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New class of chaotic mappings in symbol space

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Abstract—Symbolic dynamics studies dynamical systems on the basis of the symbol sequences obtained for a suitable partition of the state space. This approach exploits the property that system dynamics reduce to a shift operation in symbol space. This shift operator is a chaotic mapping. In this article we show that in the symbol space exist other chaotic mappings.

Keywords—Infinite symbol space, prefix metric, chaotic mapping, generator function, jump mapping.

I. Introduction

A DISCRETE dynamical system can be characterized as a function f that is composed with itself over and over again. One of the fundamental questions of dynamics concerns about the properties of the sequence

$$x, f(x), f^2(x), f^3(x), \dots, f^n(x), \dots.$$

That is, dynamical systems ask to somebody non-mathematical sounding question: where do points go and what do they do when they get there? The technique of characterizing the orbit structure of a dynamical system via infinite sequences of "symbols" is known as *symbolic dynamics*. The first exposition of symbolic dynamics as an independent subject was given by Morse and Hedlund ([17], 1938). They showed that in many circumstances such finite description of the dynamics is possible. Other ideas in symbolic dynamics come from the data storage and transmission. D.Lind and B.Marcus in 1995 published first general textbook [13] on symbolic dynamics and its applications to coding. This book, B.P.Kitchens ([12], 1998) and B.L.Hao and W.M.Zheng ([3], 1998) gives a good account of the history of symbolic dynamics and its applications. Symbolic dynamics were selected as a qualitative method used to extract some quantitative qualifiers such as entropy ([1]).

Symbolic dynamics study dynamical systems on the basis of symbol sequences obtained for a suitable partition of the state space. The basic idea behind symbolic dynamics is to divide the phase space into a finite number of regions and to label each region by an alphabetical symbol. This approach exploits the property that system dynamics reduce to a shift operation in symbol space. The technique requires knowing the current symbolic state of the system and selecting future symbols. This shift operator is chaotic mapping ([2], [3], [12], [13], [18]). Our purpose is to show that in the symbol space exist other chaotic mappings.

The remainder of the paper is organized as follows: it starts with preliminaries concerning notations and terminology that is used in the paper followed by a definition of the chaotic mapping. The jump mapping is considered in Section 3, furthermore, it is proved that part of this class mappings are chaotic but the other half mappings are non-chaotic mappings.

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II. Preliminaries

The terminology comes from combinatorics on words (for example, [16]). We give some notations at first:
$\mathbb{Z}$ — set of integers,
$\mathbb{Z}_+ = \{x \mid x \in \mathbb{Z} \,\&\, x > 0\}$,
$\mathbb{N} = \mathbb{Z}_+ \cup \{0\}$.
With A we denote a finite *alphabet*, i.e., a finite non-empty set $\{a_0, a_1, a_2, ..., a_n\}$. We assume that A contains of at least two symbols. *One-sided* (from left to right) infinite sequence or word over A is any total map $\omega : N \to A$. Set A^ω contains all infinite words. If the word $u = u_0u_1u_2... \in A^\omega$, where $u_0,\ u_1,\ u_2, ... \in A$, then finite word $u_0u_1u_2...u_n$ is called the *prefix* of u of length $n+1$.

$Pref(u) = \{\lambda,\ u_0,\ u_0u_1,\ u_0u_1u_2,\ ...,\ u_0u_1u_2...u_n,\ ...\}$
is the set of all prefixes of word u.

Definition 2.1. ([15]) The mapping $d:\ A^\omega \times A^\omega \to R$ is called a *prefix metric* in set A^ω if

$$d(u,v) = \begin{cases} 2^{-m}, & u \neq v, \\ 0, & u = v, \end{cases}$$

where $m = \max\{|\omega| \mid \omega \in Pref(u) \cap Pref(v)\}$.

The term "chaos" in reference to functions was first used in Li and Yorke's paper "Period three implies chaos" ([14], 1975). We use the following definition of R. Devaney [10]. Let (X, ρ) be metric space.

Definition 2.2. ([10]) The function $f : X \to X$ is *chaotic* if
a) the periodic points of f are dense in X,
b) f is topologically transitive,
c) f exhibits sensitive dependence on initial conditions.

We note that

Definition 2.3. Let $A,\ B \subset X$ and $A \subset B$. Then A is *dense* in B if

$$\forall x \in B\, \forall \varepsilon > 0\, \exists y \in A \quad \rho(x,y) < \varepsilon.$$

Definition 2.4. ([10]) The function $f : X \to X$ is *topologically transitive* on X if

$$\forall x, y \in X\, \forall \varepsilon > 0\, \exists z \in X\, \exists n \in \mathbb{N}\, (\rho(x,z) < \varepsilon \,\&\, \rho(f^n(z), y) < \varepsilon).$$

Definition 2.5. ([10]) The function $f : X \to X$ *exhibits sensitive dependence on initial conditions* on X if

$$\exists \delta > 0\, \forall x \in X\, \forall \varepsilon > 0\, \exists y \in X\, \exists n \in \mathbb{N} \quad (\rho(x,y) < \varepsilon \,\&\, \rho(f^n(x), z) > \delta).$$

Devaney's definition is not the only classification of a chaotic map. For example, another definition can be found in [18]. Mappings with only one property — sensitive dependence on initial conditions — also are considered as chaotic ([11]). In [4] is shown that for continuous functions the defining characteristics of chaos requires less conditions than in general case.

Theorem 2.1. ([4]) Let A be an infinite subset of metric space X and $f : A \to A$ to be continuous. If f is topologically transitive on A and the set of periodic points of f is dense in A, then f is chaotic on A.

A well known chaotic mapping in symbol space A^ω is a shift mapping ([10], [12], [13], [18]). However in symbol space exists other chaotic mappings. The basic change is to consider the process (physical or social phenomenon) not only at a set of times which are equally spaced, for example, at unit time apart (a shift mapping), but at a set of times which are not equally spaced, for example, if we cannot fixed unit time. There is a philosophy of modelling in which we study idealized systems that have properties that can be closely approximated by physical systems (by Bai-Lin Hao [2]: Symbolic dynamics may be understood as a kind of coarse-grained description of the time evolution of a dynamical system.).

III. Jump mappings

The notion of increasing mapping had been introduced in [5]. Let

$$f_\omega(x) = x_{f(0)}x_{f(1)}x_{f(2)}\dots x_{f(i)}\dots, \qquad i \in \mathbb{N}, \quad x \in A^\omega.$$

In this case the function f is called *the generator function* of mapping f_ω.

Definition 3.1. A function $f : \mathbb{N} \to \mathbb{N}$ is called *positively increasing function* if

$$0 < f(0) \text{ and } \forall i\, \forall j\ (\, i < j \Rightarrow f(i) < f(j)\,).$$

Mapping $f_\omega : A^\omega \to A^\omega$ is called an *increasing mapping* if its generator function $f : \mathbb{N} \to \mathbb{N}$ is positively increasing.

The function $f(x) = 5x$, $x \in \mathbb{N}$, is increasing function in ordinary sense. Since $0 = f(0)$ it is not positively increasing function. If we consider $f(x) = 5x+2$ as a generator function, then the corresponding generated mapping is increasing, it is $f_\omega : A^\omega \to A^\omega$, where

$$\forall s = s_0s_1s_2... \in A^\omega :\ f_\omega(s) = s_2s_7s_{12}...s_{5i+2}..., \quad i \in \mathbb{N}.$$

The well known shift map is increasing mapping in one-sided infinite symbol space A^ω, in this case the generator function is a positively increasing function $f : \mathbb{N} \to \mathbb{N}$, where $f(x) = x + 1$.

In [5] we have proved that increasing mapping $f_\omega : A^\omega \to A^\omega$ is chaotic in the set A^ω therefore as the consequence shift map is chaotic too. It is possible for increasing mapping (from two symbols 0 and 1 space) to construct corresponding mapping in unit segment that is chaotic ([7]).

In [6] and [9] we have considered another class of chaotic mappings — class of k-switch mappings. But in [8] we have considered combination of increasing mapping and k-switch mapping, this class of increasing-switch mappings is chaotic too. But if the generator function $f : \mathbb{N} \to \mathbb{N}$ of mapping $f_\omega : A^\omega \to A^\omega$ is such that $f(0) = 0$, then the generated mapping f_ω is not chaotic in the set A^ω (see [5]). Even more, if $\exists i \in \mathbb{N}\ f(i) = i$, then the generated mapping f_ω is not chaotic in the set A^ω.

Now we define a new class of mappings in symbol space A^ω.

Definition 3.2. Mapping $f_{i,k} : A^\omega \to A^\omega$ is called i,k*-jump mapping* if its generator function $f : \mathbb{N} \to \mathbb{N}$ is such that

$$f(x) = \begin{cases} x+1, & 0 \le x < i-1, \\ x+2, & i-1 \le x \le i+k-1, \\ i, & x = i+k, \\ x+1, & i+k < x, \end{cases} \tag{3.1}$$

where $i, k \in \mathbb{N}$ and $i \ge 1$ and $k > i$. If $i = 1$, then definition of generator function do not contain the first row.

In other words, firstly, this mapping is a shift and secondly, $i+1$th symbol x_i jumps to $i+k+1$ place (we remark that the word x begins with symbol x_0). For example, let $x = x_0x_1x_2... \in A^\omega$, then

$$f_{3,4}(x) = x_1x_2x_4x_5x_6x_8x_9\mathbf{x_3}x_{10}x_{11}x_{12}...,$$

$$\text{or } f_{1,5} = x_2x_3x_4x_5x_6x_7\mathbf{x_1}x_8x_9... \,.$$

Unlike previous mapping classes (increasing mappings and k-switch mappings) in this case we can observe two different behaviours of i,k-jump mapping orbits. For example, we consider mappings $f_{2,2}$ and $f_{2,3}$. Let $x = x_0x_1x_2x_3...$. In the table below is shown first four iterations of $f_{2,2}(x)$:

			x_i		x_{i+k}				
0	x_0	x_1	$\mathbf{x_2}$	x_3	x_4	x_5	x_6	x_7	...
1	x_1	x_3	$\mathbf{x_4}$	x_5	$\mathbf{x_2}$	x_6	x_7	x_8	...
2	x_3	x_5	$\mathbf{x_2}$	x_6	$\mathbf{x_4}$	x_7	x_8	x_9	...
3	x_5	x_6	$\mathbf{x_4}$	x_7	$\mathbf{x_2}$	x_8	x_9	x_{10}	...
4	x_6	x_7	$\mathbf{x_2}$	x_8	$\mathbf{x_4}$	x_9	x_{10}	x_{11}	...

In the next table is shown first nine iterations of $f_{2,3}(x)$:

			x_i			x_{i+k}			
0	x_0	x_1	$\mathbf{x_2}$	x_3	x_4	x_5	x_6	x_7	...
1	x_1	x_3	$\mathbf{x_4}$	x_5	x_6	$\mathbf{x_2}$	x_7	x_8	...
2	x_3	x_5	$\mathbf{x_6}$	$\mathbf{x_2}$	x_7	$\mathbf{x_4}$	x_8	x_9	...
3	x_5	$\mathbf{x_2}$	$\mathbf{x_7}$	$\mathbf{x_4}$	x_8	x_6	x_9	x_{10}	...
4	$\mathbf{x_2}$	$\mathbf{x_4}$	$\mathbf{x_8}$	x_6	x_9	x_7	x_{10}	x_{11}	...
5	$\mathbf{x_4}$	x_6	$\mathbf{x_9}$	x_7	x_{10}	x_8	x_{11}	x_{12}	...
6	x_6	x_7	$\mathbf{x_{10}}$	x_8	x_{11}	x_9	x_{12}	x_{13}	...
7	x_7	x_8	$\mathbf{x_{11}}$	x_9	x_{12}	x_{10}	x_{13}	x_{14}	...
8	x_8	x_9	$\mathbf{x_{12}}$	x_{10}	x_{13}	x_{11}	x_{14}	x_{15}	...
9	x_9	x_{10}	$\mathbf{x_{13}}$	x_{11}	x_{14}	x_{12}	x_{15}	x_{16}	...

If we allow that k is negative integer, for example, $f_{7,-5}(x)$:

			x_{i+k}					x_i	
0	x_0	x_1	x_2	x_3	x_4	x_5	x_6	x_7	...
1	x_1	x_2	x_7	x_3	x_4	x_5	x_6	x_8	...
2	x_2	x_7	x_8	x_3	x_4	x_5	x_6	x_9	...
3	x_7	x_8	x_9	x_3	x_4	x_5	x_6	x_{10}	...

If $\exists i \in \mathbb{N}\ f(i) = i$, then the generated mapping f_ω is not chaotic in the set A^ω (see [5]) therefore the last mapping $f_{7,-5}$ is not chaotic. In the orbit of $f_{2,3}$ all symbols "travel" and disappear with iterations of $f_{2,3}$. But in the orbit of mapping $f_{2,2}$ symbols x_2 and x_4 change places and do not exist an iteration in which these symbols disappear from the orbit. Similar behaviour is for every i,k-jump mapping if k is even number — this observation follows from second row of definition of generator function, i.e. the set $\{x_i, x_{i+2}, ..., x_{i+k}\}$ is set of symbols that disappear from the orbit of $f_{i,k}$ if k is even number.

Theorem 3.1. If k is an even number, then the i,k-jump mapping $f_{i,k} : A^\omega \to A^\omega$ is not topologically transitive in the set A^ω.

Proof: We will prove the opposite of topological transitivity:

$$\exists x\, \exists y\, \exists \varepsilon > 0\, \forall z\, \forall n \in \mathbb{N}\ (d(x,z) \geq \varepsilon\ \vee\ d(f^n_{i,k}(z), y) \geq \varepsilon).$$

Since the alphabet A contains at least two symbols we assume that $x = x_0x_1x_2... \in A^\omega$ and $y = y_0y_1y_2... \in A^\omega$ are chosen so that

$$x_i = x_{i+2} = x_{i+4} = ... = x_{i+k} \text{ and } y_{i+k} \neq x_i.$$

We chose $\varepsilon = 2^{-(i+k)}$.

Let $z \in A^\omega$ be an arbitrary word. Note that:

$$\text{if } \exists m < i+k : z_m \neq x_m, \text{ then } d(z,x) \geq 2^{-m} > 2^{-(i+k)} = \varepsilon.$$

Two cases are possible:
1) $z_j = x_j$, $j = 0,1,2,...,i+k-1$ and $z_{i+k} \neq x_{i+k}$, then by definition of prefix metric $d(z,x) = 2^{-(i+k)} \geq \varepsilon$.
2) if $z_j = x_j$, $j = 0,1,2,...,i+k$, then we cannot state that $d(z,x) \geq \varepsilon$. But in this case we have $z_i = z_{i+2} = ... = z_{i+k}$ and

$$\forall n \in \mathbb{N} \quad f^n_{i,k}(z) = z_{f^n(0)} z_{f^n(1)} ... z_{f^n(i+k)} \cdots$$
$$\text{where } z_{f^n(i+k)} = z_{i+k} = x_i \neq y_{i+k}$$

therefore

$$d(f^n_{i,k}(z), y) \geq 2^{-(i+k)} = \varepsilon.$$

■

We conclude that if k is even number then i,k-jump mapping is not chaotic. But for an arbitrary k this mapping is continuous.

Theorem 3.2. The i,k-jump mapping $f_{i,k} : A^\omega \to A^\omega$ is continuous in set A^ω.

Proof: We fix word $u \in A^\omega$ and $\varepsilon > 0$. We need to prove

$$\exists \delta > 0\, \forall v \in A^\omega\ (d(u,v) < \delta \Rightarrow d(f_{i,k}(u), f_{i,k}(v)) < \varepsilon).$$

We choose m such that $2^{-m} < \varepsilon$ and assume that $0 < \delta \leq 2^{-(m+2)}$. If $d(u,v) < \delta$, then by definition of prefix metric follows that $u_j = v_j$, $j = 0,1,...,m,m+1$. From definition of i,k-jump mapping

$$f_{i,k}(u) = u_1u_2...u_{i-1}u_{i+1}u_{i+2}...u_{i+k}u_{i+k+1}u_iu_{i+k+2}\cdots,$$
$$f_{i,k}(v) = v_1v_2...v_{i-1}v_{i+1}v_{i+2}...v_{i+k}v_{i+k+1}v_iv_{i+k+2}\cdots.$$

Independently of the choice of i and k in the both sequences $f_{i,k}(u)$ and $f_{i,k}(v)$ m symbols are equal therefore $d(f_{i,k}(u), f_{i,k}(v)) \leq 2^{-m} < \varepsilon$. ■

The authors of [4] have demonstrated that for continuous functions, the defining characteristics of chaos are topological transitivity and density of the set of periodic points (Theorem 2.1). If k is an odd number we show that these two properties possess to i,k-jump mapping.

Theorem 3.3. If k is odd, then the i,k-jump mapping $f_{i,k} : A^\omega \to A^\omega$ is topologically transitive in set A^ω.

Proof: We fix words $u, v \in A^\omega$ and $\varepsilon > 0$. We need to prove that

$$\exists z \in A^\omega\, \exists n \in \mathbb{N} \quad (d(z,u) < \varepsilon\ \&\ d(f^n_{i,k}(z), v) < \varepsilon).$$

We choose m such that $2^{-m} < \varepsilon$. Then there exists a word $z \in A^\omega$ such that $u_j = z_j$, $j = 0,1,...,m-1$, and therefore $d(z,u) \leq 2^{-m} < \varepsilon$. What should be other symbols of word z, it depends on i, k and m.

(1) If $2m \leq i$, then we choose $n = m$. If $2m \leq i$, then after m iterations of z first m symbols z_0, z_1, z_2,..., z_{m-1} will be lost from word z and another symbols z_m, z_{m+1},...,z_{2m-1} (without changes) will come in the first m places. If we define $z_m = v_0$, $z_{m+1} = v_1$, ..., $z_{2m-1} = v_{m-1}$, then $d(f^n_{i,k}(z), v) \leq 2^{-m} < \varepsilon$. Hence in this case

$$z = u_0u_1u_2...u_{m-1}v_0v_1v_2...v_{m-1}z_{2m}z_{2m+1}\cdots,$$

where $z_{2m+j} \in A^\omega$, $j = 0,1,...,$ are freely chosen.

(2) If $m \leq i$, then by m iterations of z first m symbols $z_0 = u_0$, $z_1 = u_1$, $z_2 = u_2$,..., $z_{m-1} = u_{m-1}$ will be lost from word z and another symbols z_j, $j \geq m$ will come in the first m places - these symbols must be equal with v_0, v_1, v_2,..., v_{m-1}. Clearly, this can be done since sequence $f^m_{i,k}(z)$ in each place have symbols with different indices.

For example, if $m = 3$, $i = 4$, $k = 3$, then $n = m = 3$ and first iterations are as follows:

					z_i			z_{i+k}		
0	z_0	z_1	z_2	z_3	z_4	z_5	z_6	z_7	z_8	...
1	z_1	z_2	z_3	z_5	z_6	z_7	z_8	z_4	z_9	...
2	z_2	z_3	z_5	z_7	z_8	z_4	z_9	z_6	z_{10}	...
3	z_3	z_5	z_7	z_4	z_9	z_6	z_{10}	z_8	z_{11}	...

We define $z_3 = v_0$, $z_5 = v_1$, $z_7 = v_2$, then

$$z = u_0u_1u_2v_0z_4v_1z_6v_2z_8z_9...,$$

where z_4, z_6, z_8, z_9,...$\in A^\omega$ are chosen freely, and z is the searched word.

(3) If $m > i$, then by m iterations of z not all of $z_0 = u_0$, $z_1 = u_1$,..., $z_{m-1} = u_{m-1}$ will be lost from sequence $f^m_{i,k}(z)$. In this case n is a number of iteration such that all first m symbols of word z are lost from sequence $f^m_{i,k}(z)$. Since all symbols are lost from word z with iterations of $f_{i,k}$ then exist such n. But first m symbols of $f^n_{i,k}(z)$ must be equal with v_0, v_1,..., v_{m-1}.

For example, if $i = 3$, $k = 3$, $m = 5$, then:

				z_i			z_{i+k}			
0	u_0	u_1	u_2	u_3	u_4	z_5	z_6	z_7	z_8	...
1	u_1	u_2	u_4	z_5	z_6	z_7	u_3	z_8	z_9	...
2	u_2	u_4	z_6	z_7	u_3	z_8	z_5	z_9	z_{10}	...
3	u_4	z_6	u_3	z_8	z_5	z_9	z_7	z_{10}	z_{11}	...
4	z_6	u_3	z_5	z_9	z_7	z_{10}	z_8	z_{11}	z_{12}	...
5	u_3	z_5	z_7	z_{10}	z_8	z_{11}	z_9	z_{12}	z_{13}	...
6	z_5	z_7	z_8	z_{11}	z_9	z_{12}	z_{10}	z_{13}	z_{14}	...

In this case $n = 6$ and

$$z = u_0u_1u_2u_3u_4v_0z_6v_1v_2v_4z_{10}v_3z_{12}z_{13}\cdots,$$

where z_6, z_{10}, z_{12}, $z_{13}, ... \in A^\omega$ are chosen freely, and z is searched word. ■

Theorem 3.4. If k is odd, then the set of periodic points of i, k-jump mapping $f_{i,k} : A^\omega \to A^\omega$ is dense set in set A^ω.

Proof: Let $\varepsilon > 0$. There exists m such that $2^{-m} < \varepsilon$. Assume that $u \in A^\omega$ and $u_0 u_1 ... u_{m-1}$ is a prefix of word u of length m. We prove that there exists a word $x \in A^\omega$ with the same prefix (i.e., $d(u, x) \leq 2^{-m} < \varepsilon$) and this x is a periodic point for i, k-jump mapping with period m or greater than m.

Let $x = u_0 u_1 ... u_{m-1} x_m x_{m+1} x_{m+2} ... \in A^\omega$.

It is possible two cases:

(1) If $m \leq i$, then by m iterations of x the first m symbols u_0, u_1, u_2,..., u_{m-1} will be lost from word x. By definition of periodicity we need $f^m_{i,k}(x) = x$. Clearly, this is possible since in each place of sequence $f^m_{i,k}(x)$ are symbols with different indices. For, examples, $m = 3$, $i = 5$, $k = 3$. We consider the first $m = 3$ iterations of x by $f_{5,3}$:

						x_5			x_8	
0	u_0	u_1	u_2	x_3	x_4	x_5	x_6	x_7	x_8	...
1	u_1	u_2	x_3	x_4	x_6	x_7	x_8	x_9	x_5	...
2	u_2	x_3	x_4	x_6	x_8	x_9	x_5	x_{10}	x_7	...
3	x_3	x_4	x_6	x_8	x_5	x_{10}	x_7	x_{11}	x_9	...

x is periodic point with period $m = 3$ if

$$\begin{aligned}
&u_0 = x_3 = x_8 = x_9 = x_12 = ... = x_{3j} = ... \\
&u_1 = x_4 = x_5 = x_{10} = x_{13} = ... = x_{3j+1} = ... \\
&u_2 = x_6 = x_7 = x_{11} = x_{14} = ... = x_{3j+2} = ... \\
&j = 3, 4, 5, ...
\end{aligned}$$

Therefore

$$x = u_0 u_1 u_2 \; u_0 u_1 u_1 \; u_2 u_2 u_0 \; u_0 u_1 u_2 \; u_0 u_1 u_2 \; ...$$

is periodic point with period 3 and it is the searched word.

(2) If $m > i$, then it is possible to find smallest n such that $f^n_{i,k}(x)$ do not contain prefix symbols $u_0 u_1 u_2 ... u_{m-1}$ and we can find word x with period n ($n \geq m$) similar by as above. For example, $m = 4$, $i = 2$, $k = 3$. We consider first iterations of x by $f_{2,5}$ while prefix symbols are lost:

			x_2			x_5			
0	u_0	u_1	u_2	u_3	x_4	x_5	x_6	x_7	...
1	u_1	u_3	x_4	x_5	x_6	u_2	x_7	x_8	...
2	u_3	x_5	x_6	u_2	x_7	x_4	x_8	x_9	...
3	x_5	u_2	x_7	x_4	x_8	x_6	x_9	x_{10}	...
4	u_2	x_4	x_8	x_6	x_9	x_7	x_{10}	x_{11}	...
5	x_4	x_6	x_9	x_7	x_{10}	x_8	x_{11}	x_{12}	...

From this follows that $n = 5$. Word x is periodic point with period $n = 5$ if

$$\begin{aligned}
&u_0 = x_4 = x_{10} = x_{15} = ... = x_{5j} = ... \\
&u_1 = x_6 = x_{11} = x_{16} = ... = x_{5j+1} = ... \\
&u_2 = x_9 = x_{14} = x_{19} = ... = x_{5j+4} = ... \\
&u_3 = x_7 = x_{12} = x_{17} = ... = x_{5j+2} = ... \\
&x_5 = x_8 = x_{13} = x_{18} = ... = x_{5j+3} = ... \\
&j = 2, 3, 4, 5, ...; \; x_5 \in A^\omega \text{ is chosen freely.}
\end{aligned}$$

Therefore

$$x = u_0 u_1 u_2 u_3 u_0 \; x_5 u_1 u_3 x_5 u_2 \; u_0 u_1 u_3 x_5 u_2 \; u_0 u_1 u_3 x_5 u_2 \; ...$$

is periodic point with period 5 and it is the searched word. We note that in this case we can find more than one periodic point with desired properties. ■

Now we can assert by Theorem 2.1:

Theorem 3.5. If k is odd, then the i, k-jump mapping $f_{i,k}$: $A^\omega \to A^\omega$ is chaotic in the set A^ω.

IV. Conclusion

Let

$$x(t_0), \; x(t_1), \; ..., \; x(t_n), ...$$

be the flow of discrete signals. Suppose that we have experimentally observed the subsequence

$$x(T_0), \; x(T_1), \; ..., \; x(T_n), ...$$

If

$$T_0 = t_1, \; T_1 = t_2, \; ... T_n = t_{n+1}, ...,$$

then we have a shift map. If, for example,

$$T_0 = t_1, \; T_1 = t_3, \; T_2 = t_4, \; T_3 = t_2, \; T_4 = t_5 ... T_n = t_{n+1}, ...,$$

then we have jump mapping $f_{2,1}$. Notice if we have the infinite word

$$x = x_0 x_1 x_2 ... x_n ...$$

instead of flow of discrete signals, then we have respectively the infinite word

$$y = y_0 y_1 y_2 ... y_n ...$$

instead of the experimentally observed subsequence. If for all indices t

$$y_t = \begin{cases} x_1, & t = 0, \\ x_3, & t = 1, \\ x_4, & t = 2, \\ x_2, & t = 3, \\ x_{t+1}, & t \geq 4, \end{cases}$$

then we obtain a generator map of jump mapping $f_{2,1}$

$$f(t) = \begin{cases} 1, & t = 0, \\ 3, & t = 1, \\ 4, & t = 2, \\ 2, & t = 3, \\ t + 1, & t \geq 4, \end{cases}$$

and

$$y = f_{2,1}(x) = x_{f(0)} x_{f(1)} x_{f(2)} ... x_{f(n)} ....$$

We do not claim that the function $f(t)$ is chaotic on the real space $\mathbf{R}$ but we have proved that this function as a generator creates a chaotic map $f_{2,1}$ in the symbol space A^ω. We have proved something more: if k is odd, then every generator function (3.1) creates a chaotic map $f_{i,k}$ in the symbol space A^ω. But models with chaotic mappings are not predictable in long-term.

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