Comparison of Reliability Systems Based Uncertainty

A. Aissani, and H. Benaoudia

***Abstract*—**Stochastic comparison has been an important direction of research in various area. This can be done by the use of the notion of stochastic ordering which gives qualitatitive rather than purely quantitative estimation of the system under study. In this paper we present applications of comparison based uncertainty related to entropy in Reliability analysis, for example to design better systems. These results can be used as a priori information in simulation studies.

***Keywords*—**Uncertainty, Stochastic comparison, Reliability, serie's system, imperfect repair.

I. Introduction

THIS paper describes an approach to the comparison of reliability models. This can be done by using the notion of some partial stochastic ordering (it is a binary relation which is reflexive, transitive and anti-symetric). Indeed it is often easy to make value judgements when such ordering exists. In this paper, we consider partial ordering defined on the set $\Im$ (or its suitable subsets) of all distribution functions real-valued random variables. For example, if X and Y are two variables with their distribution function F and G satisfying $F(x) \leq G(x)$ for every x, then we say that X is stochastically larger than Y (We write $F \geq_{ST} G$ or $X \geq_{ST} Y$). It must be noted that even X and Y have been defined on the same probability space, we can have the anti-symetry holding for F and G without it necessarily being the case that X and Y have are the same random variables. However, stochastic ordering between two distributions, if it holds is more informative than simply comparing their means or dispersions only. Thus the proposed approach leads to more qualitative rather than purely quantitaive estimation of the system under study. For example, such an approach can be used to design better system. Since an agent can find two situations incomparable, then one situation may be better in some stochastic sense but worse in another.

Manuscript received September 15, 2006. This work was supported in part by Ministery of High Education and Scientific Research of Algeria under Grant B 1602/02/2003.

The authors are with the University of Science and Technology Houari Boumediene (USTHB), BP 32 El Alia, Bab-Ez-Zouar, 16111, Algeria (corresponding author A. Aissani: Tel/fax: 213 21 24 71 87.; 213 21 24 76 07; e-mail: amraissani@ yahoo.fr).

In economic theory, this is known as the first-order stochastic dominance and is denoted by $F \geq_{FSD} G$. There is a growth literature on stochastic comparability (or dominance) and various stochastic orders have been introduced , most of theim can be found in the monograph by Shaked & Shanthikumar [6]. Almost all existing partial orders known in the literature are described in terms of ageing properties. In this paper we use a partial order based on uncertainty in the sense of differential entropy. Moreover, we can use the terms more variable, riskier, and more uncertain synonomously, althougt the term more variable is related to some specific variability orders.

In this paper we show conditions under which the reliabilities of two different systems are comparable in terms of their uncertainty. In particular, the reliability of an unknown system can be compared (in terms of their uncertainty) to a more simple one.

The paper is organized as follows. In the following section we describe the "Less Uncertainty" order and some of its properties. In the following section we present notations and preliminary definitions about reliability objects under study. In section 3, we describe the "less uncertainty" order (or LU-order) and some of its properties. In section 4 we give conditions for comparability of series's systems in the LU-order sense. Section 5 concerns similar results for the imperfect repair process. Finally in section 6, we give some simple inequality and bounds.

II. Definitions and Notations

S = the component (or the system) under study: it can be a technical system or a biological system.

X = Non negative random variable representing the failure time of S

$F(t) = P(X \leq t)$ Probability distribution function (PDF) of X

$\overline{F}(t) = P(X > t)$ tail of F interpreted as reliability (or survival) function

$f(t) = F'(t)$ density of F (provided it exists)

$\lambda_F(t) = \dfrac{f(t)}{\overline{F}(t)}$ = failure (or hazard) rate; also called failure risk.

$X_t = X - t$ (given $X > t$) is the residual lifetime of a component of age t

$\overline{F}(x/t) = \dfrac{\overline{F}(t+x)}{\overline{F}(t)}$ =Reliability of X_t

$\mu_F(t) = E(X_t - t / X > t)$ =Mean residual lifetime function.

Y lifetime of another component (or system) with characteristics $G(t), \overline{G}(t), g(t), \lambda_G(t), \mu_G(t)$

III. Comparison Based Uncertainty

It is well known that we can define an uncertainty measure for probability distribution function F via differential entropy [4].

$$H(f) = -\int_0^\infty f(x) Log f(x) dx = -E\big(Log f(X)\big)$$

which is commonly referred to Shannon information measure. The entropy is interpreted as the expected uncertainty contained in $f(x)$ about the predictability of an outcome of X. That is , it measures concentration of probabilities: low entropy distributions are more concentrated, hence more informative than higher ones. In this sense the entropy can be used for qualitative studies. Now in Reliability and Survival Analysis, Insurance, we have some additionnal information about the current age of the component S under study and we must reevalute the uncertainty of the remaining lifetime of S .

The uncertainty of residual lifetime distribution of component S with PDF F can be defined as [4]

$$H(f;t) = -\int_0^\infty \frac{f(x)}{\overline{F}(t)} Log \frac{f(x)}{\overline{F}(t)} dx$$

or

$$H(f;t) = 1 - \frac{1}{\overline{F}(t)} \int_t^\infty f(x) Log f(x) dx$$

which can be seen as a dynamic measure of uncertainty about S. associated to its lifetime distribution.

In other words, $H(f;t)$ measures the expected uncertainty contained in conditional density of the residual lifetime X_t of a component of age t i.e. given $X > t$. In this sense $H(f;t)$ measures the concentration of conditional probability distributions. Note finally that the dynamic entropy of a new component (of age 0) $H(f;0) = H(f)$ is the ordinary Shannon entropy and that the function $H(f;t)$ uniquely determines the reliability function $\overline{F}$ or the PDF F .

We can now define the uncertainty ordering [4]. The non negative random variable X have less uncertainty than Y , we note $X_{LU} Y$, if $H(f;t) \le H(g;t), \forall t \ge 0$. It is some denoted by $X_{LU} Y$.

If X and Y are the lifetimes of two components S and S' and if $X_{LU} Y$, then the expected uncertainty contained in the conditional density of X_t about the predictability of the residual lifetime of the first component S is less than expected uncertainty contained in the the conditional density of Y_t about the remaining lifetime of the second component S' .

Note that the usual stochastic orderings used in the literature can be interpreted in terms of ageing properties and in general there is no relation between these orderings and the above defined uncertainty order. So intuitively speaking, the better system is the system which lives longer and there is less uncertainty about its residual lifetime. This motivates the introduction [5] of several definitions of preference based on ageings and on uncertainty. This aspect is not considered here.

If the function $H(f;t)$ is decreasing (increasing) in $t \ge 0$, then the corresponding PDF F is called DURL (IURL): F has decreasing (increasing) uncertainty of residual life. If a component has a survival PDF belonging to the class DURL, then as the component ages the conditional probability density function becomes more informative. The exponential distribution is the only continuous distribution which is both DURL and IURL.

Example: Consider the survival function [5]

$$\overline{F}(x) = \begin{cases} 1, & if\ 0 \le x \le 2 \\ e^{2-x}, & if\ 2 \le x \le 3 \\ e^{-1}, & if\ 3 \le x \le 4 \\ e^{2-x}, & if\ 4 \le x \end{cases}$$

We can easily compute

$$H(f;t) = \begin{cases} 1 - \dfrac{Log 2}{e}, & if\ 0 \le t \le 2 \\ 1 - \dfrac{Log 2}{e} e^{t-2}, & if\ 2 \le t \le 3 \\ 1 - Log\, 2, & if\ 4 \le t \end{cases}$$

This function is plotted on the Fig. 1 below.

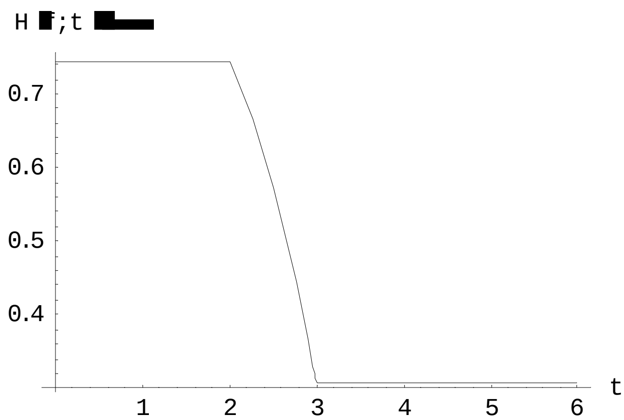

Fig. 1 Dynamic entropy $H(f;t)$ of example 1

It can be proved that indeed F is DURL.

In the sequel, we need the following results [4] which gave criterion of LU-comparability.

Let X and Y be the lifetime of two components (or systems) with distribution functions F and G.

Lemma 1. [4]. Assume that

(i) For any $t \geq 0$,

$$\frac{1}{\overline{F}(t)} Log\lambda_F(x) \geq \frac{1}{\overline{G}(t)} Log\lambda_G(x),\ \forall x \geq t$$

(ii) F is IFR

Then $X \leq_{LU} Y$.

The random variable X is said to be smaller than the random variable Y, we note $X \leq_{FR} Y$, if and only if $\frac{\overline{F}(x)}{\overline{G}(x)}$ decreases over the union of the support of X and Y or equivalently $\lambda_F(x) \geq \lambda_G(x),\ \forall x \geq 0$.

Lemma 2. Let

(i) $\frac{\lambda_F(x)}{\lambda_G(x)}$ be a non-decreasing function in x

(ii) F is DFR

(iii) $X \leq_{FR} Y$

Then $X \leq_{LU} Y$.

Lemma 3. Let $X \leq_{FR} Y$ and let F or G be DFR. Then $X \leq_{LU} Y$.

IV. Comparison of Systems in Series

Recall that a system of n components in series functions if all components function and it fails when the first of theim fails [1,2]. Assume that component $N°i$ has lifetime $X_i, = 1,2,...,n$, then the whole system lifetime is

$$Z = \min(X_1, X_2, ..., X_n)$$

If the components are statistically independent, then it is easy to compute the system reliability

$$\overline{F}_S(t) = \prod_{i=1}^{n} \overline{F}_i(t)$$

where $\overline{F}_i(t)$ is the reliability of the i-th component.

The situation is more complex in the case of lack of knowledge about statistical properties of the component reliabilities. So, we are now interested in the comparison (relatively to the order $\leq_{LU}$) of systems with such serie's configurations.

For $k = 1,2$, let S_k be two independent systems having the same configuration with components in series, the component lifetimes being $X_{i,k}$ and the corresponding failure rates $\lambda_i^{(k)}, i = 1,2,...,n$. As above, we denote by Z_k the lifetime of $S_k, k = 1,2.$

Recall that a distribution function F has an increasing (decreasing) failure rate distribution if $\lambda(t)$ is an increasing (decreasing) function. We say that F is IFR (DFR). The exponential distribution is the only continuous distribution which is both IFR and DFR.

Proposition 1. If all components of the first system are IFR and if for any $t \geq 0$

$$\frac{Log \sum_{i=1}^{n} \lambda_i^{(1)}(x)}{Log \sum_{i=1}^{n} \lambda_i^{(1)}(x)} \geq \exp\left[\frac{\int_0^x \sum_{i=1}^{n} \lambda_i^{(1)}(x)}{\int_0^x \sum_{i=1}^{n} \lambda_i^{(1)}(x)}\right], \forall x \geq t \qquad (1)$$

Then the lifetime Z_1 of the first system has less uncertainty than the lifetime Z_2 of the second one i.e. $Z_1 \leq_{LU} Z_2$.

Proof. Since the systems are in series, it follows that

$$Z_k = \min\left(X_{1,k}, X_{2,k}, ..., X_{n,k}\right),\ k = 1,2$$

It is well known that the reliability is related to the failure rate as follow

$$\overline{F}_i^{(k)}(x) = \exp\left(-\int_0^x \lambda_i^{(k)}(x)dx\right),\ i = 1,2,...,n,$$ then the inequality (1) insure that conditions of lemma 1 holds.

Proposition 2. Assume that all components of the first system are DFR and that the ratio

$$\frac{\sum_{i=1}^{n} \lambda_i^{(1)}(x)}{\sum_{i=1}^{n} \lambda_i^{(1)}(x)}$$

is non-decreasing in x. Then the lifetime of the first system has less uncertainty than the lifetime of the second one i.e. $Z_1 \leq_{LU} Z_2$.

Proof. Since all components of the first system are comparable in failure rate ordering and since this order is closed under the operation of taking minimum [6]. Then $Z_1 \leq_{FR} Z_2$. Next, the lifetime of the first system is DFR since this class is stable under minimum operation. Our assertion follows now from lemma2.

V. IMPERFECT REPAIR

The model considered here is related to the well-known model of "minimal repair at failure" [3] which has been considered in renewall Theory and reliability theory. It has been shown that this process is of increasing uncertainty according to our order of uncertainty. Here, we compare two such imperfect repair processes according to the ordering based uncertainty.

Assume that a system (or an item) is repaired at failure. With probability p it is returned to the as-good-as-new state (perfect repair), with probability $q=1-p$ it is returned to the functionning state, but it is only as good as a system (item) of age aqual to its age at failure (imperfect repair). Thus if F is the failure-time distribution until a perfect repair, then the failure-time distribution following imperfect repair for a system which fails at age s is given by

$$P(X_S > t) = \overline{F}_t(s) = \frac{\overline{F}(t+s)}{\overline{F}(t)}$$

We assume that the repair time is negligible. A perfect repair is assumed to take place at time 0. For $p=1$ we obtain the ordinary renewal process with underlying distribution F. The case $p=0$ corresponds to a non-homogenous Poisson process in which the intensity function equal the failure rate function $\lambda_F(t)$ of F. The choice $0<p<1$ leads to a non trivial process in which the time epochs of perfect repairs are regeneration points. The interval between successive regeneration points is the waiting time for a perfect repair starting with a new system.

Let F_p denote the waiting-time distribution for a perfect repair starting with a new component and λ_p denote its failure rate function. Then it can be shown [4] that (i) $\lambda_p(t) = p\lambda(t)$; (ii) $\overline{F}_p(t) = \overline{F}^{\,p}(t)$.

Consider now another scenario in which the waiting-time distribution for a perfect repair is defined by the probability q rather than p. Denote by $\overline{F}_q(t) = \overline{F}^{\,q}(t)$ and $\lambda_q(t) = q\lambda(t)$ the corresponding survival distribution and failure rate function.

Proposition 3. Consider two imperfect processes with parameters p and q with the same underlying distribution F. If

(i) $q \le p$;
(ii) F is IFR

Then the waiting time for a perfect repair p, has less uncertainty than with a probability q i.e. $X_p \le_{LU} X_q$.

Proof . It is easy to see that the first condition (i) of lemma 2 holds since the ratio $\frac{\lambda_p(x)}{\lambda_q(x)} = \frac{p}{q}$ is constant thus non-decreasing function in x. The condition (i) above insure that $X_p \le_{FR} X_q$, and the conditions of lemma 2 are satisfied.

Proposition 4. Under the condition of proposition 3, and if

(i) F is IFR;

(ii) $\forall t \ge 0$, $\frac{Log\ p + Log\lambda(x)}{Log\ q + Log\lambda(x)} \ge \frac{\overline{F}^{(p)}(x)}{\overline{F}^{(q)}(x)}$ for all $x \ge t$.

Then $X_p \le_{LU} X_q$.

These conditions are given by lemma 1.

VI. SOME USEFUL BOUNDS

Based on the above discussion, we can derive some useful conclusions about quantitative and/or qualitative properties of several stochastic models.

Proposition 5. Mixture of exponential distributions is of increasing uncertainty.

Proof. Indeed , let X_i, $i=1,2,...,n$ be independent random variables exponentially distributed at rate $\lambda_i, i=1,2,...,n$ and consider the mixture W of $X_1, X_2,..., X_n$ with density $f_W(x) = \sum_{i=1}^{n} \alpha_i f_i(x), 0 \le \alpha_i \le 1$. From [2], the random variable W with such density is DFR, thus IURL[5].

Such a bound is interesting for the estimation of ruin probability in insurance and waiting time process in Queueing theory. This is the subject of further studies.

Proposition 6. If F is DFR, then the underlying random variable X_p corresponding to the imperfect repair is such that $X \le_{LU} X_p$.

Proposition 7. If $\overline{F}_p$ is DURL, then the uncertainty of residual lifetime of the imperfect repair is bounded above by

$$H_p(t) \le 1 - Log\ p - Log\ \lambda(t)$$

This is a convenient and simple bound which comes from a similar result of [5].

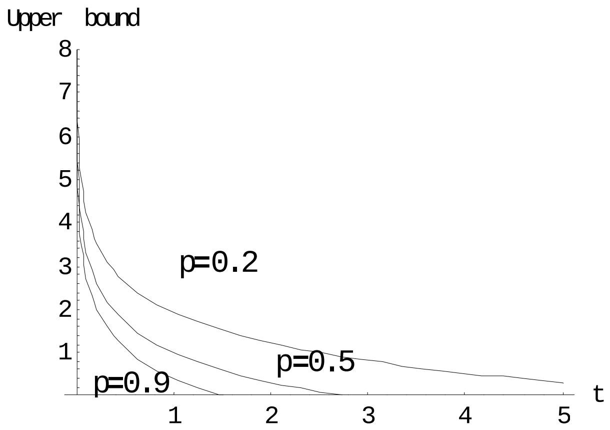

Fig. 2 Upper bound on entropy for $p = 0.2; 0.5$ and 0.9

The above Fig. 2 shows the evolution of the upper bound on $H_p(t)$ for different values of $p = 0.2$, $p = 0.5$, $p = 0.9$ when the failure rate is fixed.

REFERENCES

[1] A. Aissani,, *Modèles Stochastiques de la Théorie de Fiabilité*. Algiers, OPU, 1992 (in french).

[2] R.E. Barlow, F. Proshan, *Statistical Theory of Reliability and Life Testing*, Probability Models, New York, Holt, Rinehart and Winston, 1975.

[3] M. Brown, F. Proshan, “Imperfect Repair”, J. Applied Probability 20, 851-859, 1983.

[4] N. Ebrahimi, F. Pellery, “New Partial Ordering of Survival Function Based on the Notion of Uncertainty,” J. Appl. Probability 32, 202-211.

[5] N. Ebrahimi, “How to measure uncertainty in the residual Life Time Distribution”, Sankhya, The Indian Journal of Statistics 58 (A), 48-56.

[6] M. Shaked, J.G. Shanthikumar, *Stochastic Orders and their Applications*, Boston, Academic Press, 1994.

[7] Terje Aven, “Reliability Analysi as a tool for expressing and communicating uncertainty,” in: Recent Advances in Reliability Theory, Methodologies, Practice and Inference, Series Statistic for Industry and Technology, eds: Limnios N., M. Nikulin, Birkhauser, Boston, 2000.