

Variational Iteration Method for the Solution of Boundary Value Problems

Olayiwola M.O., Gbolagade A .W., and Akinpelu F. O.

Abstract—In this work, we present a reliable framework to solve boundary value problems with particular significance in solid mechanics. These problems are used as mathematical models in deformation of beams. The algorithm rests mainly on a relatively new technique, the Variational Iteration Method. Some examples are given to confirm the efficiency and the accuracy of the method.

Keywords—Variational iteration method, boundary value problems, convergence, restricted variation.

I. INTRODUCTION

THIS paper discussed the approximate solution of the equation of the form.

$$y^{(2n)} + f(x, y) = k, \quad n \geq 2 \quad (1)$$

Subject to the boundary condition

$$y(a) = \alpha_1, y'(a) = \alpha_2, y(b) = \beta_1, y'(b) = \beta_2 \quad (2)$$

Ma and Silva [20] adopted iterative solution for (1) representing beams on elastic foundation when $k = 0$. In the configuration of the deformed beam, the bending moment satisfies the relation $M = -EIU''$, where E is the Young modulus of elasticity and I is the inertial moment. Considering the deformation caused by a load $f = f(x)$ then $f = -v'$ and $v = M' = -EIU'''$, where v denotes the shear force. For u representing an elastic beam of length $L = 1$, which is clamped at its left side $x = 0$, and resting on an elastic bearing at its right side $x = 1$, and adding a load f along its length to cause deformation. $u = u(x)$. This lead to the boundary value problem:

$$U^{(iv)}(x) = f(x, u(x)), 0 < x < 1 \quad (3)$$

$$u(0) = u'(0) = 1, \quad u''(1) = 0, \quad u'''(1) = g(v(1))$$

M. O. Olayiwola, PhD, is with the Department of Mathematical and Physical sciences, College of Science, Engineering and Technology, Osun State University, Osogbo, Nigeria (phone: +2348055218234, +2348028063936; e-mail: Olayiwola.oyedunsi@uniosun.edu.ng).

A. W Gbolagade, Prof., was with the Department of Mathematical and Physical sciences, College of Science, Engineering and Technology, Osun State University, Osogbo, Nigeria (phone: +2348034037010).

F. O. Akinpelu, Prof., is with the Department of Pure and Applied Mathematics Ladoko Akintola University of Technology, Ogbomoso, Nigeria (phone: +2348035605721).

Solving (3) by means of iterative procedure, Ma and Silva obtained solution and argued that accuracy of result depends highly upon the integration method used in the iterative process.

Barari *et al* [3] solved the problem with variational iteration method, a case of $k = 0$.

In this work, it is aimed to apply the VIM proposed by He [12, 13, 14, 16] to different forms of (1) with boundary condition of physical significance.

II. VARIATIONAL ITERATION METHOD

To illustrate the basic concept of the technique, we consider the following general differential equation

$$Lu + Nu = g(x) \quad (4)$$

Where L is a linear operator, N a nonlinear operator and $g(x)$ is the forcing term. According to variational iteration method He [12, 13, 14, 17], Inokuti *et al* [15], we can construct a correct functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda (Lu_n(\tau) + Nu_n(\tau) - g(\tau)) d\tau \quad (5)$$

Where λ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscripts n denotes the n^{th} approximation, \bar{u}_n is considered as a restricted variation, i.e., $\delta \bar{u}_n = 0$ equation (5) is called a correction functional. The solution of the linear problems can be solved in a single iteration steps due to exact identification of Lagrange multiplier [3, 16, 20].

III. NUMERICAL EXAMPLE

Example 1:

$$u^{(iv)}(x) - u(x) = 4e^{-x} + k \quad (6)$$

$$u(0) = k + 1, u'(0) = 1, u(1) = 2e + k, \quad u'(1) = 3e + k$$

We construct a correction functional for (6), as follows:

$$U_{n+1}(t) = U_n(t) + \int_0^t \lambda \{LU_n(s) + NU_n(s) - g(s)\} ds \quad (7)$$

The variational iteration formula corresponding to (6) is therefore

$$u_{n+1}(x) = u_n(x) + \int_0^x \left\{ \frac{1}{6}(\tau - x)^3 u_n(\tau) + f(x, u(\tau)) - k \right\} d\tau \quad (8)$$

$$\text{Let } u_0(x) = ax^3 + bx^2 + cx + d \quad (9)$$

Then (7) becomes

$$u_1(x) = ax^3 + bx^2 + cx + d - \frac{1}{6} \int_0^x (\tau-x)^3 (a\tau^3 + b\tau^2 + c\tau + d + 4e\tau + k) d\tau$$

$$u_1(x) = ax^3 + bx^2 + cx + d - \frac{1}{6} \left[\int_0^x (a\tau^6 + \tau^5(b-3ax)) + \tau^4(c+3bx+3ax^2) + \tau^3(d-3cx+3d^2-ax^3+k) + \tau^2(3cx^2-3dx-3kx-bx^3) + \tau(3d^2-3kx^2-cx^3) + 4\tau^2e^{\tau} - 12^2e^{\tau}x + 12x^{\tau}e^{\tau} - 4x^3e^{\tau} - dx^3 - x^3k \right] d\tau \quad (10)$$

$$u_1(x) = \frac{ax^7}{840} + \frac{bx^6}{360} + \frac{cx^5}{120} + \frac{x^4}{24}(d+k) + x^3\left(a-\frac{2}{3}\right) + x^2(b-2) + x(c-4) + 4e^x + d - 4 \quad (11)$$

Introducing the boundary condition, we have:

$$d = k + 1 \quad (12)$$

$$c = 2 \quad (13)$$

$$\frac{7a}{840} + \frac{b}{60} + \frac{cx^5}{120} + \frac{1}{24}(d+k) + \left(a-\frac{2}{3}\right) + (b-2) + (c-4) + 4e + d - 4 = 2e + k \quad (14)$$

$$\frac{7a}{840} + \frac{b}{60} + \frac{1}{24}(d+k) + 3\left(a-\frac{2}{3}\right) + 2(b-2) + (c-4) + 4e = 3e \quad (15)$$

Solving equations (12)- (15), we have:

Case 1: k = 1, a = 0.500947014, b = 1.582497142, c = 2, d = 2

Case 2: k = 2, a = 0.334347197, b = 1.665730754, c = 2, d = 3.

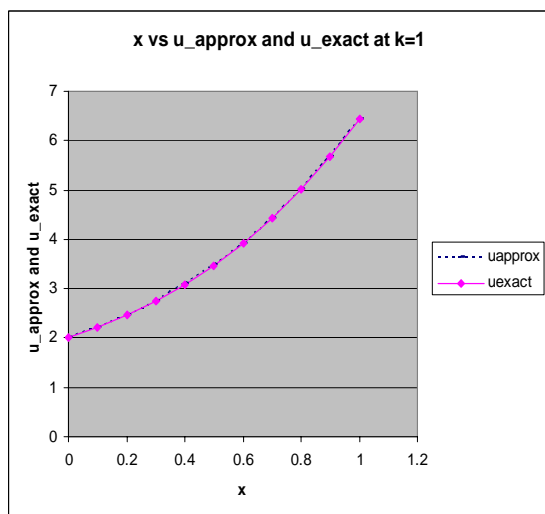


Fig. 1 (a)

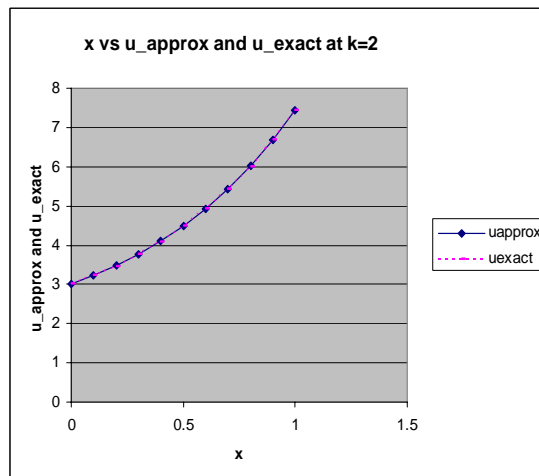


Fig. 1 (b)

Fig. 1 (a) and Fig. 1 (b) The graphs of exact solution (u_{exact}) and the approximate solution (u_{approx}) against x when $k=1$ and $k=2$ respectively

Example 2:

Consider the following boundary value problem:

$$u^4(x) = u(x) + u''(x) + e^x(x-3) + k, \quad 0 < x < 1 \quad (16)$$

$$u(0) = 1 + k, u'(0) = 0, u(1) = k, u'(1) = -e \quad (17)$$

The iteration formulation is:

$$u_{n+1}(x) = u_n(x) + \int_0^x \frac{1}{6} (\tau-x)^3 (u_n^4(\tau) - u_n(\tau) - u_n''(\tau) - e^{\tau}(\tau-3) - k) d\tau$$

$$u_1(x) = \frac{ax^7}{840} + \frac{bx^6}{360} + \left(\frac{a}{20} + \frac{c}{120}\right)x^5 + \left(\frac{b}{12} + \frac{d}{24} + \frac{k}{24}\right)x^4 + \left(a-\frac{2}{3}\right)x^2 + \left(b+\frac{5}{2}\right)x^2 + (c+6+e^x)x - 7e^x + d + 7 \quad (18)$$

Incorporating the boundary into (18), we have

$$d = 1 + k \quad (19)$$

$$c = 0 \quad (20)$$

$$\frac{a}{840} + \frac{b}{360} + \left(\frac{a}{20} + \frac{c}{120}\right) + \left(\frac{b}{12} + \frac{d}{24} + \frac{k}{24}\right) + \left(a-\frac{2}{3}\right) + \left(b+\frac{5}{2}\right) + c + d + 13 - 6e = k \quad (21)$$

$$\frac{a}{120} + \frac{b}{60} + \left(\frac{a}{4} + \frac{c}{24}\right) + \left(\frac{b}{3} + \frac{d}{6} + \frac{k}{6}\right) + (3a+2)x^2 + (2b+5) + c + 6 = 4e \quad (22)$$

The solutions of equations (19)- (22) gives:

Case 1: k = 1, c=0, d=2, b=0.4101204308, a=-0.5104111543

Case 2: k = 2, c=0, d=3, b=-0.336315472, a=-0.6655431229

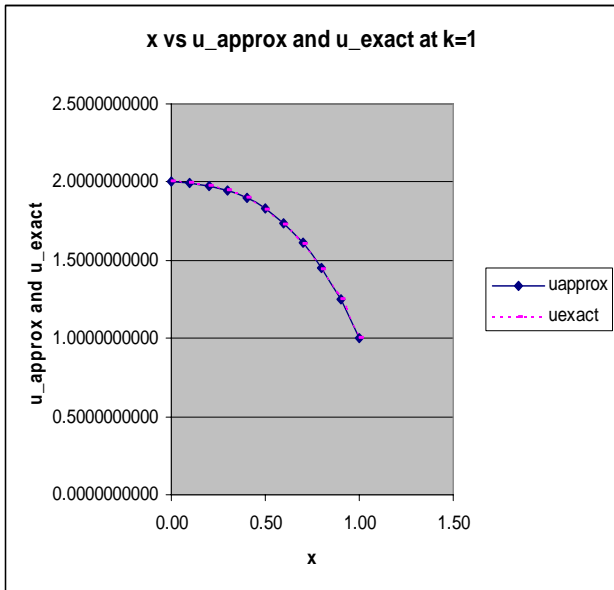


Fig. 2 (a)

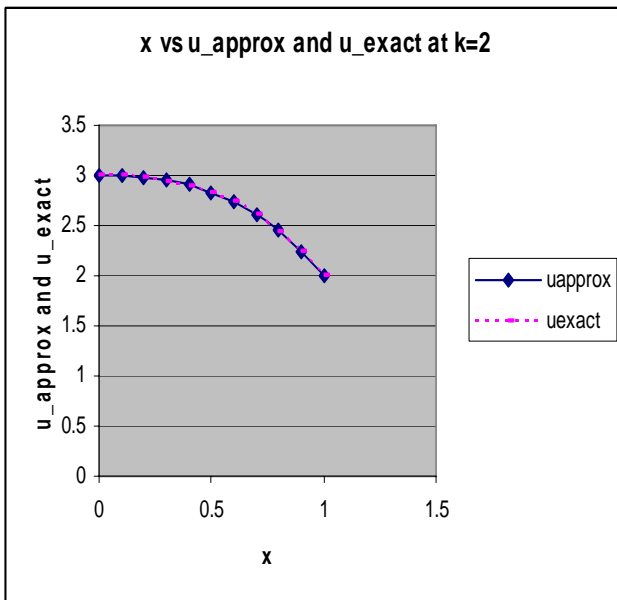


Fig. 2 (b)

Fig. 2 (a) and Fig. 2 (b) The graphs of exact solution (u_{exact}) and the approximate solution ($u_{approx.}$) against x when $k=1$ and $k=2$ respectively

Example 3:

Consider the following nonlinear boundary value problem:

$$u^{(4)}(x) = u^2(x) + g(x) + k \quad (23)$$

$$u(0) = k \quad u'(0) = 0 \quad u(1) = 1+k \quad u'(1) = 1$$

$$g(x) = -x^{10} + 4x^9 - 4x^8 - 4x^7 + 8x^6 - 4x^4 + 120x - 48$$

$$u_{n+1} = u_n + \int_0^x \lambda(u_n^{(4)}(x) - u_n^2(\tau) - g(\tau) - k) d\tau$$

$$u_1(x) = -\frac{1}{24024}x^{14} + \frac{1}{4290}x^{13} + \frac{1}{2970}x^{12} - \frac{1}{1980}x^{11} + \left(\frac{1}{5040}a^2 + \frac{1}{630}\right)x^{10} + \frac{1}{1512}abx^9 + \left(-\frac{1}{120} + \frac{b^2}{1680} + \frac{ac}{840}\right)x^8 + \left(\frac{bc}{420} + \frac{ad}{420} + \frac{ak}{420}\right)x^7 + \left(\frac{bd}{180} + \frac{bk}{180} + \frac{c^2}{360}\right)x^6 + \left(1 + \frac{cd}{60} + \frac{ck}{60}\right)x^5 + \left(\frac{d^2}{24} + \frac{k^2}{24} + 2\right)x^4 + ax^3 + bx^2 + cx + d \quad (24)$$

Introducing boundary conditions, we have

$$d = 0 \quad (25)$$

$$c = 0 \quad (26)$$

$$-\frac{1}{24024} + \frac{1}{4290} - \frac{1}{2970} - \frac{1}{1980} + \frac{a}{5040} + \frac{1}{630} + \frac{ab}{1512} + \frac{b^2}{1680} + \frac{ac}{840} - \frac{1}{420} + \frac{bc}{420} + \frac{bd}{420} + \frac{ak}{420} + \frac{bd}{180} + \frac{bk}{180} + \frac{c^2}{360} + 1 + \frac{cd}{60} + \frac{ck}{60} + \frac{d^2}{24} + \frac{k^2}{24} - 2 + a + b + c + d = 1 \quad (27)$$

$$-\frac{1}{1716} + \frac{1}{330} - \frac{12}{2970} - \frac{1}{180} + \frac{a^2}{504} + \frac{1}{63} + \frac{ab}{168} + \frac{b^2}{210} + \frac{ac}{105} - \frac{8}{420} + \frac{bc}{60} + \frac{ad}{60} + \frac{kd}{60} + \frac{bd}{30} + \frac{bk}{30} + \frac{c^2}{60} + 5 + \frac{cd}{12} + \frac{ck}{12} + \frac{d^2}{6} + \frac{k^2}{6} - 8 + 3a + 2b + c = 1 \quad (28)$$

Solving (25)-(28), we have:

Case 1: $k = 1, d = 0, c = 0, a = -0.1363676737, b = 2.082180724$

Case 2: $k = 2, d = 0, c = 0, a = -0.4395283713, b = 2.24694145$

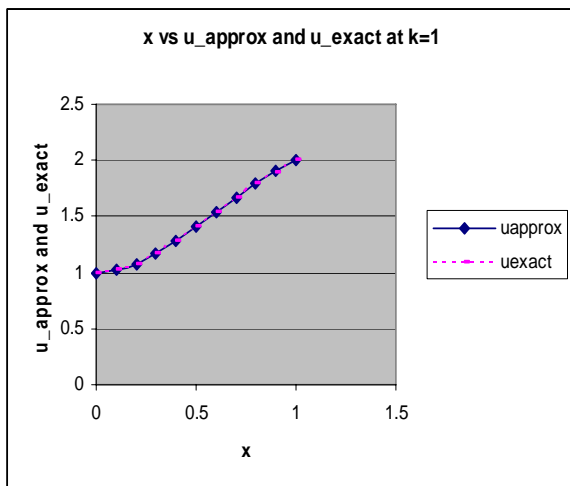


Fig. 3 (a)

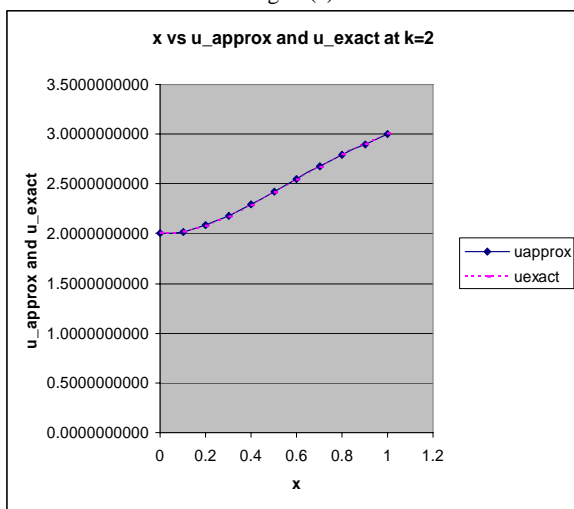


Fig. 3 (b)

Fig. 3 (a) and Fig. 3 (b) shows the graphs of exact solution (u_{exact}) and the approximate solution ($u_{approx.}$) against x when $k=1$ and $k=2$ respectively

IV. CONCLUSION

In this work, VIM has been successfully used to find the numerical solution of models which has fundamental importance in different field of engineering and applied sciences and can also be extended to those investigated in [1]-[11] and [18]-[31]. Many of the results attained in this work confirm the idea that VIM is powerful mathematical tool for solving different kinds of practical problems, having wide application in engineering.

Comparison between the approximate and exact solutions; Figs. 1a, 1b, 2a, 2b, 3a and 3b, shows that the one iteration of variational iteration method is enough. The methods also can be introduced to overcome the difficulties arising in calculating Adomian polynomials.

ACKNOWLEDGEMENT

Dr. M. O. Olayiwola is highly grateful to the Tertiary

Education Trust Fund (TetFund) through Osun State University, Osogbo, Nigeria for the provision of grants for this research.

REFERENCES

- [1] Abulwafa E.M, Abdou M.A, Mahmoud A.A. "Nonlinear fluid flows in pipe-like domain problems using VIM" *soliton and fractals* 32(4) pp. 1384-1397,2007.
- [2] Abulwafa E.M, Abdou M.A, Mahmoud A.A. "The solution of nonlinear coagulation problem with mass" *soliton and Fractals*. 29 (2) 313-330, 2006.
- [3] Barari A et al. "An approximate solution for boundary value problems in structural Engineering and Fluid Mechanics." *Mathematical problems in Engineering*. 2008
- [4] Batiha B. et al "Application of Variational Iterational method to a General Riccati Equation". *Int. Mathematical Forum*, 2, no 56, 2759-2770, 2007.
- [5] Biazar J., Ghazuini H. "He's Variational iteration method for solving linear and non-linear system of ordinary differential equations". *Applied Maths and Computation* 191, 281-287, 2007.
- [6] Bildik N., Konuralp A. "The use of variation iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear pole". *Int. J. nonlinear Science; Numerical Simulation* 7(1) 65-70, 2006.
- [7] Celik E et al. "Numerical method to solve chemical differential Allgerai equation." *Int. J. of Quantum chemistry*, 89(5), 447-451, 2002.
- [8] Diethelin K. and Ford N.J. "Analysis of fractional differential equation". *J. math and Appl.* 265 229-248, 2002.
- [9] Djidjeli K. et al "Numerical methods for special nonlinear boundary value problems of order $2m$ ". *J. of Computational and Appl. Mathematics* Vol. 47, no 1, 161-169, 2006.
- [10] Draganescu G E, capalnasan V. "Nonlinear relaxation phenomenon in polycrystalline solid" *int. J nonlinear Numerical Simulation* 4(3) 219-225, 2003.
- [11] Ganji D.D. et al. "A Comparison of Variational iteration method with Adomian decomposition method in some highly nonlinear equations". *International Journal of Science and Technology* Vol. 2, No 2, 179-188, 2007.
- [12] He. J.H "Approximate analytical solution for seepage flow with fractional derivatives in porous media". *Compt. Math. App. Mech. Eng.* 167. 57-68, 1998.
- [13] He. J.H "Approximate solution of nonlinear differential equations with convolution product nonlinearities" *Comp. Math. Apply Mech. Eng.* 167: 69-73. 1998.
- [14] He. J.H "Variational iteration method for autonomous ordinary differential system." *App. Math and Computation*, 114:115-123, 2000.
- [15] Inokuti M. et al." General use of the Lagrange multipliers in nonlinear mathematical physics in: S." *Nemat Nasser (Ed), Variational method in the mechanics of solid*, Pergamon Press, 156-162, 1978.
- [16] Ji-Huan He "Variational iteration methods some recent results and new interpretation." *J. of competition and Applied mathematics* 207 3-17, 2007.
- [17] Ji-Huan He. "Variational Iteration method: a kind of non-linear analytical technique: Some examples." *Int. Journal of Non-linear mechanics* (3494) 699-708, 1999.
- [18] Levant Yilmaz "Some Considerations on the series solution of Differential equations and its engineering Applications." *RMZ Materials and Geo-environment*, Vol. 53, No 1, 247-259, 2006.
- [19] Lu J.F. "variational Iteration method for solving two-point boundary value problems. " *Journal of computational and Applied Mathematics* 207(1) 92-95, 2007.
- [20] Ma T.F and Silva J. Da "Iterative solution for a beam equation with nonlinear boundary conditions of third order", *Applied mathematics and Computation*, Vol. 159 no 1. 11-18, 2004.
- [21] Muhammed A.N, Syed T.M "Variational Iteration Decomposition method for solving Eight-Order Boundary Value Problem." *Diff. Equation and Nonlinear mechanics*. Vol. Id 19529, 2007.
- [22] Nuran Guzek, Bayram M. "Power Series solution of nonlinear first order differential equations systems" *Trakya Univ. J.Se*, 6(1). 107-111, 2005.
- [23] Odibat Z.M, Momani S. "Application of Variation iteration method to nonlinear differential equation of fractional order". *Int. J to Nonlinear Science. Numerical Simulation* 7(1) 27-34, 2006.

- [24] Olayiwola M.O *et al* (2008): on the existence of solution of differential equation of fractional order. *J. of Modern Maths. and Statistics* 2(5) 157-159.
- [25] Olayiwola M.O, Gbolagade A.W. "On the Chaotic behaviour of Harmonically driven Oscillator". *Far East J. of Dynamical Systems* Vol. 10, Issue 1, 47-52, 2008.
- [26] Podhibny I. "Fractional Differential equations" Academic Press, San Diego, 1999.
- [27] Saharay S... Bera R.K. "Solution of an extraordinary differential equation by Adomain decomposition method." *J. of Applied Maths* 4, 331-338, 2004.
- [28] Siddiqi S.S. Twizell. E.H. "Spline solution of linear eight-order boundary value problems." *Computer methods in Apply Mechanics and engineering* Vol. 131, no 3-4, 309-325, 1996.
- [29] Tatari M, Delighan M. "on the convergence of He's Variational iteration method" *Journal of computational and applied Mathematics* 207 (1) 121-128, 2007.
- [30] Wazwaz A M. "A reliable algorithm for obtaining positive solution for nonlinear boundary value problems". *Computer Math App* 41(10-11) 1237-1244, 2001.
- [31] Wei-Xia Qian *et al*. "He's Iteration Formulation for solving Nonlinear Algebraic equations." *Journal of Physics: Conference series* 96 1-6, 2008.