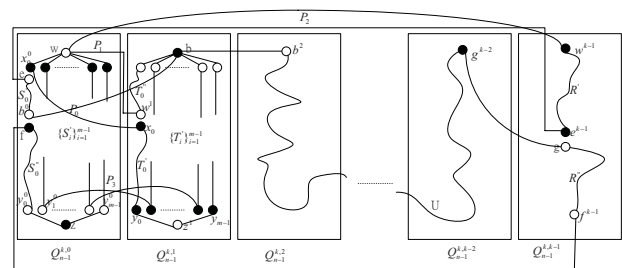

Fig. 11. The illustration for Case 2.2.2.2 of Theorem 3.

Case 3. For $|j' - j| \ge 2$. Without loss of generality, we let $j = 0$ and $2 \le j' \le \frac{k}{2}$ be even.
Because $b \in Q_{n-1}^{k,j'}$ where j' is even, b^i is a white (resp. black) vertex in $Q_{n-1}^{k,i}$ for $0 \le i \le k-1$ when i is odd (resp. even). It is easy to see that w^i is a black (resp. white) vertex in $Q_{n-1}^{k,i}$ for $0 \le i \le k-1$ when i is odd (resp. even). By the induction hypothesis, there exist m internal disjoint paths $\{R_p^i\}_{p=0}^{m-1}$ of $Q_{n-1}^{k,i}$ between w^i and b^i for $0 \le i \le j'$.

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Let $R_p^i = \langle w^i, x_p^i, U_p^i, y_p^i, b^i \rangle$ for $0 \leq p \leq m-1$ and $0 \leq i \leq j'$. According to Lemma 2, a hamiltonian path S covers all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \leq i \leq k-2$ joining from $w^{j'+1}$ to $b^{j'+1}$. There is a hamiltonian path T of Q_{n-1}^{k-1} from w^{k-1} to b^{k-1} by Theorem 1. Hence, we can write $P_p = \langle w = w^0, x_p^0, U_p^0, y_p^0, y_p^1, (U_p^1)^{-1}, x_p^1, x_p^2, U_p^2, \ldots, (U_p^{j'-1})^{-1}, x_p^{j'-1}, x_p^{j'}, U_p^{j'}, y_p^{j'}, b^{j'} = b \rangle$ for $0 \leq p \leq m-1$, $P_m = \langle w = w^0, w^1, w^2, \ldots, w^{j'}, w^{j'+1}, S, b^{j'+1}, b^{j'} = b \rangle$, and $P_{m+1} = \langle w = w^0, w^{k-1}, T, b^{k-1}, b^0, b^1, \ldots, b^{j'-1}, b^{j'} = b \rangle$. Therefore, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 12 for an illustration.
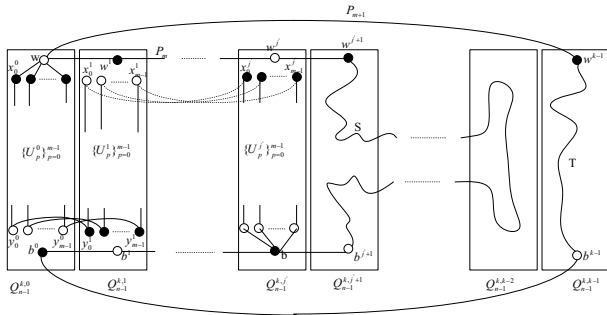

Fig. 12.   The illustration for Case 3 of Theorem 3.

Case 4. For $|j' - j| \geq 2$. Without loss of generality, we let $j = 0$ and $3 \leq j' \leq \frac{k}{2} + 1$ be odd.

Case 4.1. If $m = 1$.

Choosing a black vertex z of $Q_{n-1}^{k,0}$, by Theorem 1, there is a hamiltonian path R of $Q_{n-1}^{k,0}$ joining from w to z. In $Q_{n-1}^{k,k-1}$, there exists a hamiltonian path S of $Q_{n-1}^{k,k-1}$ between w^{k-1} and z^{k-1}. We can let $S = \langle w^{k-1}, S', e, b^{k-1}, S'', z^{k-1} \rangle$, where b^{k-1} is a black vertex of $Q_{n-1}^{k,k-1}$, so e is a white vertex of $Q_{n-1}^{k,k-1}$. By Theorem 1, there is a hamiltonian path T of $Q_{n-1}^{k,k-2}$ joining from e^{k-2} to b^{k-2}. Let $T = \langle e^{k-2}, W, f^{k-2}, b^{k-2} \rangle$. In $Q_{n-1}^{k,i}$, we also have a hamiltonian path T^i between e^i and b^i for $j' \leq i \leq k-3$, so we let $T^i = \langle e^i, W^i, f^i, b^i \rangle$. According to Lemma 1, there is a hamiltonian path U between a black vertex $w^1 \in Q_{n-1}^{k,1}$ and a white vertex $b^{j'-1} \in Q_{n-1}^{k,j'-1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $2 \leq i \leq j'-1$. We set $P_0 = \langle w, w^1, U, b^{j'-1}, b \rangle$, $P_1 = \langle w, R, z, z^{k-1}, (S'')^{-1}, b^{k-1}, b^{k-2}, \ldots, b^{j'+1}, b^{j'} = b \rangle$, and $P_2 = \langle w, w^{k-1}, S', e, e^{k-2}, W, f^{k-2}, f^{k-3}, (W^{k-3})^{-1}, e^{k-3}, e^{k-4}, W^{k-4}, f^{k-4}, \ldots, e^{j'+1}, W^{j'+1}, f^{j'+1}, f^{j'}, W^{j'}, b^{j'} = b \rangle$. Hence, there are 3 internal disjoint paths $\{P_0, P_1, P_2\}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 13 for an illustration.

Case 4.2. If $m \geq 2$.

Given a white vertex z in $Q_{n-1}^{k,j'}$ such that z is adjacent to b. So z^i is a black (resp. white) vertex and w^i is a white (reps. black) vertex of $Q_{n-1}^{k,i}$ if $0 \leq i \leq j'-1$ when i is even (resp. odd). By the induction hypothesis, there exist m internal disjoint paths $\{R_i\}_{i=0}^{m-1}$ of $Q_{n-1}^{k,0}$ between w and z^0. We write $R_0 = \langle w, x_0(1), x_0(2), \ldots, x_0(\alpha), z^0 \rangle$, and $R_p = \langle w, x_p, S_p, y_p, z^0 \rangle$ for $1 \leq p \leq m-1$. Again, by the induction hypothesis, there exist m internal disjoint paths
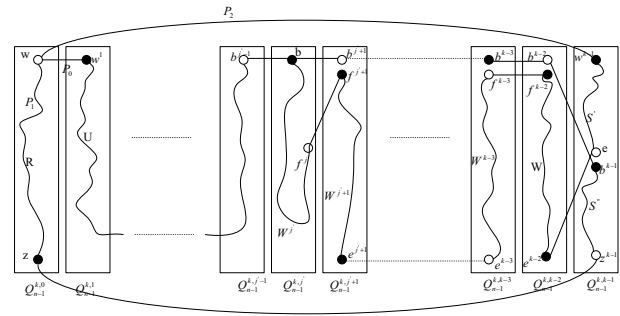

Fig. 13.   The illustration for Case 4.1 of Theorem 3.

$\{T_p^i\}_{p=0}^{m-1}$ of $Q_{n-1}^{k,i}$ between w^i and z^i for $2 \leq i \leq j'-1$. We let $T_p^i = \langle w^i, x_p^i, U_p^i, t_p^i, z^i \rangle$ for $0 \leq p \leq m-1$ and $2 \leq i \leq j'-1$. Notice that $b^{j'-1}$ is adjacent to $z^{j'-1}$, without loss of generality, we let $t_{m-1}^{j'-1} = b^{j'-1}$. In $Q_{n-1}^{k,j'}$, there are m internal disjoint paths $\{W_i\}_{i=0}^{m-1}$ from b to z by the induction hypothesis. We can write $W_p = \langle z, t_p^{j'}, Y_p, b \rangle$ for $0 \leq p \leq m-2$ and $W_{m-1} = \langle z, b \rangle$. According to Lemma 1, there is a hamiltonian path V between $w^{k-1} \in Q_{n-1}^{k,k-1}$ and $b^{j'+1} \in Q_{n-1}^{k,j'+1}$ covering all vertices of $Q_{n-1}^{k,i}$ for $j'+1 \leq i \leq k-1$. Set $P_0 = \langle w, w^{k-1}, V, b^{j'+1}, b \rangle$, $P_1 = \langle w, w^1, w^2, x_0^2, U_0^2, t_0^2, t_0^3, (U_0^3)^{-1}, x_0^3, w^3, w^4, \ldots, w^{j'-1}, x_0^{j'-1}, U_0^{j'-1}, t_0^{j'-1}, t_0^{j'}, Y_0, b \rangle$, $P_2 = \langle w, x_0(1), x_0^1(1), x_0^1(2), x_0(2), \ldots, x_0(\alpha-1), x_0^1(\alpha-1), x_0^1(\alpha), x_0(\alpha), z^0, z^1, \ldots, z^{j'}, b \rangle$, $P_3 = \langle w, x_{m-1}, S_{m-1}, y_{m-1}, y_{m-1}^1, (S_{m-1}^1)^{-1}, x_{m-1}^1, x_{m-1}^2, U_{m-1}^2, t_{m-1}^2, t_{m-1}^3, (U_{m-1}^3)^{-1}, x_{m-1}^3, \ldots, x_{m-1}^{j'-1}, U_{m-1}^{j'-1}, t_{m-1}^{j'-1} = b^{j'-1}, b \rangle$, and $P_i = \langle w, x_{i-3}, S_{i-3}, y_{i-3}, y_{i-3}^1, (S_{i-3}^1)^{-1}, x_{i-3}^1, x_{i-3}^2, U_{i-3}^2, t_{i-3}^2, t_{i-3}^3, (U_{i-3}^3)^{-1}, x_{i-3}^3, \ldots, x_{i-3}^{j'-1}, U_{i-3}^{j'-1}, t_{i-3}^{j'-1}, t_{i-3}^{j'}, Y_{i-3}, b \rangle$ for $4 \leq i \leq m+1$. So, there are $m+2$ internal disjoint paths $\{P_i\}_{i=0}^{m+1}$ whose union covers all vertices of Q_n^k between w and b. Please see Figure 14 for an illustration.
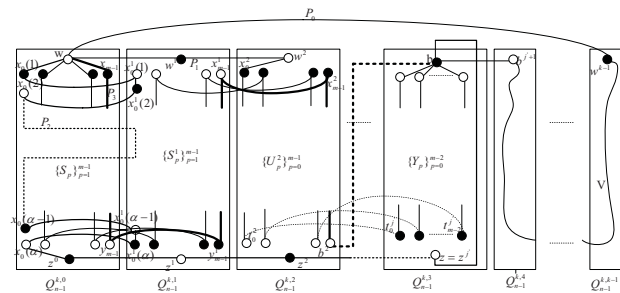

Fig. 14.   The illustration for Case 4.2 of Theorem 3.

■

APPENDIX A
PROOF OF LEMMA 3

Notice that Q_2^4 is vertex symmetric. W.L.O.G, let $w = (0,0)$. There are only two cases for b. That is, $b \in \{(1,0), (2,1)\}$.

Case 1. To prove that Q_2^4 is 3*-laceable.

Case 1.1. Let $b = (1,0)$.

The three disjoint paths $\{P_1, P_2, P_3\}$ between w and b whose

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union covers all vertices of Q_2^4 are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, and $P_3 = \langle(0,0),(3,0),(3,1), (3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2),(2,2),(2,1), (2,0),(1,0)\rangle$.

Case 1.2. Let $b = (2,1)$.

The three disjoint paths $\{R_1, R_2, R_3\}$ between w and b whose union covers all vertices of Q_2^4 are $R_1 = \langle(0,0),(1,0),(2,0), (2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, and $R_3 = \langle(0,0), (3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2), (2,2),(2,1)\rangle$.

Case 2. To prove that Q_2^4 is 4^*-laceable.

Case 2.1. Let $b = (1,0)$.

The four disjoint paths $\{P_1, P_2, P_3, P_4\}$ between w and b whose union covers all vertices of Q_2^4 are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, $P_3 = \langle(0,0),(0,3),(0,2), (1,2),(1,3),(1,0)\rangle$, and $P_4 = \langle(0,0),(3,0),(3,1),(3,2), (3,3),(2,3),(2,2),(2,1),(2,0),(1,0)\rangle$.

Case 2.2. Let $b = (2,1)$.

The four disjoint paths $\{R_1, R_2, R_3, R_4\}$ between w and b whose union covers all vertices of Q_2^4 are $R_1 = \langle(0,0),(3,0), (3,1),(2,1)\rangle$, $R_2 = \langle(0,0),(1,0),(2,0),(2,1)\rangle$, $R_3 = \langle (0,0),(0,1),(1,1),(2,1)\rangle$, and $R_4 = \langle(0,0),(0,3),(0,2), (1,2),(1,3),(2,3),(3,3),(3,2),(2,2),(2,1)\rangle$.

APPENDIX B
PROOF OF LEMMA 4

Notice that Q_2^6 is vertex symmetric. W.L.O.G, let $w = (0,0)$. There are four cases for b. That is, $b \in \{(1,0),(2,1),(3,0),(3,2)\}$.

Case 1. To prove that Q_2^6 is 3^*-laceable.

Case 1.1. Let $b = (1,0)$.

The three disjoint paths $\{P_1, P_2, P_3\}$ between w and b whose union covers all vertices of Q_2^6 are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, and $P_3 = \langle(0,0),(5,0),(5,1), (5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5),(0,5), (0,4),(1,4),(2,4),(3,4),(4,4),(4,3),(4,2),(4,1),(4,0), (3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3),(0,2),(1,2), (2,2),(2,1),(2,0),(1,0)\rangle$.

Case 1.2. Let $b = (2,1)$.

The three disjoint paths $\{R_1, R_2, R_3\}$ between w and b whose union covers all vertices of Q_2^6 are $R_1 = \langle(0,0),(1,0),(2,0), (2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, and $R_3 = \langle(0,0), (5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5), (1,5),(0,5),(0,4),(1,4),(2,4),(3,4),(4,4),(4,3),(4,2), (4,1),(4,0),(3,0),(3,1),(3,2),(3,3),(2,3),(1,3),(0,3), (0,2),(1,2),(2,2),(2,1)\rangle$.

Case 1.3. Let $b = (3,0)$.

The three disjoint paths $\{S_1, S_2, S_3\}$ between w and b whose union covers all vertices of Q_2^6 are $S_1 = \langle(0,0),(1,0),(2,0), (3,0)\rangle$, $S_2 = \langle(0,0),(5,0),(4,0),(3,0)\rangle$, and $S_3 = \langle(0,0), (0,5),(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),(4,4),(3,4), (2,4),(1,4),(0,4),(0,3),(1,3),(2,3),(3,3),(4,3),(5,3), (5,2),(5,1),(4,1),(4,2),(3,2),(2,2),(1,2),(0,2),(0,1), (1,1),(2,1),(3,1),(3,0)\rangle$.

Case 1.4. Let $b = (3,2)$.

The three disjoint paths $\{T_1, T_2, T_3\}$ between w and b whose union covers all vertices of Q_2^6 are $T_1 = \langle(0,0),(1,0),(2,0),$

$(3,0),(3,1),(3,2)\rangle$, $T_2 = \langle(0,0),(0,1),(0,2),(1,2),(1,1), (2,1),(2,2),(3,2)\rangle$, and $T_3 = \langle(0,0),(5,0),(4,0),(4,1), (5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(3,5),(2,5),(1,5), (0,5),(0,4),(0,3),(1,3),(1,4),(2,4),(2,3),(3,3),(3,4), (4,4),(4,3),(4,2),(3,2)\rangle$.

Case 2. To prove that Q_2^6 is 4^*-laceable.

Case 2.1. Let $b = (1,0)$.

The four disjoint paths $\{P_1, P_2, P_3, P_4\}$ between w and b whose union covers all vertices of Q_2^6 are $P_1 = \langle(0,0),(1,0)\rangle$, $P_2 = \langle(0,0),(0,1),(1,1),(1,0)\rangle$, $P_3 = \langle(0,0),(0,5),(0,4), (0,3),(0,2),(1,2),(1,3),(1,4),(1,5),(1,0)\rangle$, and $P_4 = \langle (0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(4,4), (4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(3,2),(3,3),(3,4), (3,5),(2,5),(2,4),(2,3),(2,2),(2,1),(2,0),(1,0)\rangle$.

Case 2.2. Let $b = (2,1)$.

The four disjoint paths $\{R_1, R_2, R_3, R_4\}$ between w and b whose union covers all vertices of Q_2^6 are $R_1 = \langle(0,0),(1,0), (2,0),(2,1)\rangle$, $R_2 = \langle(0,0),(0,1),(1,1),(2,1)\rangle$, $R_3 = \langle (0,0),(5,0),(5,1),(5,2),(5,3),(5,4),(5,5),(4,5),(4,4), (4,3),(4,2),(4,1),(4,0),(3,0),(3,1),(2,1)\rangle$, and $R_4 = \langle (0,0),(0,5),(1,5),(2,5),(3,5),(3,4),(2,4),(1,4),(0,4), (0,3),(0,2),(1,2),(1,3),(2,3),(3,3),(3,2),(2,2),(2,1)\rangle$.

Case 2.3. Let $b = (3,0)$.

The four disjoint paths $\{S_1, S_2, S_3, S_4\}$ between w and b whose union covers all vertices of Q_2^6 are $S_1 = \langle(0,0),(1,0),(2,0),(3,0)\rangle$, $S_2 = \langle(0,0),(0,1),(1,1), (2,1),(3,1),(3,0)\rangle$, $S_3 = \langle(0,0),(5,0),(5,1),(5,2),(5,3), (5,4),(5,5),(4,5),(4,4),(4,3),(4,2),(4,1),(4,0),(3,0)\rangle$, and $S_4 = \langle(0,0),(0,5),(0,4),(0,3),(0,2),(1,2),(1,3), (1,4),(1,5),(2,5),(2,4),(2,3),(2,2),(3,2),(3,3),(3,4), (3,5),(3,0)\rangle$.

Case 2.4. Let $b = (3,2)$.

The four disjoint paths $\{T_1, T_2, T_3, T_4\}$ between w and b whose union covers all vertices of Q_2^6 are $T_1 = \langle(0,0),(1,0),(2,0),(3,0),(3,1),(3,2)\rangle$, $T_2 = \langle(0,0),(0,1), (0,2),(1,2),(1,1),(2,1),(2,2),(3,2)\rangle$, $T_3 = \langle(0,0),(5,0), (4,0),(4,1),(5,1),(5,2),(4,2),(3,2)\rangle$, and $T_4 = \langle(0,0), (0,5),(1,5),(2,5),(3,5),(4,5),(5,5),(5,4),(5,3),(4,3), (4,4),(3,4),(2,4),(1,4),(0,4),(0,3),(1,3),(2,3),(3,3), (3,2)\rangle$.

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Shu-Li Chang received the B.S. degree in the Department of Applied Mathematics from Feng Chia University, Taichung, Taiwan, R.O.C. in 2003. She received her M.S. degree from the Department of Applied Mathematics from Chung Yuan Christian University, Chungli, Taiwan, R.O.C. in 2010. Her research interests include interconnection networks, and graph theory application.

Shin-Shin Kao received the B.S. degree in mathematics from National Tsing Hua University in Taiwan in 1990, and the M.S. and Ph.D. degrees from University of California, Los Angeles, U.S.A. in 1993 and 1995, respectively. She has joined the faculty of the Department of Applied Mathematics, Chung-Yuan Christian University, Chungli, Taiwan, R.O.C. since 1995 and has been chairing the department since 2006. Her research interests include combinatorial optimization, interconnection networks, and graph theory.

Yuan-Kang Shih received the B.S. degree in the Department of Mathematics from Fu Jen Catholic University, Hsinchuang, Taipei County, Taiwan, R.O.C. in 2004. He received his M.S. degree from the Department of Applied Mathematics from Chung Yuan Christian University, Chungli, Taiwan, R.O.C. in 2006. He is now a student in the Ph.D. program in the College of Computer Science in the National Chiao Tung University, Hsinchu, Taiwan, R.O.C. His research interests include interconnection networks, fault-tolerant problems, and graph theory applications.