

Extremal Properties of Generalized Class of Close-to-convex Functions

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Abstract—Let $G_{\alpha,\beta}(\gamma,\delta)$ denote the class of function $f(z)$, $f(0)=f'(0)-1=0$ which satisfied $\operatorname{Re} e^{i\delta} \{\alpha f'(z) + \beta z f''(z)\} > \gamma$ in the open unit disk $D = \{z \in \mathbb{C} : |z| < 1\}$ for some $\alpha \in \mathbb{C}$ ($\alpha \neq 0$), $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ ($0 \leq \gamma < \alpha$) where $|\delta| \leq \pi$ and $\alpha \cos \delta - \gamma > 0$. In this paper, we determine some extremal properties including distortion theorem and argument of $f'(z)$.

Keywords—Argument of $f'(z)$, Carathéodory Function, Close-to-convex Function, Distortion Theorem, Extremal Properties

I. INTRODUCTION

WE denote $G_{\alpha,\beta}(\gamma,\delta)$ the class of normalized analytic function f in the open unit disk, $D = \{z \in \mathbb{C} : |z| < 1\}$ where

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

satisfying $\operatorname{Re} e^{i\delta} \{\alpha f'(z) + \beta z f''(z)\} > \gamma$, $z \in D$ for some $\alpha \in \mathbb{C}$ ($\alpha \neq 0$), $\beta \in \mathbb{C}$ and $\gamma \in \mathbb{R}$ ($0 \leq \gamma < \alpha$).

Many of the subclasses of $G_{\alpha,\beta}(\gamma,\delta)$ have been studied by some other researchers as [1] for $G_{\alpha,\beta}(\gamma,0)$ of some $\alpha \in \mathbb{C}$, $\beta \in \mathbb{C}$ ($\beta \neq 0$) and $\gamma \in \mathbb{R}$ ($0 \leq \gamma < \alpha$), [2] for $G_{1,\beta}(\gamma,0)$ where $\alpha > 0, \beta < 1$, [3] for $G_{1,1}(\gamma,0)$, [4] for $G_{1,1}(0,0)$, [5] for $G_{1,0}(\gamma,\delta)$ where $|\delta| \leq \pi$ and $\cos \delta - \gamma > 0$, [6] for $G_{1,0}(0,\delta)$ where $|\delta| < \frac{\pi}{2}$ and [7] for $G_{1,0}(0,0)$.

There is a relationship of the class P in the form of $p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n$ with the extremal information of each selected classes. Writing

$$\frac{e^{i\delta} (\alpha f'(z) + \beta z f''(z)) - \gamma - i\alpha \sin \delta}{(\alpha \cos \delta - \gamma)} = p(z) \quad (z \in D)$$

clearly $f \in G_{\alpha,\beta}(\gamma,\delta)$ if $p \in P$, the class of functions with positive real parts.

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We make use the result of representation theorem

$$f(z) = \int_{|x|=1} \left[-e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) z - \frac{2e^{-i\delta} A}{(n\beta + \alpha)} \bar{x} \log(1-xz) \right] d\mu(x)$$

where $A = (\alpha \cos \delta - \gamma)$ given by [8] in order to determine the distortion theorem and argument of $f'(z)$ for this class of function.

II. EXTREMAL PROPERTIES

We begin by finding the radius and centre of $G_{\alpha,\beta}(\gamma,\delta)$ that will be used for later results.

Theorem 3.1 Let $f(z) \in G_{\alpha,\beta}(\gamma,\delta)$. Then $f'(z)$ maps $|z| \leq r$ into disc D_r with centre and radius

$$-e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + \frac{2e^{-i\delta} AM}{1-r^2} \quad \text{and} \quad \frac{2AMr}{1-r^2}$$

where $A = \alpha \cos \delta - \gamma$, $M = \frac{1}{n\beta + \alpha}$ respectively.

Proof. If a and b are complex numbers with $|b| < 1$ and if $0 < r < 1$, the range of the function $(1+arw)/(1+brw)$ where $|w| \leq 1$ is a disc with center and radius respectively.

$$\frac{1 - a\bar{b}r^2}{1 - |b|^2 r^2}, \quad \frac{|a-b|r}{1 - |b|^2 r^2}$$

By taking $a = \bar{B}e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right)$ and $b = xr$ where $|x| = 1$, we see that maps $|z| \leq r$ onto D_r . By convexity, any linear combination of functions of this form also maps D onto D_r . Since for some probability measure μ , we have

$$B \left\{ \frac{1 + \bar{B}e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) xz}{1 - xz} \right\}$$

$$f'(z) = \int_{|x|=1} B \left\{ \frac{1 + \overline{B}e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) xz}{(1-xz)} \right\} d\mu(x)$$

where $B = \frac{\frac{2\gamma}{\alpha} n\beta e^{-i\delta} - n\beta e^{-2i\delta} + \alpha}{n\beta + \alpha}$, so, the result follows.

Corollary 3.1 If $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$ then

$$f'(z) \prec B \left\{ \frac{1 + \overline{B}e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) z}{1-z} \right\}, z \in D$$

The simple geometry of a circle enables us to deduce from Theorem 3.2, upper and lower bounds for $\operatorname{Re} f'(z)$, $\operatorname{Im} f'(z)$, $|f'(z)|$ and $\arg f'(z)$ when $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$.

Theorem 3.2 If $f(z) \in G_{\alpha,\beta}(\gamma, \delta)$, then

$$\frac{1+B+r^2(2AR-1)-2AMr}{1-r^2} \leq \operatorname{Re} f'(z) \leq \frac{1+B+r^2(2AR-1)+2AMr}{1-r^2}$$

where $B = \frac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$ and $R = \frac{(A+\gamma)}{\alpha^2}$, and

$$\frac{-2A \left(T \left(M - \frac{1}{\alpha} \right) + r \left(M + \frac{rT}{\alpha} \right) \right)}{1-r^2} \leq \operatorname{Im} f'(z) \leq \frac{2A \left(T \left(M - \frac{1}{\alpha} \right) + r \left(M + \frac{rT}{\alpha} \right) \right)}{1-r^2}$$

where $T = \sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2}$ and all bounds are sharp for any extreme point $f(z)$ of $G_{\alpha,\beta}(\gamma, \delta)$.

Proof. By Theorem 3.1, we can write

$$\left| f'(z) - \left\{ -e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + \frac{2e^{-i\delta} AM}{1-r^2} \right\} \right| \leq \frac{2AMr}{1-r^2} \quad (1)$$

So that

$$\frac{2AMr}{1-r^2} \leq \operatorname{Re} \left\{ f'(z) + e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) - \frac{2e^{-i\delta} AM}{1-r^2} \right\} \leq \frac{2AMr}{1-r^2}$$

that gives ;

$$-\frac{2AMr}{1-r^2} \leq \operatorname{Re} f'(z) - \left\{ \frac{2AM \cos \delta - (1-r^2) \left(2 \cos^2 \delta - 1 - \frac{2\gamma}{\alpha} \cos \delta \right)}{1-r^2} \right\} \leq \frac{2AMr}{1-r^2}$$

and

$$-\frac{2AMr}{1-r^2} \leq \operatorname{Im} \left\{ f'(z) + e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) - \frac{2e^{-i\delta} AM}{1-r^2} \right\} \leq \frac{2AMr}{1-r^2}$$

that gives

$$\frac{-2A \left(\sin \delta \left(M - \frac{1}{\alpha} \right) + r \left(M + \frac{r \sin \delta}{\alpha} \right) \right)}{1-r^2} \leq \operatorname{Im} f'(z) \leq \frac{2A \left(\sin \delta \left(\frac{1}{\alpha} - M \right) + r \left(M - \frac{r \sin \delta}{\alpha} \right) \right)}{1-r^2}$$

Since $\cos \delta = \frac{A+\gamma}{\alpha}$ and $\sin \delta = \sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2}$, we can write the inequalities in this form

$$-\frac{2AMr}{1-r^2} \leq \operatorname{Re} f'(z) - \left\{ \frac{1 + 2A(A+\gamma) \left(\frac{M\alpha-1}{\alpha^2} \right) + r^2 \left(\frac{2A(A+\gamma)}{\alpha^2} - 1 \right)}{1-r^2} \right\} \leq \frac{2AMr}{1-r^2}$$

and

$$\frac{-2A \left(\sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2} \left(M - \frac{1}{\alpha} \right) + r \left(M + \frac{r \sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2}}{\alpha} \right) \right)}{1-r^2} \leq \operatorname{Im} f'(z) \leq \frac{2A \left(\sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2} \left(\frac{1}{\alpha} - M \right) + r \left(M - \frac{r \sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2}}{\alpha} \right) \right)}{1-r^2}$$

Letting $B = \frac{2A(A+\gamma)(M\alpha-1)}{\alpha^2}$, $R = \frac{(A+\gamma)}{\alpha^2}$ and

$T = \sqrt{1 - \left(\frac{A+\gamma}{\alpha} \right)^2}$, we obtain the above inequalities as required. It is clear that each inequality is sharp for some z on

$|z| = r$ when f is an extreme point.

Our next result is to obtain a distortion theorem for $f(z) \in G_{\alpha, \beta}(\gamma, \delta)$.

Theorem 3.3 If $f(z) \in G_{\alpha, \beta}(\gamma, \delta)$, then

$$|f'(z)| \leq |\Gamma(r)| + \frac{2AMr}{1-r^2} \text{ where}$$

$$C(r) = \left(1 - \frac{4\gamma A}{\alpha^2} + \frac{4AM}{\alpha(1-r^2)} \left\{ \frac{AM\alpha}{(1-r^2)} + \gamma - A \right\} \right)^{\frac{1}{2}} \quad (2)$$

Proof. Let

$$\Gamma(r) = -e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + \frac{2e^{-i\delta} AM}{1-r^2} \quad (3)$$

and from (1), we have

$$|f'(z) - \Gamma(r)| \leq \frac{2AMr}{1-r^2}$$

so that

$$|f'(z)| \leq |\Gamma(r)| + \frac{2AMr}{1-r^2} = C(r) + \frac{2AMr}{1-r^2}$$

as required.

If $\gamma \geq 0$, then f' is non-zero throughout D and has continuous argument whereas if $\gamma < 0$ and f_0 is any extreme function of $G_{\alpha, \beta}(\gamma, \delta)$, then at some points in D , f'_0 has a zero, thus, there is no argument. We next obtain bounds for $\arg f'(z)$ when $f(z) \in G_{\alpha, \beta}(\gamma, \delta)$ with restricted value of $|z|$ for the case of $\gamma < 0$. Furthermore, we will use the following property for argument: for given δ in $[-\pi, \pi]$ and as x varies in some interval $[0, c]$, so that $e^{i\delta} + x \neq 0$, $\phi_\delta(x)$ is continuous argument of $e^{i\delta} + x \neq 0$ for which $\phi_\delta(0) = \delta$. We have

$$\phi_\delta(x) = \begin{cases} \tan^{-1} \left(\frac{\sin \delta}{\cos \delta + x} \right) & x + \cos \delta > 0 \\ \pi + \tan^{-1} \left(\frac{\sin \delta}{\cos \delta + x} \right) & x + \cos \delta < 0 \\ \frac{\pi}{2} & x + \cos \delta = 0 \end{cases}$$

when $0 < \delta < \pi$ and $-\pi < \delta < 0$ for $\delta = 0, \pm\pi$.

Theorem 3.4 Let $f(z) \in G_{\alpha, \beta}(\gamma, \delta)$ and put

$$x(r) = 2 \left(\frac{AM}{1-r^2} - \frac{A}{\alpha} \right) (0 \leq r \leq 1). \text{ Let}$$

$$r_0 = \begin{cases} 1 & \gamma \geq 0 \\ \sqrt{1 - \frac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}} & \gamma < 0 \end{cases}$$

Then, for $0 < |z| = r < r_0$, and for a suitable determination of argument

$$|\arg f'(z) + \delta - \phi_\delta(x(r))| \leq \sin^{-1} \frac{2AMr}{(1-r^2)C(r)}$$

where $\phi_\delta(x)$ is defined on $[0, x(r_0))$ as above and $C(r)$ is given by (2). The result is sharp.

Proof. To make sure that $f'(z) \neq 0$, we restrict the values of $|z| = r$ by the condition

$$\left| \frac{2AM}{1-r^2} - \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) \right| > \frac{2AMr}{1-r^2}$$

Squaring both sides, we have

$$\frac{4A^2M^2}{(1-r^2)^2} + 1 + \frac{4AM}{(1-r^2)} \left(\frac{2\gamma}{\alpha} - \cos \delta \right) - \frac{4\gamma \left(\cos \delta - \frac{\gamma}{\alpha} \right)}{\alpha} > 0$$

and since $A = \alpha \cos \delta - \gamma$, hence

$$1 + \frac{4AM}{(1-r^2)} \left(AM - \frac{(A-\gamma)}{\alpha} \right) - \frac{4A\gamma}{\alpha^2} > 0$$

The inequality holds for all r in $[0, 1)$ if $\gamma \geq 0$ and for $0 \leq r < \sqrt{1 - \frac{4\alpha AM(\alpha AM - A + \gamma)}{(4A\gamma - \alpha^2)}}$ if $\gamma < 0$. This establishes the restricted on $|z|$ in the statement of the theorem.

From (1) then with $\Gamma(r)$ given by (3) and $C(r) = |\Gamma(r)|$, we have

$$C(r) = \left(1 + \frac{4A^2M}{1-r^2} \left(\frac{M}{1-r^2} - \frac{1}{\alpha} \right) + \frac{4A(\alpha \cos \delta - A)}{\alpha^2} \left(\frac{M\alpha}{1-r^2} - 1 \right) \right)^{\frac{1}{2}}$$

and deduced to $|\arg f'(z) - \arg \Gamma(r)| \leq \sin^{-1} \frac{2AMr}{(1-r^2)C(r)}$

also

$$\begin{aligned} \arg \Gamma(r) &= \arg \left(e^{-i\delta} \left(\frac{2AM}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha} \right) \right) \\ &= -\delta + \arg \left(e^{-i\delta} + 2 \left(\frac{AM}{1-r^2} - \frac{A}{\alpha} \right) \right). \end{aligned}$$

Put $x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)$, then $\arg \Gamma(r) = -\delta + \phi_\delta(x(r))$.

We obtain another theorem that replaced $\arg f'(z)$ with restricted range of $|z|$ as $\arg(f'(z) + k)$ for some real number k that satisfied $f'(z) + k \neq 0$ for $z \in D$ and $f \in G_{\alpha, \beta}(\gamma, \delta)$. By taking $|\delta| \neq \pi/2$ as any choice of k with $k \cos \delta + \gamma > 0$ will ensure that above conditions are fulfilled and this is important for the following result to be valid. In the following theorem, for a given $\delta \in [-\pi, \pi]$ and as x varies in same interval $[0, c)$, so that $(k+1)e^{i\delta} + x \neq 0$, $\psi_\delta(\delta)$ is the continuous argument of $(k+1)e^{i\delta} + x$ for which $\psi_\delta(0)$ is principal.

Theorem 3.5 For $|\delta| \neq \pi/2$, $f(z) \in G_{\alpha, \beta}(\gamma, \delta)$ and put $x(r) = 2\left(\frac{AM}{1-r^2} - \frac{A}{\alpha}\right)$ ($0 \leq r \leq 1$). Let $k\alpha \cos \delta + \gamma > 0$ where k is a real number. Then, for $\psi_\delta(x)$ defined on $[0, \infty)$,

$$|\arg(f'(z) + k) + \delta - \psi_\delta(x(r))| \leq \sin^{-1} \frac{2AMr}{(1-r^2)C_1(r)} \text{ where}$$

$$C_1(r) = \left[4AT \left(k \cos \delta + \frac{A}{\alpha}(T+1) + \frac{\gamma}{\alpha} \right) + (k+1)^2 \right], T = \frac{M}{1-r^2} - \frac{1}{\alpha}$$

Proof. Let $|\delta| \neq \pi/2$ and k satisfied $k\alpha \cos \delta + \gamma > 0$. Using (1), we have

$$|(f'(z) + k) - (\Gamma(r) + k)| \leq \frac{2AMr^2}{1-r^2}$$

where

$$\Gamma(r) = -e^{-i\delta} \left(e^{-i\delta} - \frac{2\gamma}{\alpha} \right) + \frac{2Ae^{-i\delta}AM}{1-r^2} = 1 + \frac{2Ae^{-i\delta} \left(M - \frac{1}{\alpha} + \frac{r^2}{\alpha} \right)}{1-r^2}$$

Hence

$$|\arg(f'(z) + k) - \arg(\Gamma(r) + k)| \leq \sin^{-1} \frac{2AMr}{(1-r^2)C_1(r)} \quad (4)$$

where

$$C_1(r) = |\Gamma(r) + k| = \left[(k+1)^2 + 4A^2 \left(\frac{M}{1-r^2} - \frac{1}{\alpha} \right)^2 + k \cos \delta + \cos \delta \right]^{\frac{1}{2}} + \frac{4A}{1-r^2} \left(\frac{M}{1-r^2} - \frac{1}{\alpha} \right) \left(A \left(\frac{M}{1-r^2} - \frac{1}{\alpha} \right) \right)$$

Let $T = \frac{M}{1-r^2} - \frac{1}{\alpha}$, we have

$$C_1(r) = \left[4AT \left(k \cos \delta + \frac{A}{\alpha}(T+1) + \frac{\gamma}{\alpha} \right) + (k+1)^2 \right]^{\frac{1}{2}}$$

Now

$$\begin{aligned} \arg(\Gamma(r) + k) &= \arg \left(e^{-i\delta} \left(\frac{2AM}{1-r^2} - e^{-i\delta} + \frac{2\gamma}{\alpha} + ke^{i\delta} \right) \right) \\ &= -\delta + \arg \left((k+1)e^{i\delta} + 2A \left(\frac{M}{1-r^2} - \frac{1}{\alpha} \right) \right) \\ &= -\delta + \psi_\delta(x(r)) \end{aligned}$$

and with (4) this completes the proof.

REFERENCES

- [1] S. Owa, T. Hayami, and K. Kuroki, "A note on certain analytic functions," *J. Math. Soc. Japan*, vol. 1538, pp. 74–81, 2007.
- [2] C. Y. Gao, and S. Q. Zhou, "Certain subclass of starlike functions," *Applied Math. and Computation*, vol. 187, pp. 176–182, 2007.
- [3] H. Silverman, "A class of bounded starlike functions," *Internat. J. Math. & Math. Sci. (2nd Series)*, vol. 17, pp. 249–252, 1994.
- [4] P. N. Chichra, "New subclasses of the class of close-to-convex functions," *Proceedings of the American Mathematical Society (First Series)*, vol. 62, pp. 37–43.
- [5] D. Mohamad, "On a class of functions whose derivatives map the unit disc into a half plane," *Bull. Malaysian Math. Sc. Soc. (2nd Series)*, vol. 23, pp. 163–171, 2001.
- [6] H. Silverman, and E. M. Silvia, "On α -close-to-convex function," *Publ. Math. Debrecen*, vol. 49, pp. 532–537, 1996.
- [7] T. H. MacGregor, "Function whose derivative has a positive real part," *Trans. Amer. Math. Soc.*, vol. 104, pp. 532–537, 1962.
- [8] N. Mohamed, D. Mohamad, and S. S. Cik, "Some results on generalized class of close-to-convex functions," unpublished.