

# New Classes of Salagean type Meromorphic Harmonic Functions

Hakan Bostancı, and Metin Öztürk

**Abstract**—In this paper, a necessary and sufficient coefficient are given for functions in a class of complex valued meromorphic harmonic univalent functions of the form  $f = h + \bar{g}$  using Salagean operator. Furthermore, distortion theorems, extreme points, convolution condition and convex combinations for this family of meromorphic harmonic functions are obtained.

**Keywords**—Harmonic mappings, Meromorphic functions, Salagean operator.

## I. INTRODUCTION

A continuous function  $f = u + iv$  is a complex valued harmonic function in a complex domain  $D$  if both  $u$  and  $v$  are real harmonic in  $D$ . In any simply connected domain  $D \subset \mathbb{C}$  we can write  $f = h + \bar{g}$ , where  $h$  and  $g$  are analytic in  $D$ . A necessary and sufficient condition for  $f$  to be locally univalent and sense preserving in  $D$  is that  $|h'(z)| > |g'(z)|$  in  $D$  (see [2]). In [3], Hengartner and Schober investigated functions harmonic in the exterior of the unit disc  $\tilde{U} = \{z : |z| > 1\}$ . They showed that complex valued, harmonic, sense preserving, univalent mapping  $f$  must admits the representation

$$f(z) = h(z) + \overline{g(z)} + A \log |z|,$$

where

$$h(z) = \alpha z + \sum_{k=1}^{\infty} a_k z^{-k}$$

and

$$g(z) = \beta \bar{z} + \sum_{k=1}^{\infty} b_k z^{-k}$$

for  $0 \leq |\beta| < |\alpha|$ ,  $A \in \mathbb{C}$ .

Let  $MH$  denote the class of functions

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k} \quad (1)$$

which are harmonic in the punctured unit disk  $U \setminus \{0\}$ .  $h(z)$  and  $g(z)$  are analytic in  $U \setminus \{0\}$  and  $U$ , respectively, and  $h(z)$  has a simple pole at the origin with residue 1 here.

For  $f = h + \bar{g}$  given by (1), Jahangiri [4] defined the modified Salagean operator of  $f$  as

$$D^n f(z) = D^n h(z) + (-1)^n \overline{D^n g(z)}; \quad n = 0, 1, 2, \dots, \quad (2)$$

where

$$D^n h(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k$$

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and

$$D^n g(z) = \sum_{k=1}^{\infty} k^n b_k z^k.$$

A function  $f(z) \in MH$  is said to be in the subclass  $MHS^*$  of meromorphically harmonic starlike in  $U \setminus \{0\}$  if it satisfies the condition

$$Re \left\{ -\frac{zh'(z) - \overline{zg'(z)}}{h(z) + \overline{g(z)}} \right\} > 0, \quad z \in U \setminus \{0\}.$$

Now we define a new class  $MHS_S^*(n, \alpha)$  (see [1]).

**Definition 1.1:** For  $0 \leq \alpha < 1$ , we let  $MHS_S^*(n, \alpha)$  denote the class of meromorphic harmonic functions  $f$  of the form (1) such that

$$Re \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\} > \alpha, \quad z \in U \setminus \{0\}. \quad (3)$$

We let the subclass  $\overline{MHS_S^*}(n, \alpha)$  consist of meromorphic harmonic functions  $f_n = h_n + \bar{g}_n$  in  $MHS_S^*(n, \alpha)$  so that  $h_n$  and  $g_n$  are of the form

$$h_n(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k \quad (4)$$

and

$$g_n(z) = (-1)^n \sum_{k=1}^{\infty} b_k z^k, \quad (5)$$

where  $a_k \geq 0, b_k \geq 0$ .

In this paper, we have obtained the coefficient conditions for the classes  $MHS_S^*(n, \alpha)$  and  $\overline{MHS_S^*}(n, \alpha)$ . Further a representation theorem, inclusion properties and distortion bound for the class  $\overline{MHS_S^*}(n, \alpha)$  are established.

## II. MAIN RESULTS

**Theorem 2.1:** Let  $f$  be of the form (1). If

$$\sum_{k=1}^{\infty} [(|a_{2k}| + |b_{2k}|)(2k)^{n+1} + ((2k-1+\alpha)|a_{2k-1}| \quad (6)$$

$$+ (2k-1-\alpha)|b_{2k-1}|)(2k-1)^n] \leq 1 - \alpha,$$

then  $f$  is harmonic univalent, sense preserving in  $U \setminus \{0\}$  and  $f \in MHS_S^*(n, \alpha)$ .

*Proof:* For  $0 < |z_1| \leq |z_2| < 1$  we have

$$\begin{aligned} & |f(z_1) - f(z_2)| \\ & \geq |h(z_1) - h(z_2)| - |g(z_1) - g(z_2)| \\ & \geq \frac{|z_1 - z_2|}{|z_1||z_2|} \\ & \quad - |z_1 - z_2| \sum_{k=1}^{\infty} (|a_k| + |b_k|) |z_1^{k-1} + \dots + z_2^{k-1}| \\ & > \frac{|z_1 - z_2|}{|z_1||z_2|} \left[ 1 - |z_2|^2 \sum_{k=1}^{\infty} k(|a_k| + |b_k|) \right] \\ & = \frac{|z_1 - z_2|}{|z_1||z_2|} \left[ 1 - |z_2|^2 \left( \sum_{k=1}^{\infty} 2k(|a_{2k}| + |b_{2k}|) \right. \right. \\ & \quad \left. \left. + \sum_{k=1}^{\infty} (2k-1)(|a_{2k-1}| + |b_{2k-1}|) \right) \right] \\ & > \frac{|z_1 - z_2|}{|z_1||z_2|} \left[ 1 - \sum_{k=1}^{\infty} (2k)^{n+1} (|a_{2k}| + |b_{2k}|) \right. \\ & \quad - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1+\alpha)|a_{2k-1}|] \\ & \quad \left. - \sum_{k=1}^{\infty} (2k-1)^n [(2k-1-\alpha)|b_{2k-1}|] \right]. \end{aligned}$$

This last expression is non negative by (6) and so  $f$  is univalent in  $U \setminus \{0\}$ . To show that  $f$  is sense preserving in  $U \setminus \{0\}$ , we need to show that  $|h'(z)| \geq |g'(z)|$  in  $U \setminus \{0\}$ . We have

$$\begin{aligned} |h'(z)| & \geq 1 - \sum_{k=1}^{\infty} k|a_k||z|^{k-1} \\ & = 1 - \sum_{k=1}^{\infty} k|a_k|r^{k-1} > 1 - \sum_{k=1}^{\infty} k|a_k| \\ & \geq 1 - \sum_{k=1}^{\infty} (2k)^{n+1}|a_{2k}| \\ & \quad - \sum_{k=1}^{\infty} (2k-1)^n(2k-1+\alpha)|a_{2k-1}| \\ & \geq \sum_{k=1}^{\infty} (2k)^{n+1}|b_{2k}| \\ & \quad + \sum_{k=1}^{\infty} (2k-1)^n(2k-1-\alpha)|b_{2k-1}| \\ & \geq \sum_{k=1}^{\infty} 2k|b_{2k}| + \sum_{k=1}^{\infty} (2k-1)|b_{2k-1}| \\ & > \sum_{k=1}^{\infty} k|b_k|r^{k-1} = \sum_{k=1}^{\infty} k|b_k||z|^{k-1} \geq |g'(z)|. \end{aligned}$$

Now, we will show that  $f \in MHS_S^*(n, \alpha)$ . According to (2) and (3), for  $0 \leq \alpha < 1$ , we have

$$Re \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\}$$

$$= Re \left\{ -\frac{2D^{n+1}h(z) - 2(-1)^n \overline{D^{n+1}g(z)}}{T^n(z)} \right\} \geq \alpha,$$

where

$$T^n(z) = D^n h(z) + (-1)^n \overline{D^n g(z)} - D^n h(-z) - (-1)^n \overline{D^n g(-z)}.$$

Using the fact that  $Re\{w\} \geq \alpha$  if and only if  $|1 - \alpha + w| \geq |1 + \alpha - w|$ , it suffices to show that

$$\begin{aligned} & \left| 1 - \alpha - \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right| \\ & \geq \left| 1 + \alpha + \frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right| \end{aligned}$$

which is equivalent to

$$\begin{aligned} & |2D^{n+1}f(z) - (1-\alpha)(D^n f(z) - D^n f(-z))| \\ & - |2D^{n+1}f(z) + (1+\alpha)(D^n f(z) - D^n f(-z))| \geq 0. \quad (7) \end{aligned}$$

Substituting for  $D^n f(z)$  and  $D^{n+1}f(z)$  in (7) yields

$$\begin{aligned} & \left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^{n+1} a_k z^k + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k \right. \\ & \quad \left. + (1-\alpha) \left[ \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} k^n a_k z^k \right. \right. \\ & \quad \left. \left. + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k \right. \right. \\ & \quad \left. \left. - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\ & - \left| \frac{2(-1)^n}{z} - 2 \sum_{k=1}^{\infty} k^{n+1} a_k z^k \right. \\ & \quad \left. + 2(-1)^n \sum_{k=1}^{\infty} k^{n+1} \bar{b}_k \bar{z}^k - (1+\alpha) \left[ \frac{(-1)^n}{z} \right. \right. \\ & \quad \left. \left. + \sum_{k=1}^{\infty} k^n a_k z^k + (-1)^n \sum_{k=1}^{\infty} k^n \bar{b}_k \bar{z}^k + \frac{(-1)^n}{z} \right. \right. \\ & \quad \left. \left. - \sum_{k=1}^{\infty} (-1)^k k^n a_k z^k - (-1)^n \sum_{k=1}^{\infty} (-1)^k k^n \bar{b}_k \bar{z}^k \right] \right| \\ & = \left| \frac{2(2-\alpha)(-1)^n}{z} \right. \\ & \quad \left. - \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k(1-\alpha)) k^n a_k z^k \right. \\ & \quad \left. + (-1)^n \sum_{k=1}^{\infty} (2k + (1-\alpha) - (-1)^k(1-\alpha)) k^n \bar{b}_k \bar{z}^k \right| \\ & - \left| \frac{2\alpha(-1)^n}{z} + \sum_{k=1}^{\infty} (2k + (1+\alpha) - (-1)^k(1+\alpha)) k^n a_k z^k \right. \\ & \quad \left. - (-1)^n \sum_{k=1}^{\infty} (2k - (1-\alpha) + (-1)^k(1+\alpha)) k^n \bar{b}_k \bar{z}^k \right| \\ & = \left| \frac{2(2-\alpha)(-1)^n}{z} \right. \end{aligned}$$

$$\begin{aligned}
 & -2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n a_{2k-1} z^{2k-1} \\
 & -2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k} + 2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} \bar{b}_{2k} \bar{z}^{2k} \\
 & + 2(-1)^n \sum_{k=1}^{\infty} (2k-\alpha)(2k-1)^n \bar{b}_{2k-1} \bar{z}^{2k-1} \Big| \\
 & - \left| \frac{2\alpha(-1)^n}{z} + 2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k} \right. \\
 & + 2 \sum_{k=1}^{\infty} (2k+\alpha)(2k-1)^n a_{2k-1} z^{2k-1} \\
 & - 2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} \bar{b}_{2k} \bar{z}^{2k} \\
 & \left. - 2(-1)^n \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n \bar{b}_{2k-1} \bar{z}^{2k-1} \right| \\
 \geq & \frac{2(2-\alpha)(-1)^n}{z} \\
 & - 2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n |a_{2k-1}| |z|^{2k-1} \\
 & - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |b_{2k}| |z|^{2k} \\
 & - 2 \sum_{k=1}^{\infty} (2k-\alpha)(2k-1)^n |b_{2k-1}| |z|^{2k-1} \\
 & - \frac{2\alpha(-1)^n}{z} - 2 \sum_{k=1}^{\infty} (2k+\alpha)(2k-1)^n |a_{2k-1}| |z|^{2k-1} \\
 & - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |a_{2k}| |z|^{2k} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} |b_{2k}| |z|^{2k} \\
 & - 2 \sum_{k=1}^{\infty} (2k-2+\alpha)(2k-1)^n |b_{2k-1}| |z|^{2k-1} \\
 = & \frac{4(1-\alpha)}{z} \left[ 1 - \sum_{k=1}^{\infty} \left\{ \frac{(2k)^{n+1}}{1-\alpha} (|a_{2k}| + |b_{2k}|) \right. \right. \\
 & \left. \left. + \frac{(2k-1)^n}{1-\alpha} [(2k-1+\alpha)|a_{2k-1}| \right. \right. \\
 & \left. \left. + (2k-1-\alpha)|b_{2k-1}|] \right\} \right].
 \end{aligned}$$

This last expression is non-negative by (7), and so the proof is complete. ■

**Theorem 2.2:** Let  $f_n = h_n + \bar{g}_n$  where  $h_n$  and  $g_n$  are of the form (4) and (5). Then  $f_n \in \overline{MHS}_S^*(n, \alpha)$ , if and only if

$$\begin{aligned}
 & \sum_{k=1}^{\infty} [(a_{2k} + b_{2k})(2k)^{n+1} + ((2k-1+\alpha)a_{2k-1} \\
 & + (2k-1-\alpha)b_{2k-1})(2k-1)^n] \leq 1-\alpha. \quad (8)
 \end{aligned}$$

*Proof:* Since  $\overline{MHS}_S^*(n, \alpha) \subset MHS_S^*(n, \alpha)$ , we only need to prove the (only if) part of the theorem. To this end,

for functions  $f_n = h_n + \bar{g}_n$ , we notice that condition

$$\operatorname{Re} \left\{ -\frac{2D^{n+1}f(z)}{D^n f(z) - D^n f(-z)} \right\} \geq \alpha, \quad z \in U \setminus \{0\},$$

is equivalent to

$$\operatorname{Re} \left\{ \frac{\frac{2(1-\alpha)}{z} - \sum_{k=1}^{\infty} (2k+\alpha - (-1)^k \alpha) k^n a_k z^k}{\phi(z)} \right. \\
 \left. + \frac{(-1)^n \sum_{k=1}^{\infty} (2k-\alpha + (-1)^k \alpha) k^n b_k \bar{z}^k}{\phi(z)} \right\} \geq 0$$

which implies

$$\operatorname{Re} \left\{ \frac{\frac{2(1-\alpha)}{z} - 2 \sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} z^{2k}}{\phi(z)} \right. \\
 \left. - \frac{2 \sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n a_{2k-1} z^{2k-1}}{\phi(z)} \right. \\
 \left. - \frac{2(-1)^n \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n b_{2k-1} \bar{z}^{2k-1}}{\phi(z)} \right. \\
 \left. - \frac{2(-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} b_{2k} z^{2k}}{\phi(z)} \right\} \geq 0, \quad (9)$$

where

$$\begin{aligned}
 \phi(z) = & \frac{2}{z} + 2 \sum_{k=1}^{\infty} (2k-1)^n a_{2k-1} z^{2k-1} \\
 & + 2 \sum_{k=1}^{\infty} (2k-1)^n b_{2k-1} \bar{z}^{2k-1}.
 \end{aligned}$$

The condition (9) must hold for all  $z$  in  $U \setminus \{0\}$ . By choosing  $0 < z = r < 1$ , from the left hand (9), we have

$$\begin{aligned}
 & \frac{1-\alpha - \sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n a_{2k-1} r^{2k}}{r\phi(r)/2} \\
 & - \frac{\sum_{k=1}^{\infty} (2k)^{n+1} a_{2k} r^{2k+1} + (-1)^n \sum_{k=1}^{\infty} (2k)^{n+1} b_{2k} r^{2k+1}}{r\phi(r)/2} \\
 & - \frac{(-1)^n \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n b_{2k-1} r^{2k}}{r\phi(r)/2}. \quad (10)
 \end{aligned}$$

If the condition (8) does not hold, then the number in (10) is negative for  $r$  sufficiently close to 1. Hence there exist

$z_0 = r_0$  in  $(0, 1)$ , for which the eqnarray in (10) is negative. This contradicts the required condition for and so the proof is complete. It is easily seen that  $f_n(z) \in \overline{MHS}_S^*(n, \alpha)$ . Thus we complete the of the Theorem 2.2. ■

**Theorem 2.3:** If  $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$  for  $0 < |z| = r < 1$ , then

$$\frac{1}{r} - \frac{1-\alpha}{2^{n+1}}r \leq |f_n(z)| \leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}}r.$$

*Proof:* Let  $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$ . Taking the absolute value of  $f$  we obtain

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\leq \frac{1}{r} + \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\leq \frac{1}{r} + \frac{1-\alpha}{2^{n+1}} r \end{aligned}$$

and

$$\begin{aligned} |f_n(z)| &= \left| \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_k z^k + (-1)^n \sum_{k=1}^{\infty} \overline{b_k z^k} \right| \\ &\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r^k \\ &\geq \frac{1}{r} - \sum_{k=1}^{\infty} (a_k + b_k) r \\ &\geq \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} \sum_{k=1}^{\infty} \frac{2^{n+1}}{1-\alpha} (|a_k| + |b_k|) r \\ &\geq \frac{1}{r} - \frac{1-\alpha}{2^{n+1}} r. \end{aligned}$$

■

**Corollary 2.4:** Let  $A = \left\{ w : |w| < \frac{2^{n+1}-1+\alpha}{2^{n+1}} \right\}$ .

If  $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$ , then

$$f_n(U) \subset A^t.$$

**Theorem 2.5:**  $f_n = h_n + \bar{g}_n \in \overline{MHS}_S^*(n, \alpha)$  if and only if  $f_n$  can be expressed as

$$f_n(z) = \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}), \quad (11)$$

where for  $k = 1, 2, \dots$

$$\begin{aligned} h_{n_0}(z) &= g_{n_0}(z) = \frac{1}{z} \\ h_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} z^{2k-1} \\ h_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ g_{n_{2k-1}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} \bar{z}^{2k-1} \\ g_{n_{2k}}(z) &= \frac{1}{z} + \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \end{aligned}$$

and

$$\sum_{k=0}^{\infty} (x_k + y_k) = 1, \quad x_k \geq 0 \text{ and } y_k \geq 0.$$

In particular, the extreme point of  $\overline{MHS}_S^*(n, \alpha)$  are  $\{h_{n_k}\}$  and  $\{g_{n_k}\}$ .

*Proof:* For functions  $f_n = h_n + \bar{g}_n$ , where  $h_n$  and  $g_n$  of the form (4) and (5), we have

$$\begin{aligned} f_n(z) &= \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k}) \\ &= x_0 h_{n_0} + y_0 g_{n_0} + \sum_{k=1}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} x_{2k-1} \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} x_{2k} \frac{1-\alpha}{(2k)^{n+1}} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} y_{2k-1} \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} y_{2k} \frac{1-\alpha}{(2k)^{n+1}} \bar{z}^{2k} \\ &= \sum_{k=0}^{\infty} (x_k + y_k) \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} x_{2k-1} z^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} x_{2k} z^{2k} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} y_{2k-1} \bar{z}^{2k-1} \\ &\quad + \sum_{k=1}^{\infty} \frac{1-\alpha}{(2k)^{n+1}} y_{2k} \bar{z}^{2k}. \end{aligned}$$

Then,

$$\begin{aligned} &\sum_{k=1}^{\infty} (2k-1+\alpha)(2k-1)^n \\ &\times \left\{ \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} x_{2k-1} \right\} \end{aligned}$$

$$\begin{aligned}
 & + \sum_{k=1}^{\infty} (2k-1-\alpha)(2k-1)^n \\
 & \times \left\{ \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} y_{2k-1} \right\} \\
 & + \sum_{k=1}^{\infty} (2k)^{n+1} \left\{ \frac{1-\alpha}{(2k)^{n+1}} (x_{2k} + y_{2k}) \right\} \\
 & = (1-\alpha) \sum_{k=1}^{\infty} (x_k + y_k) \\
 & = (1-\alpha)[1 - (x_0 + y_0)] \leq 1 - \alpha.
 \end{aligned}$$

So  $f_n \in \overline{MHS}_S^*(n, \alpha)$ .

Conversely, suppose that  $f_n \in \overline{MHS}_S^*(n, \alpha)$ .

Let, for  $k = 1, 2, \dots$

$$\begin{aligned}
 x_{2k} &= \frac{1-\alpha}{(2k)^{n+1}} a_{2k}, y_{2k} = \frac{1-\alpha}{(2k)^{n+1}} b_{2k} \\
 x_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1+\alpha)} a_{2k-1} \\
 y_{2k-1} &= \frac{1-\alpha}{(2k-1)^n(2k-1-\alpha)} b_{2k-1}.
 \end{aligned}$$

Then note that by the  $\overline{MHS}_S^*(n, \alpha)$ ,  $0 \leq x_k \leq 1$  and  $0 \leq y_k \leq 1$  ( $k = 1, 2, \dots$ ). We define  $0 \leq x_0 \leq 1$  and

$$y_0 = 1 - x_0 - \sum_{k=1}^{\infty} (x_k + y_k).$$

Consequently, we obtain

$$f(z) = \sum_{k=0}^{\infty} (x_k h_{n_k} + y_k g_{n_k})$$

as required. ■

**Theorem 2.6:** If  $f \in MHS_S^*(n, \alpha)$ , then the diameter  $D_f$  of  $\mathcal{C} \setminus f(U)$  satisfies

$$D_f \geq 2|1 + b_1|.$$

*Proof:* Let  $D_f(R)$  be diameter of  $f(|z| = R)$ ,  $0 < R < 1$ , and let  $D_f^*(R) = \max_{|z|=R} |f(z) - f(-z)|$ . Then  $D_f(R) \rightarrow D_f$  as  $R \rightarrow 1$  and  $D_f(R) \geq D_f^*(R)$ . Since

$$\begin{aligned}
 [D_f^*(R)]^2 &\geq \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta}) - f(-Re^{i\theta})|^2 d\theta \\
 &= 4 \left[ \frac{1}{R^2} + b_1 + \bar{b}_1 \right. \\
 &\quad \left. + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2) R^{2(2n-1)} \right] \\
 &\geq 4[1 + 2Reb_1 \\
 &\quad + \sum_{k=1}^{\infty} (|a_{2k-1}|^2 + |b_{2k-1}|^2)]
 \end{aligned}$$

we conclude that  $D_f \geq 2\sqrt{|1 + b_1|^2}$ . ■

Note that if  $f$  and  $F$  are

$$f(z) = h(z) + \overline{g(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}$$

and

$$F(z) = H(z) + \overline{G(z)} = \frac{1}{z} + \sum_{k=1}^{\infty} A_k z^k + \overline{\sum_{k=1}^{\infty} B_k z^k},$$

then the convolution (or Hadamard product) of  $f$  and  $F$  is defined to be the function

$$(f * F)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k A_k z^k + \overline{\sum_{k=1}^{\infty} b_k B_k z^k}. \quad (12)$$

**Theorem 2.7:** For  $0 \leq \beta \leq \alpha < 1$ , let  $f_n \in \overline{MHS}_S^*(n, \alpha)$  and  $F_n \in \overline{MHS}_S^*(n, \beta)$ . Then the convolution function  $f_n * F_n \in \overline{MHS}_S^*(n, \alpha) \subset \overline{MHS}_S^*(n, \beta)$ .

*Proof:* For  $f_n$  and  $F_n$  as Theorem 2.7. Then the convolution  $f_n * F_n$  is given by (12). We wish to show that the coefficients of  $f_n * F_n$  satisfy the required condition given in Theorem 2.2. For  $F_n \in \overline{MHS}_S^*(n, \beta)$ , we note that  $|A_k| \leq 1$  and  $|B_k| \leq 1$ . Since  $0 \leq \beta \leq \alpha < 1$  and  $f_n \in \overline{MHS}_S^*(n, \alpha)$  for  $f_n * F_n$ , we obtain

$$\begin{aligned}
 & \sum_{k=1}^{\infty} \left[ (a_{2k} A_{2k} + b_{2k} B_{2k}) \frac{(2k)^{n+1}}{1-\beta} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} A_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1} B_{2k-1}) \frac{(2k-1)^n}{1-\beta} \right] \\
 & \leq \sum_{k=1}^{\infty} \left[ (a_{2k} + b_{2k}) \frac{(2k)^{n+1}}{1-\beta} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1}) \frac{(2k-1)^n}{1-\beta} \right] \\
 & \leq \sum_{k=1}^{\infty} \left[ (a_{2k} + b_{2k}) \frac{(2k)^{n+1}}{1-\alpha} \right. \\
 & \quad \left. + ((2k-1+\alpha)a_{2k-1} \right. \\
 & \quad \left. + (2k-1-\alpha)b_{2k-1}) \frac{(2k-1)^n}{1-\alpha} \right] \\
 & \leq 1.
 \end{aligned}$$

Therefore  $f_n * F_n \in \overline{MHS}_S^*(n, \alpha) \subset \overline{MHS}_S^*(n, \beta)$ . ■

**Theorem 2.8:** The class  $\overline{MHS}_S^*(n, \alpha)$  is closed under convex combination.

*Proof:* Suppose that  $f_{n_i}(z) \in \overline{MHS}_S^*(n, \alpha)$  for  $i = 1, 2, 3, \dots$ , where  $f_{n_i}$  is given by

$$f_{n_i}(z) = \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} a_{i_k} z^k + \overline{(-1)^n \sum_{k=1}^{\infty} b_{i_k} z^k}.$$

Then by Theorem 2.2.

$$\sum_{k=1}^{\infty} [(a_{i_{2k}} + b_{i_{2k}})(2k)^{n+1} + ((2k-1+\alpha)a_{i_{2k-1}})] \quad (13)$$

$$+(2k-1-\alpha)b_{i_{2k-1}})(2k-1)^n] \leq 1-\alpha.$$

For

$$\sum_{i=1}^{\infty} t_i = 1, 0 \leq t_i \leq 1,$$

the convex combinations of  $f_{n_i}$  may be written as

$$\begin{aligned} \sum_{i=1}^{\infty} t_i f_{n_i}(z) &= \frac{(-1)^n}{z} + \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i a_{i_k} \right) z^{-n} \\ &\quad + (-1)^n \sum_{k=1}^{\infty} \left( \sum_{i=1}^{\infty} t_i b_{i_k} \right) z^{-n}. \end{aligned}$$

Then by (13),

$$\begin{aligned} &\sum_{k=1}^{\infty} \left[ \frac{(2k)^{n+1}}{1-\alpha} \sum_{i=1}^{\infty} t_i (a_{i_{2k}} + b_{i_{2k}}) \right. \\ &\quad + \frac{(2k-1)^n}{1-\alpha} \left( \frac{(2k-1+\alpha)}{1-\alpha} \sum_{i=1}^{\infty} t_i a_{i_{2k-1}} \right. \\ &\quad \left. \left. + \frac{(2k-1-\alpha)}{1-\alpha} \sum_{i=1}^{\infty} t_i b_{i_{2k-1}} \right) \right] \\ &\leq \sum_{i=1}^{\infty} t_i = 1. \end{aligned}$$

Thus

$$\sum_{i=1}^{\infty} t_i f_{n_i}(z) \in \overline{MHS}_S^*(n, \alpha).$$

■

**Theorem 2.9:** If  $f_n \in \overline{MHS}_S^*(n, \alpha)$ , then

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) \leq 1 + 2Re\{b_1\}.$$

Equality occurs if and only if  $C \setminus f(U)$  has area zero.

*Proof:* The area of the omitted set is

$$\begin{aligned} &\lim_{R \rightarrow 1} \lim_{2i} \frac{1}{2i} \int_{0 < |z|=R < 1} \bar{f} df \\ &= \lim_{R \rightarrow 1} \left[ \frac{1}{2i} \int_{0 < |z|=R < 1} \bar{h} h' dz + \frac{1}{2i} \int_{0 < |z|=R < 1} g \bar{g}' d\bar{z} \right. \\ &\quad \left. + \frac{1}{2i} \int_{0 < |z|=R < 1} g h' dz + \frac{1}{2i} \int_{0 < |z|=R < 1} \bar{h} \bar{g}' d\bar{z} \right] \\ &= \pi \left[ \sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) R^{2k} - \frac{1}{R^2} - 2Re\{b_1\} \right]. \end{aligned}$$

For  $0 < r < 1$  the curve  $\Gamma_r = f(C_r)$  is a simple closed curve oriented clockwise. Hence, for  $R \rightarrow 1$  we obtain

$$\sum_{k=1}^{\infty} k(|a_k|^2 - |b_k|^2) - 1 - 2Re\{b_1\} \leq 0$$

and the result follows. ■

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