Hyers-Ulam Stability of Functional Equation

\[ f(3x) = 4f(3x - 3) + f(3x - 6) \]

Soon-Mo Jung

Abstract—The functional equation \( f(3x) = 4f(3x - 3) + f(3x - 6) \) will be solved and its Hyers-Ulam stability will be also investigated in the class of functions \( f : \mathbb{R} \to X \), where \( X \) is a real Banach space.

Keywords—Functional equation, Lucas sequence of the first kind, Hyers-Ulam stability.

I. INTRODUCTION

In 1940, Ulam gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems (ref. [20]). Among those was the question concerning the stability of homomorphisms:

Let \( G_1 \) be a group and let \( G_2 \) be a metric group with a metric \( d(\cdot, \cdot) \). Given any \( \epsilon > 0 \), does there exist a \( \delta > 0 \) such that if a function \( h : G_1 \to G_2 \) satisfies the inequality \( d(h(x)y), h(x)h(y) < \delta \) for all \( x, y \in G_1 \), then there exists a homomorphism \( H : G_1 \to G_2 \) with \( d(h(x), H(x)) < \epsilon \) for all \( x \in G_1 \)?

In the following year, Hyers affirmatively answered in his paper [8] the question of Ulam for the case where \( G_1 \) and \( G_2 \) are Banach spaces. Later, the result of Hyers has been generalized by Rassias (ref. [16]). Let \( (G_1, \cdot) \) be a groupoid and let \( (G_2, +) \) be a groupoid with the metric \( d \). The equation of homomorphism

\[ f(x \cdot y) = f(x) + f(y) \]

is stable in the Hyers-Ulam sense (or has the Hyers-Ulam stability) if for every \( \delta > 0 \) there exists an \( \epsilon > 0 \) such that for every function \( h : G_1 \to G_2 \) satisfying

\[ d[h(x \cdot y), h(x) + h(y)] \leq \epsilon \]

for all \( x, y \in G_1 \) there exists a solution \( g : G_1 \to G_2 \) of the equation of homomorphism with

\[ d[h(x), g(x)] \leq \delta \]

for any \( x \in G_1 \) (see [15, Definition 1]).

This terminology is also applied to the case of other functional equations. It should be remarked that a lot of references concerning the stability of functional equations can be found in the books [3], [9], [12] (see also [1], [4], [5], [6], [7], [10], [11], [17], [18], [19]).

The \( n \)th Fibonacci number will be denoted by \( F_n \) for \( n \in \mathbb{N} \). It is well known that the Fibonacci numbers satisfy the equality \( F_{3n} = 4F_{3n-3} + F_{3n-6} \) for all \( n \geq 2 \) (see [14, p. 89]). From this famous formula, the following functional equation

\[ f(3x) = 4f(3x - 3) + f(3x - 6) \]  \hspace{1cm} (1)

may be derived.

In this paper, using the idea from [13], the functional equation (1) will be solved and its Hyers-Ulam stability will be investigated in the class of functions \( f : \mathbb{R} \to X \), where \( X \) is a real Banach space.

Throughout this paper, the positive and the negative root of the equation \( x^2 - 4x - 1 = 0 \) will be denoted by \( a \) and \( b \), respectively, i.e.,

\[ a = 2 + \sqrt{5} \quad \text{and} \quad b = 2 - \sqrt{5}. \]

Moreover, the Lucas sequence of the first kind will be denoted by \( \{U_n(4, -1)\} \) and an abbreviation \( U_n \) will be used instead of \( U_n(4, -1) \), i.e., \( U_n \) is defined by

\[ U_n = U_n(4, -1) = \frac{a^n - b^n}{a - b} \]

for all integers \( n \). It is not difficult to see that

\[ U_{n+2} = 4U_{n+1} + U_n \]  \hspace{1cm} (2)

for any integer \( n \). For any \( x \in \mathbb{R} \), \( [x] \) stands for the largest integer that does not exceed \( x \).

II. GENERAL SOLUTION TO EQ. (1)

Throughout this section, let \( X \) be a real vector space. The general solution of the functional equation (1) will be investigated.

Theorem 2.1. Let \( X \) be a real vector space. A function \( f : \mathbb{R} \to X \) is a solution of the functional equation (1) if and only if there exists a function \( h : [-3, 3) \to X \) such that

\[ f(x) = U_{[x/3]+1}h(x - 3[x/3]) + U_{[x/3]}h(x - 3[x/3] - 3). \]  \hspace{1cm} (3)

Proof. Since \( a + b = 4 \) and \( ab = -1 \), it follows from (1) that

\[
\begin{align*}
&\begin{cases}
 f(3x) - af(3x - 3) = b[f(3x - 3) - af(3x - 6)], \\
 f(3x) - bf(3x - 3) = a[f(3x - 3) - bf(3x - 6)].
\end{cases}
\end{align*}
\]
If a function $g : \mathbb{R} \rightarrow X$ is defined by $g(x) = f(3x)$ for each $x \in \mathbb{R}$, then it follows from the above equalities that

$$
\begin{align*}
g(x) - ag(x - 1) &= b[g(x - 1) - ag(x - 2)], \\
g(x) - bg(x - 1) &= a[g(x - 1) - bg(x - 2)].
\end{align*}
$$

(4)

By the mathematical induction, it can be proved that

$$
\begin{align*}
g(x) - ag(x - 1) &= b^n [g(x - n) - ag(x - n - 1)], \\
g(x) - bg(x - 1) &= a^n [g(x - n) - bg(x - n - 1)]
\end{align*}
$$

(5)

for all $x \in \mathbb{R}$ and $n \in \{0, 1, 2, \ldots\}$. Substitute $x + n$ ($n \geq 0$) for $x$ in (5) and divide the resulting equations by $b^n$ resp. $a^n$, and then substitute $-m$ for $n$ in the resulting equations to obtain the equations in (5) with $m$ in place of $n$, where $m \in \{0, -1, -2, \ldots\}$. Therefore, the equations in (5) are true for all $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Multiply the first and the second equation of (5) by $b$ and $a$, respectively. And subtract the first resulting equation from the second one to obtain

$$
g(x) = U_{n+1}g(x - n) + U_n g(x - n - 1)
$$

(6)

for any $x \in \mathbb{R}$ and $n \in \mathbb{Z}$.

Putting $n = [x]$ in (6) yields

$$
g(x) = U_{[x]+1}g(x - [x]) + U_{[x]} g(x - [x] - 1),
$$
i.e., by the definition of $g$, it holds that

$$
f(x) = U_{[x]+1} f(x - 3[x/3]) + U_{[x]} f(x - 3[x/3] - 3)
$$

for all $x \in \mathbb{R}$.

Since $0 \leq x - 3[x/3] < 3$ and $-3 \leq x - 3[x/3] - 3 < 0$, if a function $h : [-3, 3) \rightarrow X$ is defined by $h := f|_{[-3,3)}$, then $f$ is a function of the form (3).

Now, assume that $f$ is a function of the form (3), where $h : [-3, 3) \rightarrow X$ is an arbitrary function. Then, it follows from (3) that

$$
\begin{align*}
f(3x) &= U_{[x]+1} h(3x - 3[x]) + U_{[x]} h(3x - 3[x] - 3), \\
f(3x - 3) &= U_{[x]+1} h(3x - 3[x]) + U_{[x]} h(3x - 3[x] - 3), \\
f(3x - 6) &= U_{[x]+1} h(3x - 3[x]) + U_{[x]} h(3x - 3[x] - 3)
\end{align*}
$$

for any $x \in \mathbb{R}$. Thus, by (2), it holds that

$$
\begin{align*}
f(3x) - 4f(3x - 3) - f(3x - 6) &= (U_{[x]+1} - 4U_{[x]} - U_{[x]+1} - 4U_{[x]+1} - U_{[x]+1} - 4U_{[x]+1}) h(3x - 3[x]) \\
&= 0,
\end{align*}
$$

which completes the proof.

### III. Hyers-Ulam Stability of Eq. (1)

In this section, $a$ denotes the positive root of the equation $x^2 - 4x - 1 = 0$ and $b$ is its negative root. The Hyers-Ulam stability of the functional equation (1) will be proved in the following theorem.

**Theorem 3.1.** Let $(X, \| \cdot \|)$ be a real Banach space. If a function $f : \mathbb{R} \rightarrow X$ satisfies the inequality

$$
\|f(3x) - 4f(3x - 3) - f(3x - 6)\| \leq \epsilon
$$

for all $x \in \mathbb{R}$ and for some $\epsilon \geq 0$, then there exists a unique solution function $F : \mathbb{R} \rightarrow X$ of (1) such that

$$
\|f(x) - F(x)\| \leq \frac{5 + \sqrt{5}}{20} \epsilon
$$

for all $x \in \mathbb{R}$.

**Proof.** First, define a function $g : \mathbb{R} \rightarrow X$ by $g(x) = f(3x)$ for all $x \in \mathbb{R}$. Analogously to the first equation of (4), it follows from (7) that

$$
\|g(x) - ag(x - 1) - bg(x - 1) - ag(x - 2)\| \leq \epsilon
$$

for each $x \in \mathbb{R}$. Replacing $x$ with $x - k$ in the last inequality yields

$$
\begin{align*}
\|g(x - k) - ag(x - k - 1) - bg(x - k - 1) - ag(x - k - 2)\| &
\leq \epsilon
\end{align*}
$$

and further

$$
\begin{align*}
\|b^k[g(x - k) - ag(x - k - 1)] - b^{k+1}[g(x - k - 1) - ag(x - k - 2)]\| &
\leq \|b^k\| \epsilon
\end{align*}
$$

(9)

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. By (9), it obviously holds that

$$
\begin{align*}
\|g(x) - ag(x - 1) - b^n[g(x - n) - ag(x - n - 1)]\| &
\leq \sum_{k=0}^{n-1} \|b^k[g(x - k) - ag(x - k - 1)] - b^{k+1}[g(x - k - 1) - ag(x - k - 2)]\|
\end{align*}
$$

(10)

$$
\begin{align*}
\sum_{k=0}^{n-1} \|b^k\| \epsilon &
\end{align*}
$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

For any $x \in \mathbb{R}$, (9) implies that the sequence \{\(b^k[g(x - n) - ag(x - n - 1)]\)\} is a Cauchy sequence. (Note that $|b| < 1$). Therefore, a function $G_t : \mathbb{R} \rightarrow X$ can be defined by

$$
G_t(x) = \lim_{n \to \infty} b^n[g(x - n) - ag(x - n - 1)],
$$
since $X$ is complete. It follows from the definition of $G_1$ that

$$
4G_1(x - 1) + G_1(x - 2) = 4b^{-1} \lim_{n \to \infty} b^{n+1}|g(x - (n + 1)) - ag(x - (n + 1) - 1)| + b^{-2} \lim_{n \to \infty} b^{n+2}|g(x - (n + 2)) - ag(x - (n + 2) - 1)| = 4b^{-1}G_1(x) + b^{-2}G_1(x) = G_1(x)
$$

for all $x \in \mathbb{R}$, since $b^2 = 4b + 1$. If $n$ goes to infinity, then (10) yields that

$$
\|g(x) - ag(x - 1) - G_1(x)\| \leq \frac{3 + \sqrt{5}}{4} \varepsilon
$$

for every $x \in \mathbb{R}$.

On the other hand, it also follows from (7) that

$$
\|g(x) - bg(x - 1) - a[g(x - 1) - bg(x - 2)]\| \leq \varepsilon
$$

(see the second equation in (4)). Analogously to (9), replacing $x$ by $x + k$ in the above inequality and then dividing by $a^k$ both sides of the resulting inequality yield

$$
\left\|\frac{a^{-k}[g(x + k) - bg(x + k - 1)]}{a^{-k+1}[g(x + k - 1) - bg(x + k - 2)]} - a^{-k}\varepsilon\right\| \leq a^{-k}\varepsilon
$$

for all $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. It further follows from (13) that

$$
\left\|\frac{a^{-n}[g(x + n) - bg(x + n - 1)]}{\|g(x) - bg(x - 1)\|} - a^{-n}\varepsilon\right\| \leq \frac{1}{n} \sum_{k=1}^{n} \left\|\frac{a^{-k}[g(x + k) - bg(x + k - 1)]}{a^{-k+1}[g(x + k - 1) - bg(x + k - 2)]} - a^{-k}\varepsilon\right\|
$$

for $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

On account of (13), the sequence $\{a^{-n}[g(x + n) - bg(x + n - 1)]\}$ is a Cauchy sequence for any fixed $x \in \mathbb{R}$. Hence, a function $G_2 : \mathbb{R} \to X$ can be defined by

$$
G_2(x) = \lim_{n \to \infty} a^{-n}[g(x + n) - bg(x + n - 1)].
$$

It follows from the definition of $G_2$ that

$$
4G_2(x - 1) + G_2(x - 2) = 4a^{-1} \lim_{n \to \infty} a^{-(n+1)}[g(x + (n + 1) - 1)] - bg(x + (n + 1) - 1)] + a^{-2} \lim_{n \to \infty} a^{-(n-2)}[g(x + n - 2) - bg(x + (n - 2) - 1)] = 4a^{-1}G_2(x) + a^{-2}G_2(x) = G_2(x)
$$

for any $x \in \mathbb{R}$. By letting $n$ go to infinity, (14) yields

$$
\|G_2(x) - g(x) + bg(x - 1)\| \leq \frac{\sqrt{5} - 1}{4} \varepsilon
$$

for $x \in \mathbb{R}$.

From (12) and (16), it follows that

$$
g(x) - \left[\frac{b}{b-a}G_1(x) - \frac{a}{b-a}G_2(x)\right] = \frac{1}{|b-a|}\| (b-a)g(x) - [bg_1(x) - aG_2(x)] \| \leq \frac{1}{|b-a|}\| bg_1(x) - abg(x) - bg_1(x) \|\]

$$
+ \frac{1}{|a-b|}\| bg_1(x) - abg(x) + abg(x - 1) \| \leq \frac{5 + \sqrt{5}}{20} \varepsilon
$$

for all $x \in \mathbb{R}$. Now define a function $F : \mathbb{R} \to X$ by

$$
F(x) = \left[\frac{b}{b-a}G_1\left(\frac{x}{3}\right) - \frac{a}{b-a}G_2\left(\frac{x}{3}\right)\right]
$$

for all $x \in \mathbb{R}$. Then, it follows from (11) and (15) that

$$
4F(3x - 3) + F(3x - 6) = \frac{4b}{b-a}G_1(x - 1) - \frac{a}{b-a}G_2(x - 1)
$$

$$
+ \frac{b}{b-a}G_1(x - 2) - \frac{a}{b-a}G_2(x - 2)
$$

$$
= \frac{b}{b-a}[4G_1(x - 1) + G_2(x - 2)]
$$

$$
+ \frac{a}{b-a}[4G_2(x - 1) + G_2(x - 2)]
$$

$$
= \frac{b}{b-a}G_1(x) - \frac{a}{b-a}G_2(x)
$$

$$
= F(3x)
$$

for each $x \in \mathbb{R}$, i.e., $F$ is a solution of (1). Moreover, the inequality (8) follows from (17).

The uniqueness of $F$ will be proved. Assume that $F_1, F_2 : \mathbb{R} \to X$ are solutions of (1) and that there exist positive constants $C_1$ and $C_2$ with

$$
\|f(x) - F_1(x)\| \leq C_1 \quad \text{and} \quad \|f(x) - F_2(x)\| \leq C_2
$$

for all $x \in \mathbb{R}$. According to Theorem 2.1, there exist functions $h_1, h_2 : [-3, 3] \to X$ such that

$$
F_1(x) = U_{1/\beta+1}h_1(x - 3[x/3]) + U_{1/\beta}h_1(x - 3[x/3] - 3),
$$

$$
F_2(x) = U_{1/\beta+1}h_2(x - 3[x/3]) + U_{1/\beta}h_2(x - 3[x/3] - 3)
$$

for any $x \in \mathbb{R}$, since $F_1$ and $F_2$ are solutions of (1). Fix a $t$ with $0 \leq t < 3$. It then follows from (18) and (19) that

$$
\|U_{n+1}[h_1(t - h_2(t)] + U_n[h_1(t - 3) - h_2(t - 3)]\|
$$

$$
= \|[U_{n+1}h_1(t) + U_nh_1(t - 3)]
$$

$$
- [U_{n+1}h_2(t) + U_nh_2(t - 3)]\|
$$

$$
= \|F_1(3n + t) - F_2(3n + t)\|
$$

$$
\leq \|F_1(3n + t) - f(3n + t)| + \|f(3n + t) - F_2(3n + t)|\|
$$

$$
\leq C_1 + C_2
$$
for each \( n \in \mathbb{Z} \), i.e.,
\[
\left\| \frac{a^{n+1} - b^{n+1}}{a - b} [h_1(t) - h_2(t)] + \frac{a^n - b^n}{a - b} [h_1(t - 3) - h_2(t - 3)] \right\| \leq C_1 + C_2
\]
for every \( n \in \mathbb{Z} \). Dividing both sides by \( a^n \) yields that
\[
\left\| \frac{a - (b/a)^n b}{a - b} [h_1(t) - h_2(t)] + \frac{1 - (b/a)^n}{a - b} [h_1(t - 3) - h_2(t - 3)] \right\| \leq C_1 + C_2 \cdot \frac{a^n}{a^n}.
\]
Let \( n \to \infty \) to get
\[
[a[h_1(t) - h_2(t)] + [h_1(t - 3) - h_2(t - 3)] = 0. \tag{21}
\]
Analogously, divide both sides of (20) by \( |b|^n \) and let \( n \to -\infty \) to get
\[
b[h_1(t) - h_2(t)] + [h_1(t - 3) - h_2(t - 3)] = 0. \tag{22}
\]
From (21) and (22), it follows that
\[
\begin{pmatrix}
a & 1 \\
b & 1
\end{pmatrix}
\begin{pmatrix}
h_1(t) - h_2(t) \\
h_1(t - 3) - h_2(t - 3)
\end{pmatrix}
= \begin{pmatrix} 0 \\ 0 \end{pmatrix}.
\]
Because \( a - b \neq 0 \), it should hold that
\[
h_1(t) - h_2(t) = h_1(t - 3) - h_2(t - 3) = 0
\]
for any \( t \in [0, 3] \), i.e., \( h_1(t) = h_2(t) \) for all \( t \in [-3, 3] \). Therefore, it is true that \( F_1(x) = F_2(x) \) for any \( x \in \mathbb{R} \).

**Remark 1.** The presented proof of uniqueness of \( F \) is due to an idea of Professor Changsun Choi. It should be remarked that the uniqueness of \( F \) can be obtained directly from [2, Proposition 1].

**ACKNOWLEDGMENT**

This work was supported by National Research Foundation of Korea Grant funded by the Korean Government (No. 2009-0071026).

**REFERENCES**


