A finite-time consensus protocol of the multi-agent systems

Xin-Lei Feng, Ting-Zhu Huang

Abstract—According to conjugate gradient algorithm, a new consensus protocol algorithm of discrete-time multi-agent systems is presented, which can achieve finite-time consensus. Finally, a numerical example is given to illustrate our theoretical result.

Keywords—Consensus protocols; Graph theory; Multi-agent systems; Conjugate gradient algorithm; Finite-time.

I. INTRODUCTION

RECENTLY, multi-agent systems have received significant attention due to their potential impact in numerous civilian, homeland security, and military applications, etc. Consensus plays an important role in achieving distributed coordination. The basic idea of consensus is that a team of vehicles reaches an agreement on a common value by negotiating with their neighbors. Consensus algorithms are studied for both single-integrator kinematics [1-3] and high-order-integrator dynamics [4-8].

Formal study of consensus problems in groups of experts originated in management science and statistics in 1960s. Distributed computation over networks has a tradition in systems and control theory starting with the pioneering work of Borkar, Varaiya and Tsitsiklis. Vicsek et al provided a formal analysis of emergence of alignment in the simplified model of flocking [9]. This paper have an important influence on developing of the multi-agent systems consensus theory. On the study of consensus of continuous-time system, the classical model of consensus is provided by Olfati-Saber and Murray [10].

The idea of finite-time convergence has been introduced to finite-time consensus for multiple dynamic agents in [11-15]. [11] introduces the normalized and signed gradient dynamical systems associated with a differentiable function, and identify conditions that guarantee finite-time convergence. [12] obtains that finite-time consensus tracking of multi-agent systems can be achieved on the terminal sliding-mode surface. Under both the global information and the local information, [13] develops a new finite-time formation control framework for multi-agent systems with a large population of members. [14] give outs that the agents of the group under a particular type of nonlinear interaction can reach the consensus state in finite time in the scenarios with fixed and switching undirected topologies, and also extend the topology of the group is directed and satisfies a detailed balance condition on coupling weights. Recently, [15] investigate the finite-time consensus problems if the sum of time intervals is sufficiently large.

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In this note, by using conjugate gradient algorithm idea, we present a new consensus protocol algorithm of discrete-time multi-agent systems which can achieve consensus in finite-time. Likewise, the numerical example verifies our theoretical result. The note is organized as follows.

In Section 2, we introduces basic concepts and preliminary results, while Section 3, we present our finite-time consensus algorithm, and give out main result. In Section 4, numerical example is presented to illustrate our theoretical result. Conclusions are drawn in Section 5.

II. PRELIMINARIES

A directed graph (digraph) $G = (V, E)$ of order n consists of a set of vertices $v = \{1, \ldots, n\}$ and a set of edges $V = E \times E$. (i, j) is a arc of G if and only if $(i, j) \in G$. Accordingly, agent i is a neighbor of agent j. All neighbors of agent i is denoted by $\mathcal{N}_i$. Suppose that there are n nodes in the graph. The adjacency matrix $A \in R^{n \times n}$ is defined as $a_{ii} = 0, a_{ij} = 1$, if $(j, i) \in E$ and 0 otherwise. A graph with the property that $(i, j) \in E$ implies $(j, i) \in E$ is said to be undirected. The Laplacian matrix $L \in R^{n \times n}$ is defined as $l_{ii} = \sum_{j \neq i} a_{ij}, l_{ij} = -a_{ij}$, for $i \neq j$. Moreover, matrix L is symmetric if an undirected graph has symmetric weights, i.e., $a_{ij} = a_{ji}$. A directed path is a sequence of edges in a directed graph with the form $(v_1, v_2), (v_2, v_3), \cdots$, where $v_i \in E$. A directed graph has a directed spanning tree if there exists at least one agent that has a directed path to all other agents.

Lemma 2.1. [16] *(i). All the eigenvalues of Laplacian matrix L have nonnegative real parts; (ii). Zero is an eigenvalue of L with 1_n (where 1_n is the $n \times 1$ column vector of all ones) as the corresponding right eigenvector. Furthermore, zero is a simple eigenvalue of L if and only if graph G has a directed spanning tree.*

Definition 2.1 (balanced digraphs [10]) *A digraph G is called balanced if $\sum_{j \neq i} a_{ij} = \sum_{j \neq i} a_{ji}$ for all $i \in v$.*

Lemma 2.2. [17] *Consider a network of agents $x_i(k + 1) = x_i(k) + u_i(k)$ with topology G applying the distributed consensus algorithm*

$$x_i(k + 1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k))$$

where $0 < \varepsilon < 1/\triangle$ and $\triangle$ is the maximum degree of the network. Let G be a strongly connected digraph. Then
i)A consensus is asymptotically reached for all initial states;

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ii) The group decision value is $\alpha = \Sigma_i \omega_i x_i(0)$ with $\Sigma_i \omega_i = 1$;

iii) If the digraph is balanced (or P is doubly- stochastic), an average-consensus is asymptotically reached and $\alpha = (\sum_i x_i(0))/n$.

Definition 2.2 *Let A be a Hermitian positive definite matrix of size n. For n nonzero vectors $p_1, \cdots, p_m \in R^n$, if*

$$(A p_i, p_j) = 0 \quad (i \neq j, \ i, j = 1, 2, \cdots, m),$$

then $p_1, \cdots, p_m$ is called $A-$conjugate, where $(\cdot, \cdot)$ denote vector inner product.

Lemma 2.3[18] *Let A be a Hermitian positive definite matrix of size n. Then the conjugate gradient algorithm finds the solution of $Ax = b$ within n iterations in the absence of roundoff errors.*

III. A ALGORITHM OF FINITE-TIME CONSENSUS

In the following, we will give out our main result.

Theorem 3.1. *Consider a network of n agents with topology G which is a connected undirected graph, the vectors $p_1, \cdots, p_n$ are $C-$conjugate, for any initial value $x^{(0)} \in R^n$ and $t_k = (b - C x^{k-1}, p_k)/(C p_k, p_k)$, according to discrete-time dynamic protocol algorithm*

$$x(k) = x(k-1) + t_k p_k \quad (k = 1, 2, \cdots, n),$$

then the average consensus of the discrete time multi-agent system can be reached in finite-time n, where $C = \rho I_n + L$, $b = \rho x^*$, $0 < \rho < 1$, and x^* is the convergence value of average consensus.

Proof. According to Lemma 2.2, applying the distributed consensus algorithm

$$x_i(k+1) = x_i(k) + \varepsilon \sum_{j \in \mathcal{N}_i} a_{ij}(x_j(k) - x_i(k)), \quad (1)$$

then a consensus is asymptotically reached for all initial states. Let the convergence value be equal to x^*. Since G is a connected undirected graph, G is also a balanced digraph. So an average consensus is asymptotically reached and $x_i^* = \frac{1}{n}\sum_{k=1}^{n} x_k(0)$, $i = 1, \cdots, n$. Let $k \longrightarrow \infty$, then equation (3.1) can be rewritten as

$$x^* = x^* - \varepsilon L x^*. \quad (2)$$

Furthermore, (3.2) can be simplified for

$$L x^* = 0. \quad (3)$$

In order to use Lemma 2.3, here, we introduce into an arbitrarily small positive constant ρ, which satisfies $0 < \rho < 1$. Let $C = \rho I_n + L$ and $b = \rho x^*$. Moreover, equation (3.3) is equivalent to the following equation

$$C x^* = (\rho I_n + L) x^* = \rho x^* = b. \quad (4)$$

According to the above proof, we can easily get that C is a symmetrical positive definite matrix. By Lemma 2.3, the
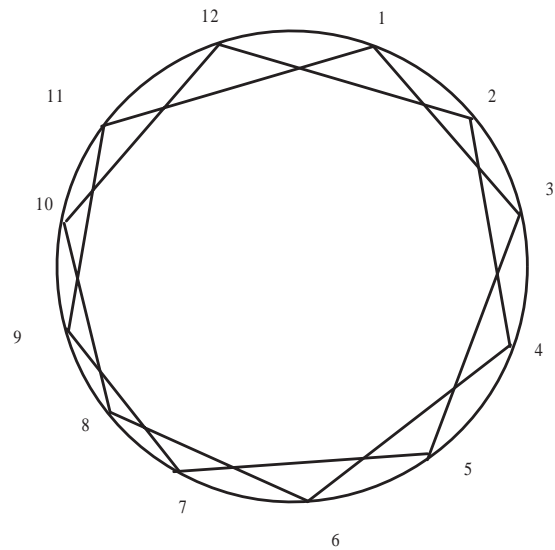

Fig. 1. Network topology G of the agents

consensus of the multi-agent system can be reached in a maximum of n step. Let an iterative step be an time unit, then the consensus of the discrete multi-agent system can be reached in finite-time n. So the proof is completed.

Remark 3.1 Our result is obtained under ideal conditions, which means that the datas of every iterative steps are accurate. We know that this may be not hold some time, and often there exist some errors. But these errors can be accepted as long as they reach our demand.

IV. SIMULATION

In this section, a simulation result is presented to illustrate the proposed consensus algorithm introduced in Section 3.

Example 4.1 We consider a system of twelve agents with the topology G in Figure 1. The corresponding Laplacian matrix is chosen as

$$
\begin{pmatrix}
7 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 \\
-1 & 8 & -1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -3 \\
-2 & -1 & 8 & -1 & -4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & -1 & 10 & -1 & -5 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -4 & -1 & 8 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -5 & -1 & 13 & -1 & -6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -2 & -1 & 7 & -1 & -3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -6 & -1 & 13 & -1 & -5 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -3 & -1 & 7 & -1 & -2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -5 & -1 & 15 & -1 & -8 \\
-3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & -1 & 7 & -1 \\
-1 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -8 & -1 & 13
\end{pmatrix}
$$

and

$$x(0)^T = \begin{pmatrix} 1 & 2 & -3 & 1 & 2 & 4 & 1 & 3 & 2 & 3 & 1 & 3 \end{pmatrix}^T.$$

According to the algorithm of Theorem 3.1, the discrete multi-agent system converge in 12 seconds. Trajectory of the agents are shown in the Figure 2. The final consensus states of the agents are reached with errors $O(10^{-12})$. The errors can be accepted for us.
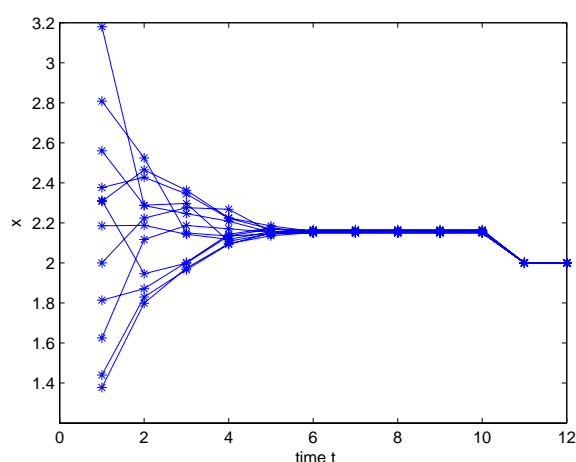
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Fig. 2. Trajectory of the agents

V. CONCLUSION

In this note, a new consensus protocol algorithm of discrete-time multi-agent systems is presented, which can achieve finite-time consensus. In fact, we mainly provide a idea about finite-time consensus. Finally, by using the idea, We hope that better results can be achieved in the future.

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