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Transient Currents in a Double Conductor Line above a Conducting Half-Space

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***Abstract*—**Transient eddy current problem is solved in the present paper by the method of the Laplace transform for the case of a double conductor line located parallel to a conducting half-space. The Fourier sine and cosine integral transforms are used in order to find the Laplace transform of the solution. The inverse Laplace transform of the solution is found in closed form. The integrated electromotive force per unit length of the double conductor line is calculated in the form of an improper integral.

***Keywords*—**Transient eddy currents, Laplace transform, double conductor line.

I. Introduction

THEORY of an eddy current method for the case where a coil is excited by an alternating current is well-developed in the literature [1]-[4]. Current excitation in the form of a pulse represents an alternative to traditional eddy current methods. There are two basic methods that are usually used to analyze non-periodic time-dependent signals in eddy current testing: fast Fourier transform and Laplace transform. The difficulty in using the Laplace transform is that the inverse transform is not always available in closed form. However, several authors [5]-[8] have obtained analytical solutions for problems where a single-turn coil or a coil with finite dimensions is located above a conducing half-space assuming that the excitation current is in the form of a step current or exponential source current.

In the present paper we consider transient eddy current problem for the case where an excitation coil is assumed to be of the form of a double conductor line formed by two infinitely long wires located parallel to a conducting half-space. The inverse Laplace transform is found in closed form for the case of an excitation current in the form of a unit step function.

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II. Mathematical formulation of the problem

Different types of eddy current coils are used in applications. One example is a coil in the form of a rectangular frame. If the ratio of the sides of the frame is 4:1 or larger then such a coil can be modeled by a double conductor line [9] with a relatively small error. Suppose that two horizontal infinitely long parallel wires are situated above a conducting half-space with electrical conductivity σ and relative magnetic permeability μ (see Fig. 1).

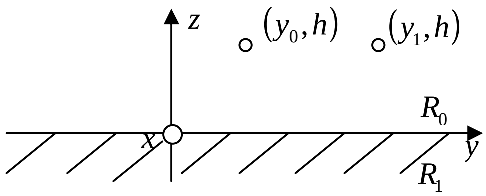

Fig. 1 A double conductor line above a conducting half-space

Assume that the current in the wires is

$$I(t) = \pm I_0 \varphi(t) \tag{1}$$

at the points (y_0, h) and (y_1, h), respectively, where I_0 is a constant, and $\varphi(t)$ is the Heaviside function of the form

$$\varphi(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{2}$$

We assume that the electric field $\vec{E}_0$ and $\vec{E}_1$ in regions $R_0 = \{z > 0\}$ and $R_1 = \{z < 0\}$, respectively, has only one non-zero component in the $x-$direction:

$$\vec{E}_i = E_i(y,z,t)\bar{e}_x, \quad i = 0,1, \tag{3}$$

where $\vec{e}_x$ is the unit vector in the positive $x-$direction. The system of Maxwell's equations in this case can be transformed to one equation for the electric field which has the following form in regions R_0 and R_1:

$$\frac{\partial^2 E_0}{\partial y^2}+\frac{\partial^2 E_0}{\partial z^2}=\mu_0\frac{\partial I}{\partial t}\delta(z-h)[\delta(y-y_0) -\delta(y-y_1)], \tag{4}$$

$$\frac{\partial^2 E_1}{\partial y^2}+\frac{\partial^2 E_1}{\partial z^2}-\mu_0\mu\frac{\partial E_1}{\partial t}=0, \tag{5}$$

where $\delta(\xi)$ is the Dirac delta-function.

The boundary conditions are

$$E_0\,|_{z=0}=E_1\,|_{z=0},\quad \frac{\partial E_0}{\partial z}\,|_{z=0}=\frac{1}{\mu}\frac{\partial E_1}{\partial z}\,|_{z=0}\,. \tag{6}$$

In addition, the following conditions hold at infinity:

$$E_0, E_1, \frac{\partial E_0}{\partial y}, \frac{\partial E_1}{\partial y}\to 0 \text{ as } \sqrt{y^2+z^2}\to\infty. \tag{7}$$

The initial conditions are

$$E_0\,|_{t=0}=0,\quad E_1\,|_{t=0}=0. \tag{8}$$

III. Laplace Transform of the Solution

Applying the Laplace transform to (4)-(7) we obtain

$$\frac{\partial^2 \overline{E}_0}{\partial y^2}+\frac{\partial^2 \overline{E}_0}{\partial z^2}=\bar f(y,z,s), \tag{9}$$

$$\frac{\partial^2 \overline{E}_1}{\partial y^2}+\frac{\partial^2 \overline{E}_1}{\partial z^2}-\mu_0\mu s\overline{E}_1=0, \tag{10}$$

$$\overline{E}_0\,|_{z=0}=\overline{E}_1\,|_{z=0},\quad \frac{\partial \overline{E}_0}{\partial z}\,|_{z=0}=\frac{1}{\mu}\frac{\partial \overline{E}_1}{\partial z}\,|_{z=0}, \tag{11}$$

$$\overline{E}_0, \overline{E}_1, \frac{\partial \overline{E}_0}{\partial y}, \frac{\partial \overline{E}_1}{\partial y}\to 0 \text{ as } \sqrt{y^2+z^2}\to\infty, \tag{12}$$

where $\overline{E}_0, \overline{E}_1$ and $\bar I$ are the Laplace transforms of the functions E_0, E_1 and I, respectively, s is the parameter of the Laplace transform, and

$$\bar f(y,z,s)=\mu_0 s\bar I(s)\delta(z-h)[\delta(y-y_0) -\delta(y-y_1)]. \tag{13}$$

It is convenient to represent the function $\bar f(y,z,s)$ in (13) as the sum of even and odd functions of the form

$$\bar f(y,z,s)=\frac{1}{2}[\bar f(y,z,s)+\bar f(-y,z,s)] +\frac{1}{2}[\bar f(y,z,s)-\bar f(-y,z,s)]=\bar f_{even}(y,z,s) +\bar f_{odd}(y,z,s), \tag{14}$$

where

$$\bar f_{even}(y,z,s)=\frac{\mu_0 s\bar I(s)}{2}\delta(z-h)[\delta(y-y_0) -\delta(y-y_1)], \tag{15}$$
$$\bar f_{odd}(y,z,s)=\frac{\mu_0 s\bar I(s)}{2}\delta(z-h)[\delta(y-y_0) -\delta(y-y_1)].$$

The following properties of the delta-function are used in order to derive (14) and (15): (a) $\delta(-y)=\delta(y)$ and (b) $\delta(y+y_0)=0, \delta(y+y_1)=0$ if $y\ge 0$ and $y_0>0, y_1>0.$

Thus, the solution to (9)-(12) is sought in the form

$$\overline{E}_i(y,z,s)=\overline{E}_{i\,even}(y,z,s)+\overline{E}_{i\,odd}(y,z,s), i=0,1. \tag{16}$$

The Fourier cosine transform of the form

$$\overline{E}_i^{(c)}(\lambda,z,s)=\int_0^\infty \overline{E}_i(y,z,s)\cos\lambda y\,dy,\quad i=0,1 \tag{17}$$

is used to find the even component of the solution. Applying the Fourier cosine transform (17) to (9)-(12) where the function $\bar f(y,z,s)$ in (9) is replaced by $\bar f_{even}(y,z,s)$ we obtain

$$\frac{d^2\overline{E}_0^{(c)}}{dz^2}-\lambda^2\overline{E}_0^{(c)}=\frac{\mu_0 s\bar I(s)}{2}\delta(z-h)[\cos\lambda y_0 -\cos\lambda y_1], \tag{18}$$

$$\frac{d^2\overline{E}_1^{(c)}}{dz^2}-q^2\overline{E}_1^{(c)}=0, \tag{19}$$

$$\overline{E}_0^{(c)}\,|_{z=0}=\overline{E}_1^{(c)}\,|_{z=0},\quad \frac{d\overline{E}_0^{(c)}}{dz}\,|_{z=0}=\frac{1}{\mu}\frac{d\overline{E}_1^{(c)}}{dz}\,|_{z=0}, \tag{20}$$

where $q=\sqrt{\lambda^2+\mu_0\mu\sigma s}.$

In order to solve (18) we consider two sub-regions, $R_{00}=\{0<z<h\}$ and $R_{01}=\{z.>h\}$, of region R_0. The solutions to (18) in R_{00} and R_{01} are denoted by $\overline{E}_{00}^{(c)}$ and $\overline{E}_{01}^{(c)}$, respectively.

The general solution to (18) in R_{00} is

$$\overline{E}_{00}^{(c)}(\lambda, z, s) = C_1 e^{\lambda z} + C_2 e^{-\lambda z}. \tag{21}$$

The general solution to (18) in R_{01} which is bounded as $z \to +\infty$ has the form

$$\overline{E}_{01}^{(c)}(\lambda, z, s) = C_3 e^{-\lambda z}. \tag{22}$$

Finally, the general solution to (19) which remains bounded as $z \to -\infty$ can be written as follows

$$\overline{E}_{1}^{(c)}(\lambda, z, s) = C_4 e^{qz}. \tag{23}$$

Since there are four unknown constants in (21)-(23) and only two boundary conditions (20), we need two additional conditions at $z = h.$ One additional condition represents continuity of the electric field at $z = h$ and has the form

$$\overline{E}_{00}^{(c)}\,|_{z=h} = \overline{E}_{01}^{(c)}\,|_{z=h}\ . \tag{24}$$

Integrating (18) with respect to z from $h-\varepsilon$ to $h+\varepsilon$ and taking the limit as $\varepsilon \to +0$ we obtain the second additional condition in the form

$$\frac{d\overline{E}_{01}^{(c)}}{dz}\,|_{z=h} - \frac{d\overline{E}_{00}^{(c)}}{dz}\,|_{z=h} = \frac{\mu_0 s\bar{I}(s)}{2}\left(\cos\lambda y_0 - \cos\lambda y_1\right). \tag{25}$$

The constants C_1, C_2, C_3 and C_4 in (21)-(23) are determined from (20), (24)-(25) and have the form

$$C_1 = \frac{\mu_0 s\bar{I}(s)}{4\lambda}(\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda h}, \tag{26}$$

$$C_2 = \frac{\mu_0 s\bar{I}(s)}{4\lambda}\frac{(\lambda - q)}{(\lambda + q)}(\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda h}, \tag{27}$$

$$C_3 = C_2 + C_1 e^{2\lambda h}, \tag{28}$$

$$C_4 = \frac{\mu_0 s\bar{I}(s)}{2(\lambda + q)}(\cos\lambda y_0 - \cos\lambda y_1)e^{-\lambda h}. \tag{29}$$

Using the inverse Fourier cosine transform of the form

$$\overline{E}_{i\ even}(y, z, s) = \frac{2}{\pi}\int_0^\infty \overline{E}_i^{(c)}(\lambda, z, s)\cos\lambda y d\lambda, \quad i = 0,1 \tag{30}$$

we obtain the solution $\overline{E}_{i\,even}(y, z, s)$ in regions R_0 and R_1:

$$\overline{E}_{0\ even}(y, z, s) = \overline{E}_{0\ even}^{(free)}(y, z, s) + \overline{E}_{0\ even}^{(ind)}(y, z, s), \tag{31}$$

$$\overline{E}_{1\ even}(y, z, s) = \frac{\mu_0 s\bar{I}(s)}{\pi} \times\int_0^\infty \frac{e^{qz-\lambda h}}{\lambda + q}(\cos\lambda y_1 - \cos\lambda y_0)\cos\lambda y d\lambda, \tag{32}$$

where

$$\overline{E}_{0\ even}^{(free)}(y, z, s) = \frac{\mu_0 s\bar{I}(s)}{2\pi} \times\int_0^\infty \frac{e^{-\lambda|z-h|}}{\lambda}(\cos\lambda y_1 - \cos\lambda y_0)\cos\lambda y d\lambda, \tag{33}$$

$$\overline{E}_{0\ even}^{(ind)}(y, z, s) = \frac{\mu_0 s\bar{I}(s)}{2\pi} \times\int_0^\infty \frac{\mu\lambda - q}{(\mu\lambda + q)\lambda}e^{-\lambda(z+h)}(\cos\lambda y_1 - \cos\lambda y_0)\cos\lambda y d\lambda. \tag{34}$$

Here $\overline{E}_{0\ even}^{(free)}(y, z, s)$ is the even component of the electric field in free space while $\overline{E}_{0\ even}^{(ind)}(y, z, s)$ is the even component of the induced electric field due to the presence of the conducting half-space.

The Fourier sine transform of the form

$$\overline{E}_i^{(s)}(\lambda, z, s) = \int_0^\infty \overline{E}_i(y, z, s)\sin\lambda y dy, \quad i = 0,1 \tag{35}$$

and the inverse Fourier sine transform

$$\overline{E}_{i\ even}(y, z, s) = \frac{2}{\pi}\int_0^\infty \overline{E}_i^{(s)}(\lambda, z, s)\sin\lambda y d\lambda, \quad i = 0,1 \tag{36}$$

can be used in order to find the odd components of the solution. It can easily be shown that the odd components of the solution, $\overline{E}_{0odd}(y, z, s)$ and $\overline{E}_{1odd}(y, z, s)$ are given by (31) and (32) where cosine is replaced by sine.

Finally, the solution to (9)-(12) is obtained as the sum of the even and odd components and can be written as follows

$$\overline{E}_0(y,z,s) = \overline{E}_0^{(free)}(y,z,s) + \overline{E}_0^{(ind)}(y,z,s), \tag{37}$$

$$\overline{E}_1(y,z,s) = \frac{\mu_0 s\bar{I}(s)}{\pi} \times \int_0^\infty \frac{e^{qz-\lambda h}}{\lambda+q}[\cos\lambda(y-y_1) - \cos\lambda(y-y_0)]d\lambda, \tag{38}$$

where

$$\overline{E}_0^{(free)}(y,z,s) = \frac{\mu_0 s\bar{I}(s)}{2\pi} \times \int_0^\infty \frac{e^{-\lambda|z-h|}}{\lambda}[\cos\lambda(y-y_1) - \cos\lambda(y-y_0)]d\lambda \tag{39}$$

is the electric field of the double conductor line in free space and

$$\overline{E}_{0n}^{(ind)}(y,z,s) = \frac{\mu_0 s\bar{I}(s)}{2\pi} \times \int_0^\infty \frac{\mu\lambda - q}{(\mu\lambda+q)\lambda} e^{-\lambda(z+h)}[\cos\lambda(y-y_1) - \cos\lambda(y-y_0)]d\lambda \tag{40}$$

is the induced electric field in free space due to the presence of the conducting half-space.

In order to determine the inverse Laplace transform of (40) we rewrite the expression $\frac{\mu\lambda - q}{\mu\lambda + q}$ in the form

$$\frac{\mu\lambda - q}{\mu\lambda + q} = \frac{2\mu}{\mu + \sqrt{1+\tau s}} - 1,$$

where $\tau = \mu_0\mu\sigma/\lambda^2$. The inverse Laplace transform of the solution is found in the next section.

IV. Transient Solution

Consider the case of a step current in the form (1). The Laplace transform of (1) is $\bar{I}(s) = I_0/s$ so that

$$\Phi(\lambda,s) = s\bar{I}(s)\frac{\mu\lambda - q}{\mu\lambda + q} = I_0\left(\frac{2\mu}{\mu + \sqrt{1+\tau s}} - 1\right). \tag{41}$$

The inverse Laplace transform of (41) can be written as follows (see [10]):

$$\Phi(\lambda,t) = \frac{2\mu I_0}{\tau}e^{-t/\tau} \times \left[\sqrt{\frac{\tau}{\pi t}} - \mu e^{\mu^2 t/\tau}\operatorname{erfc}\left(\mu\sqrt{\frac{t}{\tau}}\right)\right] - I_0\delta(t). \tag{42}$$

Using (40)-(42) we obtain the inverse Laplace transform of the induced electric field in region R_0 in the form

$$E_{0n}^{(ind)}(y,z,t) = \frac{\mu_0 I_0}{2\pi} \times \int_0^\infty \frac{1}{\lambda}e^{-\lambda(z+h)}[\cos\lambda(y-y_1) - \cos\lambda(y-y_0)] \times \left\{\frac{2\mu I_0}{\tau}e^{-t/\tau}\left[\sqrt{\frac{\tau}{\pi t}} - \mu e^{\mu^2 t/\tau}\operatorname{erfc}\left(\mu\sqrt{\frac{t}{\tau}}\right)\right] - I_0\delta(t)\right\}d\lambda \tag{43}$$

The Laplace transform of the induced electromotive force (EMF) in a double conductor line per unit length due to the presence of the conducting half-space is given by (see [8]):

$$v(s) = \frac{1}{\bar{I}(s)}\int_C \vec{E}_0^{(ind)}(y,z,s)\cdot\vec{I}(s)dl, \tag{44}$$

where C is the contour of the double conductor line of length one unit in the $x-$direction. Using (40)-(44) we obtain the induced EMF in the form

$$v(t) = \frac{\mu_0}{\pi}\int_0^\infty e^{-2\lambda h}[\cos\lambda(y_1 - y_0) - 1] \times \left\{\frac{2\mu}{\tau}e^{-t/\tau}\left[\sqrt{\frac{\tau}{\pi t}} - \mu e^{\mu^2 t/\tau}\operatorname{erfc}\left(\mu\sqrt{\frac{t}{\tau}}\right)\right] - \delta(t)\right\}\frac{d\lambda}{\lambda}. \tag{45}$$

Using (45) we calculate the integral of the EMF with respect to time. The result is

$$V(t) = \frac{\mu_0}{\pi}\int_0^\infty e^{-2\lambda h}[\cos\lambda(y_1 - y_0) - 1] \times \left\{\frac{2\mu}{\mu^2 - 1}\left[\mu - erf\left(\sqrt{\frac{t}{\tau}}\right) - \mu e^{(\mu^2-1)t/\tau} erfc\left(\mu\sqrt{\frac{t}{\tau}}\right)\right] - \varphi(t)\right\}\frac{d\lambda}{\lambda}. \tag{46}$$

V. CONCLUSIONS

Closed-form analytical solution of a transient eddy current problem is found in the present paper. Excitation coil is in the form of a double conductor line formed by two parallel infinitely long wires located above a conducting half-space. The current in the coil is modeled by a unit step function of time. The Laplace transform of the solution is found by the method of Fourier sine and cosine integral transforms. The inverse Laplace transform of the solution is found in the form of an improper integral.

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