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Quartic Nonpolynomial Spline Solutions for Third Order Two-Point Boundary Value Problem

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***Abstract*—**In this paper, we develop quartic nonpolynomial spline method for the numerical solution of third order two point boundary value problems. It is shown that the new method gives approximations, which are better than those produced by other spline methods. Convergence analysis of the method is discussed through standard procedures. Two numerical examples are given to illustrate the applicability and efficiency of the novel method.

***Keywords*—**Quartic nonpolynomial spline, Two-point boundary value problem.

I. Introduction

We consider the following third order two point boundary value problems

$$y^{(3)} + f(x)y = g(x), \quad x \in [a,b], \tag{1}$$

subject to the boundary conditions:

$$y(a) - A_1 = y^{(1)}(a) - A_2 = y^{(1)}(b) - A_3 = 0, \tag{2}$$

where $A_i, i = 1,2,3$ are finite real constants. The functions $f(x)$ and $g(x)$ are continuous on the interval [a, b].

The analytical solution of the problem (1-2) can not be obtained for arbitrary choices of $f(x)$ and $g(x)$.

The numerical analysis literature contains a few other methods developed to find an approximate solution of this problem. Al Said et al. [1] have solved a system of third order two point boundary value problems using cubic splines. Noor et al. [4] generated second order method based on quartic splines. Other authors [2,3] generated finite difference using fourth degree B-spline and quintic polynomial spline for this problem subject to other boundary conditions.

The aim of this paper is to construct a new spline method based on a nonpolynomial spline function that has a polynomial part and a trigonometric part to develop numerical methods for obtaining smooth approximations for the solution of the problem (1) subject to the boundary conditions (2). Recently, new methods based on a nonpolynomial spline function that has a polynomial part and a trigonometric part is used to develop numerical methods for obtaining numerical approximation for some partial differential equations, see for example [6,7]. An extension of the work of Ramadan et al [5], we propose in this paper quartic nonpolynomial spline method for the numerical solution of third order two point boundary value problem (1-2).

The paper is organized as follows: In section II, we derive our method. The method is formulated in a matrix form in section III. Convergence analysis for our order methods is established in section IV. Numerical results are presented to illustrate the applicability and accuracy in section V. Finally, in section VI, we concluded the numerical results of the proposed methods.

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II. Derivation of the Method

We introduce a finite set of grid points x_i by dividing the interval $[a,b]$ into *(n+1)* equal parts where:

$$x_i = a + ih, \quad i = 0,1,2,\ldots,n, \quad x_0 = a, \quad x_n = b, \quad h = \frac{b-a}{n}. \tag{3}$$

Let $u(x)$ be the exact solution of the system (1-2) and S_i be an approximation to $u_i = u(x_i)$ obtained by the segment $Q(x)$ passing through the points (x_i, S_i) and (x_{i+1}, S_{i+1}). Each nonpolynomial spline segment $Q(x)$ has the form:

$$Q_i(x) = a_i \cos k(x - x_i) + b_i \sin k(x - x_i) + c_i (x - x_i)^2 + d_i (x - x_i) + e_i$$
$$i = 0,1,2,\ldots,n \tag{4}$$

where a_i, b_i, c_i, d_i and e_i are constants and k is the frequency of the trigonometric functions which will be used to raise the accuracy of the method and Eq. (4) reduces to quartic polynomial spline function in $[a,b]$ when $k \to 0$.

First, we develop expressions for the six coefficients of (4) in terms of S_i, S_{i+1}, D_i, D_{i+1}, F_i, and F_{i+1} where

$$\begin{aligned} &(i) \quad Q_i(x_i) = S_i, && Q_i(x_{i+1}) = S_{i+1}, \\ &(ii) \quad Q_i^{(1)}(x_i) = D_i, && Q_i^{(1)}(x_{i+1}) = D_{i+1}, \\ &(iii) \quad Q_i^{(3)}(x_i) = F_i, && Q_i^{(3)}(x_{i+1}) = F_{i+1}. \end{aligned} \tag{5}$$

We obtain via a straightforward calculation the following expressions:

$$a_i = \frac{h^3}{\theta^3 \sin(\theta)} (F_{i+1} - F_i \cos(\theta)) \qquad b_i = -h^3 \frac{F_i}{\theta^3},$$

$$c_i = \frac{1}{h^2}(S_{i+1} - S_i) - \frac{h(\cos(\theta) - 1)}{\theta^3 \sin(\theta)}(F_{i+1} + F_i) - \frac{D_i}{h} - \frac{h}{\theta^2} F_i,$$

$$d_i = D_i + \frac{h^2}{\theta^2} F_i,\; e_i = S_i - \frac{h^3}{\theta^3 \sin(\theta)}(F_{i+1} - \cos(\theta) F_i) \qquad (6)$$

where $\theta = kh$ and $i = 0,1,\ldots,n-1$.

Using the continuity conditions of the first and second derivatives at the point (x_i, S_i), that is $Q_{i-1}^{(m)}(x_i) = Q_i^{(m)}(x_i)$, *m=1,2*, we get the following relations for $i = 0,1,\ldots,n$

$$D_i + D_{i-1} = \frac{2}{h}(S_i - S_{i-1}) - \frac{h^2}{\theta^2}(F_i + F_{i+1}) - \frac{2h^2}{\theta^3 \sin(\theta)}(F_i + F_{i+1})(\cos(\theta) - 1) \qquad (7)$$

$$D_i - D_{i-1} = \frac{h^2 \tan(\theta)}{\theta} F_i - h^2 \left(\frac{\cos(\theta)-1}{\theta^3 \sin(\theta)}\right)(F_{i-1} - F_{i+1}) - \frac{h^2}{2\theta \sin(\theta)}(F_{i-1} + F_{i+1})$$
$$-\frac{h^2}{\theta^2}(F_i - F_{i-1}) + \frac{1}{h^2}(S_{i-1} - 2S_i + S_{i+1}) \qquad (8)$$

Adding Eqs. (7) and (8), we get

$$D_i = \frac{1}{2h}(S_{i+1} - S_{i-1}) - \frac{h^2}{2\theta^3 \sin(\theta)}(F_{i-1} + 2F_i + F_{i+1})(\cos(\theta) - 1)$$
$$-\frac{h^2}{4\theta \sin(\theta)}(F_{i-1} + F_{i+1}) + h^2\left(\frac{\cot(\theta)}{2\theta} - \frac{1}{\theta^2}\right) F_i,$$
$$i = 0,1,\ldots,n \qquad (9)$$

Similarly, we get D_{i-1} by reducing the indices of Eq. (9) by one, we get:

$$D_{i-1} = \frac{1}{2h}(S_{i-1} - S_{i-2}) - \frac{h^2}{2\theta^3 \sin(\theta)}(F_{i-2} + 2F_{i-1} + F_i)(\cos(\theta) - 1)$$
$$-\frac{h^2}{4\theta \sin(\theta)}(F_{i-2} + F_i) + h^2\left(\frac{\cot(\theta)}{2\theta} - \frac{1}{\theta^2}\right) F_{i-1},$$
$$i = 0,1,\ldots,n \qquad (10)$$

Now from Eqs. (9-10) into Eq. (7), we get

$$-S_{i-2} + 3S_{i-1} - 3S_i + S_{i+1} = h^3(\alpha F_{i-2} + \beta F_{i-1} + \beta F_i + \alpha F_{i+1})$$
$$i = 0,1,\ldots,n-1 \qquad (11)$$

where $F_i = -f_i S_i + g_i$, with $f_i = f(x_i)$ and $g_i = g(x_i)$, and

$$\alpha = \left(\frac{1}{2\theta \sin(\theta)} + \frac{\cos(\theta) - 1}{\theta^3 \sin(\theta)}\right),$$

$$\beta = \left(-\frac{(\cos(\theta) - 1)}{\theta^3 \sin(\theta)} + \frac{(1 - 2\cos(\theta))}{2\theta \sin(\theta)}\right)$$

Eq. (11) gives (n-2) linear algebraic equations in the *(n)* unknowns S_i, $i = 0,1,\ldots,n$. We need two more equations for the direct computation of S_i. These two equations are developed by Taylor series and the method of undetermined coefficients, which are found in [1,4].

$$3S_0 - 4S_1 + S_2 = -2hD_0 + h^3(\rho_0 F_0 + \rho_1 F_1 + \rho_2 F_2),\; i=1 \qquad (12)$$

and

$$3S_{n-2} - 8S_{n-1} - 5S_n = h^3(\sigma_{n-2} F_{n-2} + \sigma_{n-1} F_{n-1} + \sigma_n F_n),\; i=n \qquad (13)$$

where the constants $\rho_i = (\frac{1}{4}, \frac{1}{3}, \frac{1}{12})$ and $\sigma_i = (-\frac{1}{6}, -1, -\frac{5}{2})$.

The local truncation errors t_i, i=1,2,…,n. associated with the scheme (11 - 13) can be obtained as follows:

$$t_i = \begin{cases} \frac{-1}{10} h^5 y_i^{(5)} + O(h^6), i = 1 \\ (1 - 2\alpha - 2\beta) h^3 y_i^{(3)} + (-\frac{1}{2} + \alpha + \beta) h^4 y_i^{(4)} + (\frac{1}{4} - \frac{5}{2}\alpha - \frac{1}{2}\beta) h^5 y_i^{(5)} \\ + (\frac{-1}{12} + \frac{7}{6}\alpha + \frac{1}{6}\beta) h^6 y_i^{(6)} + (\frac{1}{40} - \frac{17}{24}\alpha - \frac{1}{24}\beta) h^7 y_i^{(7)} + O(h^8),\; i = 2,3,\ldots,n-1 \\ \frac{-1}{60} h^5 y_i^{(5)} + O(h^6), i = n. \end{cases} \qquad (14)$$

Remarks

The scheme (11-13) gives rise to a family of methods of different orders as follows:

(1) As $k \to 0$ the scheme (11-13) reduces to Noor [4] scheme based on quartic polynomial spline.

(2) For $\alpha = \frac{1}{12}$ and $\beta = \frac{5}{12}$ then our scheme reduces to Al-Said [1] scheme based on cubic polynomial spline.

III. Spline Solution

The spline solution of (1.1) with the boundary condition (1.2) is based on the linear equations given by (11-13).

Let $U=(u_i)$, $S=(S_i)$, $C=(C_i)$, $T=(t_i)$, $E=(e_i)=Y-S$ be n-dimensional column vectors. Where U, S, T and E are Exact, approximate, truncation error and error n-column vectors respectively. Then we can write the standard matrix equations in the form

$$\text{(i) } MU = C + T,\quad \text{(ii) } MS = C,\quad \text{(iii) } ME = T. \qquad (15)$$

We also have

$$M = M_0 + h^4 GF,\; F = diag(f_i), \qquad (16)$$

where M_0 and G are four-band matrices. The four-band matrix M_0 has the form

$$M_0 = \begin{pmatrix} -4 & 1 & & & & & \\ 3 & -3 & 1 & & & & \\ -1 & 3 & -3 & 1 & & & \\ & & & \ddots & & & \\ & & & 1 & 3 & -3 & 1 \\ & & & & 3 & -8 & 5 \end{pmatrix} \qquad (17)$$

and the four-band matrix B has the form

$$B=\begin{pmatrix} 1/3 & 1/12 & & & & & \\ \beta & \beta & \alpha & & & & \\ \alpha & \beta & \beta & \alpha & & & \\ & & & \ddots & & & \\ & & & \alpha & \beta & \beta & \alpha \\ & & & & -1/8 & -1 & -15/6 \end{pmatrix} \quad (18)$$

For the vector C , we have

$$C_i=\begin{cases} -3A_1-2hA_2+h^3\left(1/4(g_0-f_0A_1)+g_1/3+g_2/12\right), i=1 \\ A_1+h^3\left(\alpha(g_0-f_0A_1+g_3)+\beta(g_1+g_2)\right), i=2 \\ h^3(\alpha(g_{i-2}+g_{i+1})+\beta(g_{i-1}+g_i)), i=3,4,...,n-1 \\ -2hA_3+h^3\left(g_{n-1}/6+g_{n-2}+15g_n/6\right), i=n \end{cases} \quad (19)$$

IV. Convergence Analysis

Our main purpose now is to derive a bound on $\|E\|_\infty$. We now turn back to the error equation (iii) in (15) and rewrite it in the form

$$E=M^{-1}T=(M_0+h^3GF)^{-1}T$$
$$=(I+h^3M_0^{-1}GF)^{-1}M_0^{-1}T$$

it follows from [5] that:

$$\|E\|_\infty \le \frac{\|M_0^{-1}\|_\infty \|T\|_\infty}{1-h^3\|M_0^{-1}\|_\infty \|G\|_\infty \|F\|_\infty} \quad (20)$$

Provided that $h^3\|M_0^{-1}\|_\infty \|G\|_\infty \|F\|_\infty$,

From equation (6) we have:

$$\|T\|_\infty = h^5\left(\frac{1}{4}-\frac{5}{2}\alpha-\frac{1}{2}\beta\right)M_5 , \quad M_5=\max_{a\le x\le b}\left|y^{(5)}(x)\right|.$$

From [4], we get that:

$$\|N_0^{-1}\|_\infty \le Lh^{-3} \quad (21)$$

where L is constant independent h . Then from Eq. (20) we have

$$\|E\|_\infty = \frac{LM_5\left(\frac{1}{4}-\frac{5}{2}\alpha-\frac{1}{2}\beta\right)h^2}{1-L|f|\|G\|_\infty} \le O(h^2) \quad (22)$$

We summarize the above results in the next theorem.

Theorem 4.1 Let $u(x)$ be the exact solution of the continuous boundary value problem (1-2) and let u_i, $i=1,2,...,n$, satisfy the discrete boundary value problem (ii) in (15). Further, if $e_i=y_i-S_i$, then $\|E\|_\infty \cong O(h^2)$ is a second order method, which is given by (22), neglecting all errors due to round off.

V. Numerical Examples and Concluding Remarks

We now consider two numerical examples illustrating the comparative performance of our method (ii) in (15) over other existing methods. All calculations are implemented by MATLAB 6.

Example 1

Consider the following BVPs

$$\begin{aligned} &y^{(3)}+y=(7-x^2)\cos x+(x^2-6x-1)\sin x \\ &y(0)=y^{(1)}(0)+1=y^{(1)}(1)+2\sin 1=0 \end{aligned} \quad (23)$$

whose analytical solution of (23) is

$$y(x)=(x^2-1)\sin x \quad (24)$$

Example 2

The following boundary value problem is considered in [3]

$$\begin{aligned} &y^{(3)}-xy=(x^3-2x^2-5x-3)e^x, \\ &y(0)=y^{(1)}(0)-1=y^{(1)}(1)+e^1=0, \end{aligned} \quad (25)$$

TABLE I
THE OBSERVED MAXIMUM ERRORS FOR EXAMPLE 1

h	Our Method $\beta=\frac{1}{2}-\alpha$			
	$\alpha=0$	$\alpha=1/30$	$\alpha=1/48$	$\alpha=-1/50$
2^{-3}	1.6501-4*	1.3253-3	8.9018-4	8.8839-3
2^{-4}	9.8380-6	3.3831-4	2.1513-4	2.5523-3
2^{-5}	5.8773-7	8.4874-5	5.3267-5	6.5685-4
2^{-6}	3.5687-8	2.1231-5	1.3283-5	1.6529-4
2^{-7}	2.1968-9	5.3085-6	3.3186-6	4.1387-5

* 1.6501-4=1.6501x10^{-4}

TABLE II
THE OBSERVED MAXIMUM ERRORS FOR EXAMPLE 2

h	Our Method $\beta=\frac{1}{2}-\alpha$			
	$\alpha=0$	$\alpha=1/30$	$\alpha=1/48$	$\alpha=-1/50$
2^{-3}	7.7161-5	2.1938-3	1.4001-3	1.6457-2
2^{-4}	6.4065-6	6.7210-4	4.2247-4	5.1879-3
2^{-5}	4.8121-7	1.7788-4	1.1135-4	1.3833-3
2^{-6}	3.3201-8	4.5160-5	2.8237-5	3.5196-4
2^{-7}	2.1839-9	1.1336-5	7.0855-6	8.8398-5

TABLE III
THE OBSERVED MAXIMUM ERRORS FOR EXAMPLE 2 AND THE METHODS IN [4]

h	$\alpha=0$	$\alpha=1/30$	$\alpha=1/48$	$\alpha=-1/50$	Quartic spline [4]
2^{-3}	7.7161-5	2.1938-3	1.4001-3	1.6457-2	2.6783-3
2^{-4}	6.4065-6	6.7210-4	4.2247-4	5.1879-3	8.3597-4
2^{-5}	4.8121-7	1.7788-4	1.1135-4	1.3833-3	2.2207-4
2^{-6}	3.3201-8	4.5160-5	2.8237-5	3.5196-4	5.6432-5
2^{-7}	2.1839-9	1.1336-5	7.0855-6	8.8398-5	1.4168-5

The analytical solution of (25) is:

$$y(x) = x(1-x)e^{x} \ . \qquad (26)$$

The numerical results for our methods are summarized in Tables I-II and compared with other existing polynomial spline methods in Table III.

Tables I-II show the numerical results for a class of methods based on quartic nonpolynomial spline. These tables reveal that our methods have accurate results. While in Table III our methods are better than (in terms of accuracy) other methods (cubic and quartic spline methods).

VI. Conclusion

A class of methods is presented for solving third order two-point boundary value problem using quartic nonpolynomial spline. These methods are shown to be optimal second order, which are better than other methods. The obtained numerical results show that the proposed methods maintain a very remarkable high accuracy which make them are very encouraging for dealing with the solution of two-point boundary value problems.

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