Research on the Correlation of the Fluctuating Density Gradient of the Compressible Flows

Yasuo Obikane

Abstract—This work is to study a roll of the fluctuating density gradient in the compressible flows for the computational fluid dynamics (CFD). A new anisotropy tensor with the fluctuating density gradient is introduced, and is used for an invariant modeling technique to model the turbulent density gradient correlation equation derived from the continuity equation. The modeling equation is decomposed into three groups: group proportional to the mean velocity, and that proportional to the mean strain rate, and that proportional to the mean density. The characteristics of the correlation in a wake are extracted from the results by the two dimensional direct simulation, and shows the strong correlation with the vorticity in the wake near the body. Thus, it can be concluded that the correlation of the density gradient is a significant parameter to describe the quick generation of the turbulent property in the compressible flows.

Keywords—Turbulence Modeling , Density Gradient Correlation, Compressible

I. INTRODUCTION

MANY difficulty of the computational fluid dynamics (CFD) are shown up in the compressible flows as reviewed [1]. Especially, it is annoying that the compressible computations require much more amount of computation time than the case of the incompressible flows. Though obtaining the force acting on the body by direct simulations for incompressible flows is getting to be realistic in industry, for complex compressible flows such as chemical reaction or interface flows it is even harder to formulate the problems because of the multi-scale, and multi-physics which sometimes includes aero acoustics problems. So, it is necessary to develop some practical multi-physics-modeling equation for turbulence or intermittent flows. In the present paper, the correlation of the fluctuating density gradient is studied since the gradient is one of the important structual parameters of the propagation of sound and interfaces in multi-phase flows. Its distribution will show the direction of propagation and the concentration of the elastic energy through the domain. In the present paper,the primitive variables are used for the modeling equation since tracing of the results is easier in practice.

II. DEFINITION OF NON-PRIMITIVE FLUCTUATING VARIABLES

The differential system of the compressible Navier Stokes equation is written as

$$\frac{\partial}{\partial t} X = \begin{pmatrix} \rho u_i \\ \rho u_i u_j + \delta_{ij} P - \tau_{ij} \\ (\epsilon + p)u_i - u_j \tau_{ij} + q_i \end{pmatrix}_{,i}$$

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$$\tag{1}$$

where i and j are 1,2,3, and "camma i" indicates the spetial derivative w.r.t. ith-direction.

$$X = \begin{pmatrix} \rho \\ \rho u_i \\ \epsilon \end{pmatrix}$$

$$\tag{2}$$

where

$$\epsilon = \rho(e + \frac{1}{2}u_i u_i), \tag{3}$$

and

$$p = \rho R T = \frac{R}{C_v}(\epsilon - \frac{1}{2}m_k m_k/\rho), \tag{4}$$

and

$$m_i = \rho u_i. \tag{5}$$

The representation of the flux with the non-primitive variables are

$$F_i = \begin{pmatrix} m_i \\ m_i m_j/\rho + \delta_{ij}\frac{R}{C_v}(\epsilon - \frac{1}{2}m_k m_k/\rho) - \tau_{ij} \\ (1 + \frac{R}{C_v})\epsilon m_i/\rho - \frac{R}{C_v}\frac{1}{2}m_k m_k m_i/\rho^2 - m_j \tau_{ij}/\rho + q_i \end{pmatrix}$$

$$\tag{6}$$

The m_i variables can be decomposed as

$$\begin{aligned} m_i &= (\hat{\rho} + \acute{\rho})(\hat{u}_i + \acute{u}_i) \\ &= (\hat{\rho} + \acute{\rho})(\hat{u}_i + \acute{u}_i) \\ &= \hat{\rho}\hat{u}_i + \hat{\rho}\acute{u}_i + \acute{\rho}\hat{u}_i + \acute{\rho}\acute{u}_i \quad (7) \\ &= \hat{m}_i + \acute{m}_i \quad (8) \end{aligned}$$

where the variables are defined as

$$\begin{aligned} \hat{m}_i &= \hat{\rho}\hat{u}_i & (9) \\ \acute{m}_i &= m_i(1) + m_i(2) + m_i(3) & (10) \\ m_i(1) &= \hat{\rho}\acute{u}_i & (11) \\ m_i(2) &= \acute{\rho}\hat{u}_i & (12) \\ m_i(3) &= \acute{\rho}\acute{u}_i & (13) \end{aligned}$$

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Here, the definition should be remarked. The averages are following:

$$\overline{m_i} = \overline{\hat{m_i}} + \overline{\acute{m_i}} \tag{14}$$

$$\overline{\hat{m_i}} = \overline{\hat{\rho}\hat{u_i}} \tag{15}$$

$$\overline{\acute{m_i}} = \overline{m_i(1)} + \overline{m_i(2)} + \overline{m_i(3)} \tag{16}$$

$$\overline{m_i(1)} = \overline{\hat{\rho}\acute{u_i}} = 0 \tag{17}$$

$$\overline{m_i(2)} = \overline{\acute{\rho}\hat{u_i}} = 0 \tag{18}$$

$$\overline{m_i(3)} = \overline{\acute{\rho}\acute{u_i}} \tag{19}$$

$$\overline{\acute{m_i}} = \overline{m_i(3)} \tag{20}$$

Thus, the present averaged mass flux is defined as

$$\overline{\rho u_i} = \overline{m_i} = \hat{m_i} + \hat{m_i(3)} = \hat{\rho}\hat{u_i} + \overline{\acute{\rho}\acute{u_i}} \tag{21}$$

The mean value is a product of the mean values.

III. Derivation of the correlation equation of the fluctuating density gradient(CFDG) $\overline{\rho_{,p}\rho_{,q}}$

The correlation of the gradient of the fluctuating density $\rho_{,i}$ is denoted as $\overline{\rho_{,p}\rho_{,q}}$. It is proportional to the square of the main wave number $<k>$, so the moment has order of $<k^2>$ at the maximum. It will be a good parameter to describe the wave property. Here, the moment equation for $\overline{\rho_{,p}\rho_{,q}}$ is derived as follows:

$$\frac{\partial}{\partial t}A = (B_i)_{,i} + (R_i)_{,i}, \tag{22}$$

where $A = \rho$, and $(B_i)_{,i} = -(\hat{\rho}u_i + \rho\hat{u_i} + \hat{\rho}\hat{u_i})_{,i}$, and $R_i = \overline{\acute{\rho}\acute{u_i}}$. By taking the gradient p on both side, and multiplying the gradient A of q to the equation, the following equation is obtained:

$$A,q\frac{\partial}{\partial t}A,p = A,qB,ip + A,qR,ii \tag{23}$$

By taking average, the second term in the right hand side vanishes, and then taking the symmetry, the following form is obtained:

$$\frac{\partial}{\partial t}\overline{A,qA,p} = \overline{A,qB,ip} + \overline{A,pB,iq}$$
$$= \overline{(AB_i),ipq} - \overline{A,pqB,i} - \overline{AB,ipq} \tag{24}$$

Then, the above equation can be represented by the primitive variables as

$$\frac{\partial}{\partial t}\overline{\rho_{,q}\rho_{,k}} = \overline{\rho_{,q}u_i}\hat{\rho}_{,ik} + \overline{\rho_{,q}u_{i,k}}\hat{\rho}_{,i} + \overline{\rho_{,q}u_{i,i}}\hat{\rho}_{,k}$$
$$+ \overline{\rho_{,q}u_{i,ik}}\hat{\rho} + \overline{\rho_{,q}\rho_{,ik}}\hat{u}_i + \overline{\rho_{,q}\rho_{,i}}\hat{u}_{i,k}$$
$$+ \overline{\rho_{,k}u_i}\hat{\rho}_{,iq} + \overline{\rho_{,k}u_{i,q}}\hat{\rho}_{,i} + \overline{\rho_{,k}u_{i,i}}\hat{\rho}_{,q}$$
$$+ \overline{\rho_{,k}u_{i,iq}}\hat{\rho} + \overline{\rho_{,k}\rho_{,iq}}\hat{u}_i + \overline{\rho_{,k}\rho_{,i}}\hat{u}_{i,q} \tag{25}$$

The terms depend on the moments as $\hat{\rho}, \hat{u}_i, \hat{u}_{i,k}, \hat{u}_{k,i}, \hat{\rho}_{,i}, \hat{\rho}_{,k}, \hat{\rho}_{,q}, \hat{\rho}_{ik}, \hat{\rho}_{iq}$. The moment will be used for the invariant modeling. For the simplicity, the terms depending on the mean density gradients are dropped. So, the approximated equation is as follows.

$$\frac{\partial}{\partial t}\overline{\rho_{,q}\rho_{,k}} = \overline{\rho_{,q}u_{i,ik}}\hat{\rho} + (\overline{\rho_{,q}\rho_{,k}})_{,i}\hat{u}_i + \overline{\rho_{,q}\rho_{,i}}\hat{u}_{i,k}$$
$$+ \overline{\rho_{,k}u_{i,iq}}\hat{\rho} + \overline{\rho_{,k}\rho_{,i}}\hat{u}_{i,q} \tag{26}$$

For two dimensional case,

$$\frac{\partial}{\partial t}\overline{\rho_{,1}\rho_{,2}} = (\overline{\rho_{,1}u_{1,12}} + \overline{\rho_{,1}u_{2,22}} + \overline{\rho_{,2}u_{1,11}} + \overline{\rho_{,2}u_{2,21}})\hat{\rho}$$
$$+ (\overline{\rho_{,1}\rho_{,2}})_{,1}\hat{u}_1 + (\overline{\rho_{,1}\rho_{,2}})_{,2}\hat{u}_2$$
$$+ \overline{\rho_{,1}\rho_{,2}}(\hat{u}_{1,1} + \hat{u}_{2,2}) + \overline{\rho_{,2}\rho_{,2}}\hat{u}_{2,1} + \overline{\rho_{,1}\rho_{,1}}\hat{u}_{1,2} \tag{27}$$

For the frozen turbulence approximation in the direction 1, replacing the convection term with a derivative of the convective time τ, then

$$\frac{\partial}{\partial \tau}\overline{\rho_{,1}\rho_{,2}} = (\overline{\rho_{,1}u_{1,12}} + \overline{\rho_{,1}u_{2,22}} + \overline{\rho_{,2}u_{1,11}} + \overline{\rho_{,2}u_{2,21}})\hat{\rho}$$
$$+ \overline{\rho_{,1}\rho_{,2}}(\hat{u}_{1,1} + \hat{u}_{2,2}) + \overline{\rho_{,2}\rho_{,2}}\hat{u}_{2,1} + \overline{\rho_{,1}\rho_{,1}}\hat{u}_{1,2} \tag{28}$$

In the equations (26) and (27) it can be noted that the productions are done by the mean shear and the mean distortion in the equation, especially the sign of the shear will contribute the sign of the correlation of the density gradient. i.e. the minus shear direction will give a minus value of the correlation. Here, it should be remarked that there is no explicit viscous effect, but it contains the density and velocity correlation that reflects the viscosity implicitly.

IV. Definition of the new Anisotropy Tensors and their Invariants

The eqation contains many moments, so some anisotropy tensors shown in the moment equation is defined as follows:

$$r_{ij} = \frac{\overline{\rho u_i \rho u_j}}{\overline{\rho u_i \rho u_i}} - 1/3\delta_{ij} \tag{29}$$

The anisotropic tensor form b_{ij} for incompressible flows is defined as

$$b_{ij} = \frac{\overline{u_i u_j}}{\overline{u_i u_i}} - 1/3\delta_{ij} \tag{30}$$

By the analogy, the fluctuating density gradient is used in adhoc to impose the directivity as follows:

$$d_{ij} = \overline{(\rho_{,i}\rho_{,j})}/\overline{(\rho_{,i}\rho_{,i})} - 1/3\delta_{ij} \tag{31}$$

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where the isotropy of the tensor d_{ij} takes some values as the wave is one direction as:

$$d_{11} = 2/3 \qquad (32)$$
$$d_{22} = d_{33} = -1/3 \qquad (33)$$

If the second rank tensor is denoted as d_{ij}, then the invariants are expressed as following:

The first invariant: $I_d = d_{ii}$

$$(34)$$

The second invariant: $II_d = (d_{ii}d_{jj} - d_{ii}^2)/2$

$$(35)$$

The third invariant: $III_d = (d_{ii}d_{jj}d_{kk} - 3d_{ii}d_{jj}^2 + 2d_{ii}^3)/3!$

$$(36)$$

Thus, any anisotropy function α is expressed with both I_d, II_d, III_d, and I_b, II_b, III_b.

V. APPROXIMATE INVARIANT FORM FOR THE FLOZEN TURBULENCE MODEL

The equation can be reformed the frozen equation in the strong shear and deformation in the previous section in adhoc as

$$
\begin{aligned}
\frac{\partial}{\partial \tau}\overline{\rho_{,p}\rho_{,k}} &= (\overline{\rho_{,q}u_{i,ik}} + \overline{\rho_{,k}u_{i,iq}})\hat{\rho} + \overline{\rho_{,q}\rho_{,i}}\hat{u}_{i,k} + \overline{\rho_{,k}\rho_{,i}}\hat{u}_{i,q} \\
&= (\alpha_d(I_d, II_d, III_d)d_{ij} + \alpha_b(I_b, II_b, III_b)b_{ij})\hat{\rho} \\
&+ \overline{\rho_{,q}\rho_{,i}}\hat{u}_{i,k} + \overline{\rho_{,k}\rho_{,i}}\hat{u}_{i,q}
\end{aligned}
$$

$$(37)$$

The invariant terms α_d, α_b vanishes if the flow is incompressible. Here, the analysis is assumed in the each invarinat, so it can be expand ed the following expression as

$$\alpha_d(I_d, II_d, III_d) = A11I_d + A12II_d + A13III_d + A14$$

$$(38)$$

$$\alpha_b(I_b, II_b, III_b) = A21I_b + A22II_b + A23III_b + A24$$

$$(39)$$

where the matrix elements Aij can be determined experimentally, numerically, or theoretically. The existence of the invariant constants is still an unsolved problem. However, some constants can be estimated by a direct numerical simulation in the following section.

VI. COMPUTATIONAL RESULTS

The numerical solutions are obtained in a wake past two plates in the wind tunnel with speed 45m/sec by using the two dimensional compressible Navier Stokes code for low Mach numbers [9], [10]. The results are shown in Fig.1. It can be seen the clear differences between the pattern of the off diagonal component $\overline{\rho_{,1}\rho_{,2}}$ from the pattern of the diagonal component $\overline{\rho_{,1}\rho_{,1}}$ and $\overline{\rho_{,2}\rho_{,2}}$ shown in Fig.1. The pattern of

the off diagonal component shows asymmetry in the wake though that of the diagonal one is almost symmetric. These results agree with the interpretation of the frozen turbulence model discussed in the previous section. The strong coupling is observed in the $\overline{\rho_{,i}\rho_{,j}}$ with the vortical wake that have a longer trail in Figure 2.

The spatial distribution of invariant II and the third are shown in Figure 2. The production of the correlation show some topography in the Figure 2, and the oblique lines dispatched from the bodies may show the directivity of the wave. The coupling is observed in the angle of the edge of the wake that has longer trails in Figure 3, 8. The order of the second invariant II was about -0.1 to -0.25, and the third invariants was -0.05 to -0.03 in Figure 3,4. The variations of the invariants II and III in figure 4, 8 imply the possibility for using the variables for the modeling.

VII. CONCLUSION

As the objectives of the study is to examine the possibility to use the correlation of the density gradient model in the compressible flow problems, the results infer that the correlation can be used for the interpretation of the compressible turbulent flows, and it can be a key parameter to show the different aspect of the compressibility. Besides, it is expected to proceed further developments of the method with the invariant II and III.
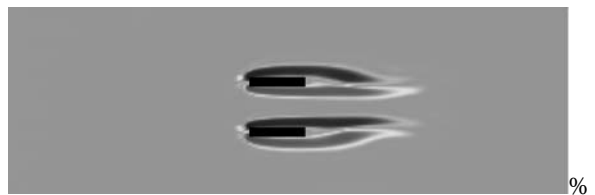

Fig. 1 The second Invarinat of d_{ij} (Early Stage)

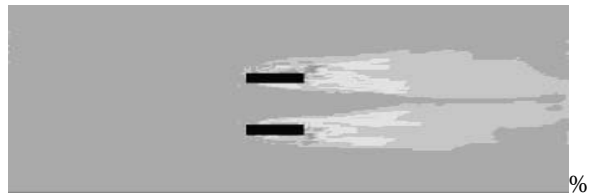

Fig. 2 Vorticity Distribution (Early Stage)

Fig. 3 The second Invarinat of d_{ij} (Developed)

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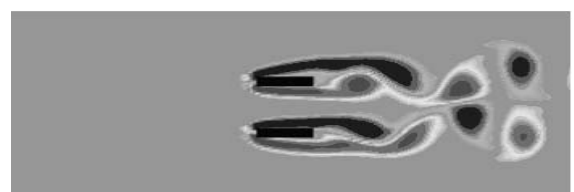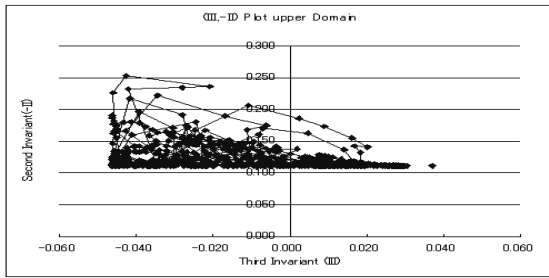Fig. 4 Vorticity Distibution (Developed)

Fig. 5 Second and Third Invariant in the upper domain (Well Developed)
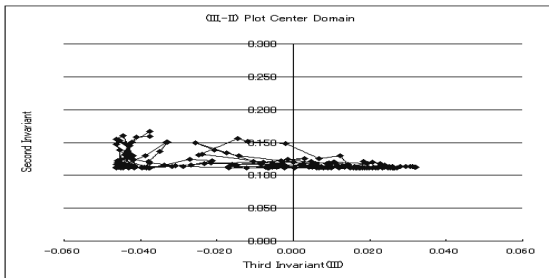

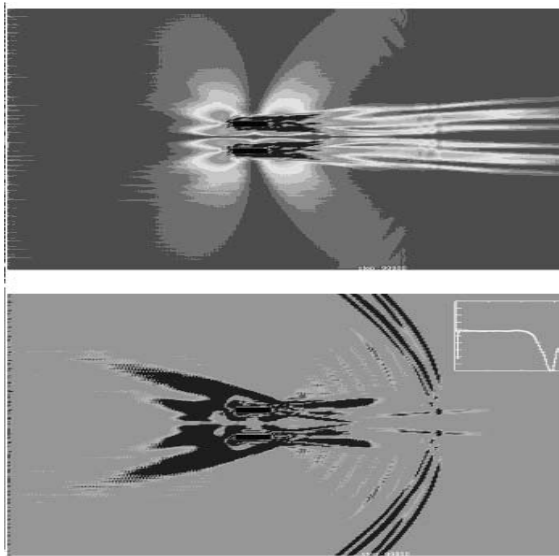Fig. 6 Second and Third Invariant in the channel of the flat plats(Well Developed)

Fig. 7 Second and Third Invariant in the channel of the flat plats(Well Developed in a wider Domain)

Fig. 8 The seoncd and Third Invariant, and Vorcicity Distibution (Well Developed)

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