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On Symmetry Analysis and Exact Wave Solutions of New Modified Novikov Equation

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Abstract—In this paper, we study a new modified Novikov equation for its classical and nonclassical symmetries and use the symmetries to reduce it to a nonlinear ordinary differential equation (ODE). With the aid of solutions of the nonlinear ODE by using the modified (G'/G)-expansion method proposed recently, multiple exact traveling wave solutions are obtained and the traveling wave solutions are expressed by the hyperbolic functions, trigonometric functions and rational functions.

Keywords—New Modified Novikov Equation, Lie Classical Method, Nonclassical Method, Modified (G'/G)-Expansion Method, Traveling Wave Solutions

I. Introduction

Nonlinear partial differential equations (NLPDEs) are widely used as models to describe many important dynamical systems in various fields of sciences, particularly in fluid mechanics, solid state physics, plasma physics and nonlinear optics. Exact solutions of NLPDEs of mathematical physics have attracted significant interest in the literature. The knowledge of these solutions of NLPDEs facilitates the verification of numerical solvers and aids in the stability analysis of the solutions. Traveling waves, whether their solution expressions are in explicit or implicit forms, are very interesting from the point of view of applications. Over the last years, much work has been done on the construction of exact solitary wave solutions and periodic wave solutions of nonlinear physical equations. The investigation of new exact solutions of NLPDEs may help one to find new phenomena. Many methods have been developed by mathematicians and physicists to find special solutions of NLPDEs, such as the inverse scattering method [1], the tanh-function method [2], the extended tanh-function method [3], Exp-function method [4], sine-cosine method [5] and the homogeneous balance method [6]. Recently, Wang et al. [7] proposed a new method called the (G'/G)-expansion method to construct traveling wave solutions for NLPDEs. The method is based on the homogeneous balance principle and linear ordinary differential equation (LODE) theory. It is assumed that the traveling wave solutions can be expressed by a polynomial in (G'/G) and that G'' satisfies a second order LODE $G'' + \lambda G' + \mu G = 0$. The degree of the polynomial can be determined by the homogeneous balance between the highest order derivative and nonlinear terms appearing in the given NLPDEs. The coefficients of the polynomial can be obtained by solving a set of algebraic equations. More details are in Section 4. As we mentioned above, a considerable research work has been devoted to finding the exact solutions of nonlinear partial differential equations, among which there is a special attention to the study of integrable non-evolutionary partial differential equations of the form

$$(1-{D_x}^2)u_t = G(u, u_x, u_{xx}, u_{xxx}, ...), \;\; u = u(x,t), \;\; D_x = \frac{\partial}{\partial x}, \tag{1}$$

where G is a function of u and its derivatives with respect to x. The most famous examples of this type of equations are the Camassa-Holm (CH) equation [8]

$$(1-{D_x}^2)u_t = 3uu_x - 2u_x u_{xx} - uu_{xxx}, \tag{2}$$

and the Degasperis-Procesi equation (DP) [9]

$$(1-{D_x}^2)u_t = 4uu_x - 3u_x u_{xx} - uu_{xxx}, \tag{3}$$

which are found to possess remarkable properties of integrable equations. Wazwaz [10,11] further advanced the studies of the DP and CH equations. Very recently, a new partial differential equation

$$u_t - u_{xxt} = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^2 u_x, \tag{4}$$

was discovered by Vladimir Novikov in a symmetry classification of nonlocal PDEs with cubic nonlinearity [12]. Novikov found the first a few symmetries for Eq. (4) and subsequently he found a scalar Lax pair for it, which proves that the equation is integrable. Integrability and multipeakon solutions of this equation have been studied in [13] and [14]. Like the Camassa-Holm and the Degasperis-Procesi equations, this new equation admits peakon solutions, but it has nonlinear terms that are cubic, rather than quadratic. Infinitely many conserved quantities are found in [12] as well as a bi-Hamiltonian structure. In the work of [14], the explicit multipeakon solutions of Novikov equation were obtained by using the matrix Lax pair found in [13].

In this paper, we will use the combination of Lie Classical method, Nonclassical method and modified (G'/G)-expansion method which is different from the described method in [7,15] to construct more traveling wave solutions for the following equation based on the Novikov equation of the form

$$u_t - u_{xxt} = u^2 u_{xxx} + 3uu_x u_{xx} - 4u^4 u_x, \tag{5}$$

namely the modified Novikov (mN) equation that is also a type (1) equation, but different to Novikov equation in that it has a nonlinear term $u^4 u_x$ instead of $u^2 u_x$.

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The rest of this paper is structured as follows. Resorting to the classical Lie group theory, we try to find out the symmetries of the mN equation in the next section and by using the symmetries, equation (5) is reduced to a nonlinear ODE system. We have also studied the mN equation for its nonclassical symmetries as in Section 3. In Section 4, by using the modified (G'/G)-expansion method, more explicit traveling wave solutions of Eq. (5) are found out, which are expressed by the hyperbolic functions, the trigonometric functions and rational functions. In Section 5, a brief conclusion and discussion are made for the results obtained in this paper.

II. Classical Lie Symmetry Analysis

In modern mathematics with ramifications to several areas of mathematics, physics and other sciences, the study of symmetry analysis of high-dimensional differential equations, such as finding symmetries, symmetry groups of transformation, symmetry reductions, and construction group invariant solutions, etc., has been very important. Nowadays, there are three basic and effective methods of finding symmetry reductions for the given nonlinear systems: the classical Lie group method [16-21], the non-classical Lie group method [22, 23] and the Clarkson and Kruskals direct method [24]. And Lou has improved the direct method [25, 26] which is based on Lax pairs. In this section, we will use Lie classical method to investigate the symmetry reductions of mN equation. To Eq. (5), we consider the one-parameter local Lie group of transformations as

$$\begin{aligned} u^* &= u + \epsilon\eta(x,t,u) + O(\epsilon^2), \\ x^* &= x + \epsilon\xi(x,t,u) + O(\epsilon^2), \\ t^* &= t + \epsilon\tau(x,t,u) + O(\epsilon^2), \end{aligned} \tag{6}$$

where ϵ is a group parameter, $\eta(x,t,u), \xi(x,t,u), \tau(x,t,u)$ are infinitesimals and the associated infinitesimal generator is

$$X = \eta(x,t,u)\frac{\partial}{\partial u} + \xi(x,t,u)\frac{\partial}{\partial x} + \tau(x,t,u)\frac{\partial}{\partial t}. \tag{7}$$

It is required that the set of Eq. (5) be invariant under the transformations (6) and this yields a system of overdetermined linear equations for the infinitesimals $\eta(x,t,u), \xi(x,t,u), \tau(x,t,u)$ which are as follows:

$$\begin{aligned} &(i)\ \tau_x = 0,\ \tau_u = 0, \\ &(ii)\ \xi_u = 0, \\ &(iii)\ \eta_{uu} = 0, \\ &(iv)\ 2\xi_x - \eta_{xxu} = 0, \\ &(v)\ \xi_t + 3u^2\xi_x - \tau_t u^2 - \eta_{xxu}u^2 - 2u\eta = 0, \\ &(vi)\ -3\tau_t u - 3\eta - 3\eta_{xxu}u - 3u\eta_u + 9u\xi_x = 0, \\ &(vii)\ -3u\eta_x + 3u^2\xi_{xx} + 2\xi_{xt} - \eta_{tu} - 3u^2\eta_{xu} = 0, \\ &(viii)\ -2\eta_{xu} + \xi_{xx} = 0, \\ &(ix)\ -4u^4\xi_x - \xi_t + u^2\xi_{xxx} + 4u^4\tau_t - 3\eta_{xxu}u^2 \\ &-2\eta_{xtu} + \xi_{xxt} - 3u\eta_{xx} + 4u^4\eta_{xxu} + 16u^3\eta = 0, \\ &(x)\ \eta_t - u^2\eta_{xxx} - \eta_{xxt} + 4u^4\eta_x = 0. \end{aligned} \tag{8}$$

The general solution of this large system provides following forms for the infinitesimal elements η, ξ and τ:

$$\xi = l,\ \tau = k,\ \eta = 0, \tag{9}$$

where l and k are arbitrary constants. Now we can reduce the Eq. (5) to ODE using following characteristic equation:

$$\frac{dx}{\xi} = \frac{dt}{\tau} = \frac{du}{\eta}. \tag{10}$$

On solving characteristic equation, we have the following similarity variables for Eq. (5):

$$u(x,t) = F(\zeta),\ \ \zeta = kx - lt, \tag{11}$$

which reduce Eq. (5) to the following ODE:

$$\begin{aligned} &-lF'(\zeta) + k^2 lF'''(\zeta) - k^3F(\zeta)^2F'''(\zeta) - 3k^3F(\zeta)F'(\zeta)F''(\zeta) \\ &+4kF(\zeta)^4F'(\zeta) = 0, \end{aligned} \tag{12}$$

where the prime means differentiation with respect to ζ.

Remark: In this case we get trivial symmetries so we will get only traveling wave solutions of the equation (5). Therefore we conclude that the non-constant similarity reduction of the mN Equation equation obtainable using classical Lie method is the traveling wave solution given by (11) and for the solutions of reduced ODE (12), we will take the aid of modified (G'/G)-expansion method.

III. Nonclassical Symmetries

Motivated by the fact that symmetry reductions for many PDEs cannot be obtained by using classical symmetries, there have been several generalizations of the classical Lie group method for symmetry reductions. The nonclassical method was first introduced in [22] to study the symmetry reductions of the heat equation. A description of the method can be found in [22, 23]. The basic idea of the nonclassical method is that PDE (5) is augmented with the invariant surface condition

$$\xi u_x + \tau u_t - \eta = 0, \tag{13}$$

which is associated with the vector field (7). Requiring that both (5) and (13) be invariant under the transformation with infinitesimal generator (7), we can obtain an overdetermined system of nonlinear equations for the infinitesimals $\eta(x,t,u), \xi(x,t,u), \tau(x,t,u)$. The number of determining equations arising is smaller in the nonclassical method than in the classical method because there are fewer linearly independent expressions in the derivatives. Because all solutions of the classical determining equations necessarily satisfy the nonclassical determining equations, the solution set can be larger in the nonclassical case. We can distinguish two different cases.

Case 1: In the case $\tau \neq 0$, we can set $\tau = 1$ without loss of generality. The nonclassical method applied to (5) yields following determining equations for the infinitesimals:

$$\begin{aligned}
&(i)\ \xi_u=0,\ \eta_{uu}=0,\\
&(ii)\ \xi\eta_{xxu}+\xi_t-2u\eta-2\xi\xi_x+3u^2\xi_x-\eta_{xxu}u^2=0,\\
&(iii)\ -3\eta+3\eta_{xxu}\xi_u+9u\xi_x-3u\eta_u-3\eta_{xxu}u=0,\\
&(iv)\ 2\xi\eta_{xu}+3u^2\xi_{xx}-3u^2\eta_{xu}+2\xi_x\eta_{xxu}-\eta_u\eta_{xxu}-3u\eta_x\\
&+2\xi_x\eta_u-4{\xi_x}^2-\xi\xi_{xx}+2\xi_{xt}-\eta_{tu}=0,\\
&(v)\ -6u\eta_{xu}+3u\xi_{xx}=0,\\
&(vi)\ 4u^4\eta_x+\xi_{xx}\eta_x-u^2\eta_{xxx}+\eta_t-2\eta_x\eta_{xu}-\eta_{xx}\eta_{xxu}\\
&+2\xi_x\eta_{xx}-\eta_{xxt}=0
\end{aligned} \tag{14}$$

Solving these equations, we get

$$\tau=1,\ \ \xi=C_1,\ \ \eta=0, \tag{15}$$

where C_1 is arbitrary constant. Consequently, after solving the determining system, we can assert that for $\tau\neq 0$, we recover only the classical symmetries and that the nonclassical symmetries of the mN equation have been completely classified. We can state that Eq. (5) does not admit proper nonclassical symmetries with $\tau=1$.

Case 2: In this case, we set $\tau=0, \xi=1$ and so the invariant surface condition reduces to $u_x=\eta(x,t,u)$. Hence we obtain the differential consequences as follows:

$$\begin{aligned}
&u_{xx}=\eta_x+\eta_u u_x,\\
&u_{xxx}=\eta_{xx}+2\eta\eta_{xu}+\eta^2\eta_{uu}+\eta_u u_{xx},\\
&u_{xxt}=\eta_{xt}+\eta_{xu}u_t+\eta\eta_{tu}+\eta\eta_{uu}u_t+\eta_u(\eta_t+\eta_u u_t).
\end{aligned} \tag{16}$$

Applying nonclassical method to eq. (5) and equating coefficients of powers of u_t to zero then generates the following determining equations:

$$\begin{aligned}
&(i)\ \ -\eta_{xxu}\eta_{xu}-\eta\eta_{uu}\eta_{xu}\eta_u-2\eta_{xuu}\eta{\eta_u}^2-\eta{\eta_{uu}}^2\eta_x\\
&-\eta_{uu}\eta_{xu}\eta_x-\eta\eta_{xxu}\eta_{uu}-\eta^2{\eta_{uu}}^2\eta_u\\
&-2\eta_{xu}\eta_u-2\eta^2\eta_{xuu}\eta_{uu}-2\eta\eta_{uu}\eta_u-\eta\eta_{uu}{\eta_u}^3\\
&-\eta^3\eta_{uuu}\eta_{uu}-2\eta\eta_{xuu}\eta_{xu}-\eta_{xxu}{\eta_u}^2-\eta_{uu}{\eta_u}^2\eta_x\\
&-\eta^2\eta_{uuu}\eta_{xu}-\eta^2\eta_{uuu}\eta_u^2=0,
\end{aligned}$$

$$\begin{aligned}
&(ii)\ \ -3\,u^2\eta_{xu}\eta_x-5\eta^2u\eta_u^2-\eta\eta_{xxu}u^2{\eta_u}^2-\eta\eta_{uu}{\eta_u}^2\eta_t\\
&-\eta\eta_{uu}\eta_{xt}\eta_u-\eta_{xxu}u^2\eta_u\eta_x-\eta_{uu}u^2\eta_u{\eta_x}^2-\eta^4\eta_{uuu}u^2\eta_{uu}\\
&-\eta^3{\eta_{uu}}^2u^2\eta_u-\eta_{uu}\eta_u\eta_x\eta_t-\eta^3\eta_{uuu}u^2{\eta_u}^2-\eta^2{\eta_{uu}}^2u^2\eta_x\\
&-\eta^2\eta_{uuu}\eta_u\eta_t-\eta^2\eta_{uu}u^2{\eta_u}^3-\eta^2\eta_{uu}\eta_{tu}\eta_u+4\eta^2\eta_{uu}u^4\eta_u\\
&-3\eta^2\eta_{xxu}u\eta_u-2\eta\eta_{xxu}u^2\eta_{xu}-2\eta\eta_{xuu}\eta_u\eta_t-6\eta^3\eta_{xuu}u\eta_u\\
&-3\eta u^2\eta_{uu}\eta_x-2\eta\eta_{xuu}\eta_{xt}-\eta\eta_{uu}\eta_{tu}\eta_x-\eta_{xxu}\eta_{xt}\\
&-2\eta^3\eta_{xuu}u^2\eta_{uu}-6\eta^2\eta_{xuu}u\eta_x-4\eta^2\eta_{xuu}u^2\eta_{xu}\\
&-\eta_{tu}\eta_x+4u^4\eta_x+16\,u^3\eta^2+\eta_t-\eta_{xxt}-2\eta_{xu}\eta_t-3\eta^2\eta_x\\
&-2\eta\eta_{xtu}-3\eta\eta_{uu}u{\eta_x}^2-3\eta\eta_{xxu}u\eta_x-2\eta\eta_{xuu}u^2\eta_u\eta_x\\
&-8u\eta\eta_u\eta_x-2\eta\eta_{uu}u^2\eta_{xu}\eta_x-6\eta^2\eta_{uu}u\eta_x\eta_u-\eta\eta_{tu}\eta_u\\
&-10\eta^2u\eta_{xu}+4\eta\eta_{xxu}u^4-3\eta^2\eta_{uu}u^2\eta_u-2\eta^2\eta_{uu}u^2\eta_{xu}\eta_u\\
&-\eta^2\eta_{tuu}-3\eta\eta_{xxu}u^2-5u\eta\eta_{xx}-\eta^3u^2\eta_{uuu}-u^2\eta_{xxx}\\
&-2\eta\eta_{uu}\eta_t+4\eta^3\eta_{uuu}u^4-\eta^3\eta_{uuu}\eta_{tu}-\eta_{xxu}u^2\eta_{xx}\\
&-\eta_{uu}\eta_{xt}\eta_x-\eta\eta_{uu}u^2\eta_{xx}\eta_u-\eta^2\eta_{uuu}u^2\eta_u\eta_x-\eta^2\eta_{xxu}u^2\eta_{uu}\\
&-3\eta^2u^2\eta_{xuu}-\eta\eta_{xxu}\eta_{tu}+8\eta^2\eta_{xuu}u^4-2\eta^2\eta_{xuu}\eta_{tu}\\
&-3\eta^3\eta_{uu}u{\eta_u}^2-2\eta^3\eta_{uuu}u^2\eta_{xu}-3u{\eta_x}^2-5\eta^3u\eta_{uu}\\
&-2\eta\eta_{uu}u^2{\eta_u}^2\eta_x-2\eta\eta_{xuu}u^2\eta_{xx}-\eta^2\eta_{uuu}\eta_{xt}-\eta_{xxu}\eta_u\eta_t\\
&-3\eta^3\eta_{uuu}u\eta_x-2\eta^2\eta_{xuu}u^2{\eta_u}^2-3\eta^3\eta_u-3\eta^4\eta_{uuu}u\eta_u\\
&-\eta^2\eta_{uuu}u^2\eta_{xx}-\eta_{uu}u^2\eta_{xx}\eta_x+4\eta\eta_{uu}u^4\eta_x\\
&-3\eta u^2\eta_{xu}\eta_u=0.
\end{aligned} \tag{17}$$

On solving the above system of highly nonlinear equations, we get $\tau=0,\ \xi=1,\ \eta=0$. Hence, we do not get any new symmetries in this case. Thus, we proved that for mN equation, the nonclassical method provides us only improper symmetries that can also be derived by using Lie classical method.

IV. Exact Traveling Wave Solutions for Eq. (5) using Modified (G'/G)-Expansion method to Reduced ODE (12)

Assume that the solution of Eq. (12) can be expressed by a polynomial in (G'/G) as follows:

$$F(\zeta)=a_0+\sum_{i=1}^{m}\left\{a_i\left(\frac{G'(\zeta)}{G(\zeta)}\right)^i+b_i\left(\frac{G'(\zeta)}{G(\zeta)}\right)^{-i}\right\}, \tag{18}$$

where a_0, a_i, b_i are constants and the positive integer m can be determined by considering the homogeneous balance of the highest order derivatives and highest order nonlinear appearing in ODE (12). The function $G(\zeta)$ is the solution of the auxiliary LODE

$$G''(\zeta)+\mu G(\zeta)=0, \tag{19}$$

where μ is a constant to be determined and the nonzero parameter μ in Eq. (19) plays an essential role in the determination of the type of the solutions. Indeed,

1) If $\mu<0$ then we find the hyperbolic-type solutions and we have

$$\frac{G'}{G}=\sqrt{-\mu}\left(\frac{A\sinh\sqrt{-\mu}\zeta+B\cosh\sqrt{-\mu}\zeta}{A\cosh\sqrt{-\mu}\zeta+B\sinh\sqrt{-\mu}\zeta}\right), \tag{20}$$

2) If $\mu>0$ then we find trigonometric-type solutions and we have

$$\frac{G'}{G}=\sqrt{\mu}\left(\frac{A\cos\sqrt{\mu}\zeta-B\sin\sqrt{\mu}\zeta}{A\sin\sqrt{\mu}\zeta+B\cos\sqrt{\mu}\zeta}\right), \tag{21}$$

where A, B are arbitrary constants. Noticing the homogeneous balance of highest order derivatives and nonlinear terms appearing in Eq. (12), we get $m=\frac{1}{2}$. We then substitute $F(\zeta)=\phi(\zeta)^{\frac{1}{2}}$ into PDE (12) to get another PDE

$$\begin{aligned}
&4l\phi'(\zeta)\phi(\zeta)^2-3k^2l\phi'(\zeta)^3+6k^2l\phi'(\zeta)\phi''(\zeta)\phi(\zeta)-4k^2l\phi'''(\zeta)\phi(\zeta)^2\\
&+4k^3\phi(\zeta)^3\phi'''(\zeta)-16k\phi(\zeta)^4\phi'(\zeta)=0.
\end{aligned} \tag{22}$$

It is easy to prove that the balancing number of PDE (22) is a positive integer $m=2$. Consequently, it follows from (18) that

$$\begin{aligned}
&\phi(\zeta)=a_0+a_1\left(\frac{G'(\xi)}{G(\xi)}\right)+a_2\left(\frac{G'(\xi)}{G(\xi)}\right)^2+b_1\left(\frac{G'(\xi)}{G(\xi)}\right)^{-1}\\
&+b_2\left(\frac{G'(\xi)}{G(\xi)}\right)^{-2}.
\end{aligned} \tag{23}$$

On substituting (23) into ODE (22) and using linear ODE (19), collecting all terms with the same powers of (G'/G) together and equating their coefficients to zero, yield a system of algebraic equations for $a_0, a_1, a_2, b_1, b_2, \mu, k, l$ as follows:

$$\left(\frac{G'}{G}\right)^{-11}: \; -32\,kb_2{}^5\mu + 96\,k^3b_2{}^4\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{11}: \; 32\,ka_2{}^5 - 96\,k^3a_2{}^4 = 0,$$

$$\left(\frac{G'}{G}\right)^{-10}: \; -144\,kb_2{}^4b_1\,\mu + 312\,k^3b_1\,b_2{}^3\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{10}: \; 144\,ka_2{}^4a_1 - 312\,k^3a_1\,a_2{}^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{-9}: \; -256\,kb_1{}^2b_2{}^3\mu - 48\,k^2lb_2{}^3\mu^3 + 360\,k^3b_1{}^2b_2{}^2\mu^3$$
$$+160\,k^3b_2{}^4\mu^2 - 32\,kb_2{}^5 + 288\,k^3a_0\,b_2{}^3\mu^3 - 128\,ka_0\,b_2{}^4\mu = 0,$$

$$\left(\frac{G'}{G}\right)^{9}: \; -160\,k^3a_2{}^4\mu + 256\,ka_1{}^2a_2{}^3 + 48\,k^2la_2{}^3$$
$$+128\,ka_0\,a_2{}^4 + 32\,ka_2{}^5\mu - 360\,k^3a_1{}^2a_2{}^2 - 288\,k^3a_0\,a_2{}^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{-8}: \; -120\,k^2lb_1\,\mu^3b_2{}^2 - 144\,kb_2{}^4b_1 - 112\,ka_1\,b_2{}^4\mu$$
$$+288\,k^3a_1\,b_2{}^3\mu^3 + 168\,k^3b_1{}^3b_2\,\mu^3 - 448\,ka_0\,b_1\,b_2{}^3\mu$$
$$-224\,kb_1{}^3b_2{}^2\mu + 512\,k^3b_1\,b_2{}^3\mu^2 + 648\,k^3a_0\,b_2{}^2b_1\,\mu^3 = 0,$$

$$\left(\frac{G'}{G}\right)^{8}: \; 144\,ka_2{}^4a_1\,\mu - 288\,k^3a_2{}^3b_1 + 120\,k^2la_1\,a_2{}^2$$
$$-648\,k^3a_0\,a_2{}^2a_1 - 168\,k^3a_1{}^3a_2 + 224\,ka_1{}^3a_2{}^2 + 448\,ka_0\,a_1\,a_2{}^3$$
$$+112\,ka_2{}^4b_1 - 512\,k^3a_1\,a_2{}^3\mu = 0,$$

$$\left(\frac{G'}{G}\right)^{-7}: \; -128\,ka_0\,b_2{}^4 - 90\,k^2lb_1{}^2\mu^3b_2 + 8\,lb_2{}^3\mu$$
$$+64\,k^3b_2{}^4\mu - 96\,kb_2{}^4a_2\,\mu - 576\,ka_0\,b_1{}^2b_2{}^2\mu - 256\,kb_1{}^2b_2{}^3$$
$$+648\,k^3a_1\,b_2{}^2b_1\,\mu^3 + 432\,k^3a_0\,b_1{}^2b_2\,\mu^3 - 64\,k^2lb_2{}^3\mu^2$$
$$+288\,k^3a_2\,b_2{}^3\mu^3 + 288\,k^3a_0{}^2b_2{}^2\mu^3 + 72\,k^2la_2\,\mu^3b_2{}^2$$
$$+72\,k^2lb_2{}^2\mu^3a_1 - 120\,k^2lb_2{}^2\mu^3a_0 + 24\,k^3b_1{}^4\mu^3$$
$$+480\,k^3a_0\,b_2{}^3\mu^2 - 192\,ka_0{}^2b_2{}^3\mu - 384\,ka_1\,b_2{}^3b_1\,\mu$$
$$-96\,kb_1{}^4b_2\,\mu + 576\,k^3b_1{}^2b_2{}^2\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{7}: \; -288\,k^3a_2{}^3b_2 + 90\,k^2la_1{}^2a_2 + 120\,k^2la_2{}^2a_0$$
$$-64\,k^3a_2{}^4\mu^2 + 576\,ka_0\,a_1{}^2a_2{}^2 + 256\,ka_1{}^2a_2{}^3\mu$$
$$-576\,k^3a_1{}^2a_2{}^2\mu + 128\,ka_0\,a_2{}^4\mu + 192\,ka_0{}^2a_2{}^3$$
$$+96\,ka_1{}^4a_2 + 96\,ka_2{}^4b_2 + 64\,k^2la_2{}^3\mu - 8\,la_2{}^3$$
$$-648\,k^3a_2{}^2b_1\,a_1 - 288\,k^3a_0{}^2a_2{}^2 - 24\,k^3a_1{}^4$$
$$+384\,ka_1\,a_2{}^3b_1 - 480\,k^3a_0\,a_2{}^3\mu - 432\,k^3a_0\,a_1{}^2a_2 = 0,$$

$$\left(\frac{G'}{G}\right)^{-6}: \; -224\,kb_1{}^3b_2{}^2 - 16\,kb_1{}^5\mu + 648\,k^3a_2\,b_2{}^2b_1\,\mu^3$$
$$-480\,ka_1\,b_1{}^2b_2{}^2\mu - 192\,k^2lb_2{}^2\mu^3a_1 - 180\,k^2lb_1\,\mu^3b_2\,a_0$$
$$-160\,k^2lb_1\,\mu^2b_2{}^2 + 256\,k^3b_1{}^3b_2\,\mu^2 - 480\,ka_0{}^2b_1\,b_2{}^2\mu$$
$$-320\,ka_0\,b_1{}^3b_2\,\mu + 576\,k^3a_0\,a_1\,b_2{}^2\mu^3 + 72\,k^3a_0\,b_1{}^3\mu^3$$
$$-15\,k^2lb_1{}^3\mu^3 - 112\,ka_1\,b_2{}^4 + 20\,lb_1\,\mu\,b_2{}^2 - 320\,ka_0\,a_1\,b_2{}^3\mu$$
$$+360\,k^3a_0{}^2b_2\,b_1\,\mu^3 + 200\,k^3b_1\,b_2{}^3\mu - 320\,kb_1\,b_2{}^3a_2\,\mu$$
$$+432\,k^3a_1\,b_1{}^2b_2\,\mu^3 - 448\,ka_0\,b_1\,b_2{}^3 + 1056\,k^3a_0\,b_2{}^2b_1\,\mu^2$$
$$+472\,k^3a_1\,b_2{}^3\mu^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{6}: \; 320\,ka_1\,a_2{}^3b_2 + 480\,ka_1{}^2a_2{}^2b_1 - 360\,k^3a_0{}^2a_1\,a_2$$
$$+120\,k^2la_2{}^2b_1 + 320\,ka_0\,a_1{}^3a_2 - 1056\,k^3a_0\,a_2{}^2a_1\,\mu$$
$$+320\,ka_0\,a_2{}^3b_1 - 472\,k^3a_2{}^3b_1\,\mu + 180\,k^2la_1\,a_2\,a_0$$
$$-256\,k^3a_1{}^3a_2\,\mu + 224\,ka_1{}^3a_2{}^2\mu - 72\,k^3a_0\,a_1{}^3$$
$$+112\,ka_2{}^4b_1\,\mu - 200\,k^3a_1\,a_2{}^3\mu^2 + 448\,ka_0\,a_1\,a_2{}^3\mu$$
$$-648\,k^3a_2{}^2b_2\,a_1 + 15\,k^2la_1{}^3 + 160\,k^2la_1\,\mu\,a_2{}^2$$
$$-432\,k^3a_1{}^2b_1\,a_2 - 20\,la_1\,a_2{}^2 + 480\,ka_0{}^2a_1\,a_2{}^2$$
$$-576\,k^3a_0\,a_2{}^2b_1 + 16\,ka_1{}^5 = 0,$$

$$\left(\frac{G'}{G}\right)^{-5}: \; -192\,ka_0{}^2b_2{}^3 - 96\,kb_1{}^4b_2 + 32\,k^3b_1{}^4\mu^2$$
$$-96\,kb_2{}^4a_2 + 8\,lb_2{}^3 + 16\,lb_2{}^2\mu\,a_0 + 16\,lb_1{}^2\mu\,b_2$$
$$+192\,k^2la_2{}^2b_2 + 288\,k^3a_1{}^2b_2{}^2\mu^3 + 416\,k^3b_2{}^3a_2\,\mu^2$$
$$+72\,k^3a_1\,b_1{}^3\mu^3 - 128\,ka_1{}^2b_2{}^3\mu - 384\,ka_1\,b_2{}^3b_1$$
$$-16\,k^2lb_2{}^3\mu + 96\,k^3a_0{}^3b_2\,\mu^3 + 192\,k^3a_0\,b_2{}^3\mu$$
$$+216\,k^3b_1{}^2b_2{}^2\mu + 72\,k^3a_0{}^2b_1{}^2\mu^3 + 480\,k^3a_0{}^2b_2{}^2\mu^2$$
$$-576\,ka_0\,b_1{}^2b_2{}^2 - 128\,ka_0{}^3b_2{}^2\mu - 64\,ka_0\,b_1{}^4\mu$$
$$+16\,lb_2{}^2\mu\,a_1 + 16\,lb_2{}^2\mu\,a_2 - 192\,k^2lb_1\,\mu^3b_2\,a_1$$
$$+720\,k^3a_0\,a_1\,b_1\,b_2\,\mu^3 + 1032\,k^3a_1\,b_2{}^2b_1\,\mu^2 + 576\,k^3a_0\,a_2\,b_2{}^2\mu^3$$
$$-768\,ka_0\,a_1\,b_1\,b_2{}^2\mu - 256\,ka_1\,b_1{}^3b_2\,\mu - 256\,ka_0\,b_2{}^3a_2\,\mu$$
$$-384\,kb_1{}^2b_2{}^2a_2\,\mu - 152\,k^2lb_2{}^2\mu^2a_0 - 122\,k^2lb_1{}^2\mu^2b_2$$
$$-96\,k^2lb_2\,\mu^3a_0{}^2 + 672\,k^3a_0\,b_1{}^2b_2\,\mu^2 - 384\,ka_0{}^2b_1{}^2b_2\,\mu$$
$$-168\,k^2la_2\,\mu^3b_2{}^2 + 432\,k^3a_2\,b_1{}^2b_2\,\mu^3 - 36\,k^2lb_1{}^2\mu^3a_0 = 0,$$

$$\left(\frac{G'}{G}\right)^{5}: \; 128\,ka_2{}^3b_1{}^2 - 96\,k^2la_2{}^2b_2 + 16\,k^2la_2{}^3\mu^2$$
$$+36\,k^2la_1{}^2a_0 + 96\,k^2la_2\,a_0{}^2 - 432\,k^3a_1{}^2b_2\,a_2$$
$$-576\,k^3a_0\,a_2{}^2b_2 - 416\,k^3a_2{}^3b_2\,\mu - 216\,k^3a_1{}^2a_2{}^2\mu^2$$
$$-480\,k^3a_0{}^2a_2{}^2\mu - 192\,k^3a_0\,a_2{}^3\mu^2 + 96\,ka_2{}^4b_2\,\mu$$
$$+384\,ka_1{}^2a_2{}^2b_2 + 256\,ka_0\,a_2{}^3b_2 + 256\,ka_1{}^3a_2\,b_1$$
$$+96\,ka_1{}^4a_2\,\mu + 384\,ka_0{}^2a_1{}^2a_2 + 192\,ka_0{}^2a_2{}^3\mu$$
$$+122\,k^2la_1{}^2\mu\,a_2 + 192\,k^2la_1\,a_2\,b_1 + 152\,k^2la_2{}^2\mu\,a_0$$
$$-720\,k^3a_0\,a_1\,b_1\,a_2 - 1032\,k^3a_1\,a_2{}^2b_1\,\mu - 672\,k^3a_0\,a_1{}^2a_2\,\mu$$
$$+384\,ka_1\,a_2{}^3b_1\,\mu - 72\,k^3a_0{}^2a_1{}^2 - 72\,k^3a_1{}^3b_1$$
$$-16\,la_1{}^2a_2 - 8\,la_2{}^3\mu - 16\,la_2{}^2a_0 + 128\,ka_0{}^3a_2{}^2$$
$$-288\,k^3a_2{}^2b_1{}^2 - 32\,k^3a_1{}^4\mu - 96\,k^3a_0{}^3a_2$$
$$+64\,ka_0\,a_1{}^4 + 768\,ka_0\,a_1\,a_2{}^2b_1 + 576\,ka_0\,a_1{}^2a_2{}^2\mu = 0,$$

$$\left(\frac{G'}{G}\right)^{-4}: \; 4\,lb_1{}^3\mu + 20\,lb_1\,b_2{}^2 + 24\,k^3a_0{}^3b_1\,\mu^3$$
$$-320\,ka_0\,b_1{}^3b_2 - 96\,ka_0{}^2b_1{}^3\mu - 480\,ka_0{}^2b_1\,b_2{}^2$$
$$+96\,k^3a_0\,b_1{}^3\mu^2 + 88\,k^3b_1{}^3b_2\,\mu - 4\,lb_2{}^2\mu\,a_1$$
$$+72\,k^3a_2\,b_1{}^3\mu^3 + 160\,k^3a_1\,b_2{}^3\mu - 48\,ka_1\,b_1{}^4\mu$$
$$-320\,kb_1\,b_2{}^3a_2 - 320\,ka_0\,a_1\,b_2{}^3 - 17\,k^2lb_1{}^3\mu^2$$
$$-480\,ka_1\,b_1{}^2b_2{}^2 - 16\,kb_1{}^5 - 39\,k^2lb_1{}^2\mu^3a_1$$
$$-168\,k^2la_2\,\mu^3b_1\,b_2 - 228\,k^2la_1\,\mu^3b_2\,a_0 + 864\,k^3b_1\,b_2{}^2a_2\,\mu^2$$
$$+360\,k^3a_1{}^2b_1\,b_2\,\mu^3 + 720\,k^3a_0\,a_2\,b_1\,b_2\,\mu^3 + 288\,k^3a_0{}^2a_1\,b_2\,\mu^3$$
$$+936\,k^3a_0\,a_1\,b_2{}^2\mu^2 + 144\,k^3a_0\,a_1\,b_1{}^2\mu^3 + 648\,k^3a_1\,b_1{}^2b_2\,\mu^2$$
$$-576\,ka_0\,b_1\,b_2{}^2a_2\,\mu - 288\,ka_1{}^2b_1\,b_2{}^2\mu - 192\,kb_1{}^3b_2\,a_2\,\mu$$
$$-192\,ka_1\,a_2\,b_2{}^3\mu - 576\,ka_0\,a_1\,b_1{}^2b_2\,\mu - 288\,ka_0{}^2a_1\,b_2{}^2\mu$$
$$+24\,lb_1\,\mu\,a_0\,b_2 - 40\,k^2lb_1\,\mu\,b_2{}^2 - 252\,k^2lb_1\,\mu^2b_2\,a_0$$
$$+576\,k^3a_1\,a_2\,b_2{}^2\mu^3 + 408\,k^3a_0\,b_2{}^2b_1\,\mu + 576\,k^3a_0{}^2b_2\,b_1\,\mu^2$$
$$-192\,ka_0{}^3b_1\,b_2\,\mu - 96\,k^2lb_2{}^2\mu^2a_1 - 24\,k^2lb_1\,\mu^3a_0{}^2 = 0,$$

$$\left(\frac{G'}{G}\right)^{4}: \; -4\,la_1{}^3 + 48\,ka_1{}^4b_1 - 12\,lb_1\,a_2{}^2 + 96\,ka_0{}^2a_1{}^3$$
$$-20\,la_1\,\mu\,a_2{}^2 - 24\,la_1\,a_0\,a_2 + 320\,ka_0\,a_1{}^3a_2\,\mu$$
$$+39\,k^2la_1{}^2b_1 + 17\,k^2la_1{}^3\mu + 24\,k^2la_1\,a_0{}^2$$
$$-360\,k^3a_1\,b_1{}^2a_2 - 160\,k^3a_2{}^3b_1\,\mu^2 - 144\,k^3a_0\,a_1{}^2b_1$$
$$-288\,k^3a_0{}^2b_1\,a_2 - 96\,k^3a_0\,a_1{}^3\mu - 88\,k^3a_1{}^3a_2\,\mu^2$$
$$+480\,ka_0{}^2a_1\,a_2{}^2\mu + 192\,ka_2{}^3b_1\,b_2 + 288\,ka_1\,a_2{}^2b_1{}^2$$
$$+192\,ka_1{}^3b_2\,a_2 + 288\,ka_0{}^2a_2{}^2b_1 + 192\,ka_0{}^3a_1\,a_2$$
$$-576\,k^3a_2{}^2b_1\,b_2 + 96\,k^2la_2{}^2\mu\,b_1 + 40\,k^2la_1\,\mu^2a_2{}^2$$
$$+168\,k^2la_2\,b_2\,a_1 + 228\,k^2lb_1\,a_2\,a_0 + 252\,k^2la_1\,\mu\,a_2\,a_0$$
$$-864\,k^3a_2{}^2b_2\,a_1\,\mu - 720\,k^3a_0\,a_1\,b_2\,a_2 - 24\,k^3a_0{}^3a_1$$
$$+16\,ka_1{}^5\mu - 72\,k^3a_1{}^3b_2 - 648\,k^3a_1{}^2a_2\,b_1\,\mu$$
$$-936\,k^3a_0\,a_2{}^2b_1\,\mu - 408\,k^3a_0\,a_2{}^2a_1\,\mu^2 - 576\,k^3a_0{}^2a_1\,a_2\,\mu$$
$$+320\,ka_1\,a_2{}^3b_2\,\mu + 576\,ka_0\,a_2{}^2b_2\,a_1 + 480\,ka_1{}^2a_2{}^2b_1\,\mu$$
$$+320\,ka_0\,a_2{}^3b_1\,\mu + 576\,ka_0\,a_1{}^2a_2\,b_1 = 0,$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{-3}:\ 8\,k^3b_1{}^4\mu - 8\,lb_2{}^2\mu\,a_2 + 32\,k^3b_2{}^3a_2\,\mu\\
&+456\,k^3a_1{}^2b_2{}^2\mu^2 + 72\,k^3a_1{}^2b_1{}^2\mu^3 - 150\,k^2lb_2\,\mu^3a_1{}^2\\
&+88\,k^3a_1\,b_1{}^3\mu^2 - 256\,ka_0\,b_2{}^3a_2 - 384\,kb_1{}^2b_2{}^2a_2\\
&-64\,ka_2{}^2b_2{}^3\mu - 256\,ka_1\,b_1{}^3b_2 - 32\,kb_1{}^4a_2\,\mu\\
&+8\,lb_2\,\mu\,a_0{}^2 + 8\,lb_1{}^2\mu\,a_0 + 288\,k^3a_2{}^2b_2{}^2\mu^3\\
&+192\,k^3a_0{}^2b_2{}^2\mu + 96\,k^3a_0{}^2b_1{}^2\mu^2 + 160\,k^3a_0{}^3b_2\,\mu^2\\
&-32\,ka_0{}^4b_2\,\mu - 64\,ka_0{}^3b_1{}^2\mu - 384\,ka_0{}^2b_1{}^2b_2\\
&+16\,lb_1\,\mu\,a_1\,b_2 + 80\,k^2lb_2{}^2\mu^2a_2 - 30\,k^2lb_1{}^2\mu^3a_2\\
&-240\,k^2la_2\,\mu^3b_2\,a_0 - 200\,k^2la_1\,b_1\,\mu^2b_2 - 128\,ka_1{}^2b_2{}^3\\
&-128\,ka_0{}^3b_2{}^2 + 16\,lb_2{}^2a_0 + 16\,lb_1{}^2b_2\\
&-60\,k^2la_1\,\mu^3b_1\,a_0 + 288\,k^3a_0\,a_1{}^2b_2\,\mu^3 + 288\,k^3a_0{}^2a_2\,b_2\,\mu^3\\
&+768\,k^3a_0\,b_2{}^2a_2\,\mu^2 - 64\,ka_0\,b_1{}^4 + 144\,k^3a_0\,a_2\,b_1{}^2\mu^3\\
&+312\,k^3a_1\,b_2{}^2b_1\,\mu + 480\,k^3b_1{}^2b_2\,a_2\,\mu^2 + 720\,k^3a_1\,a_2\,b_1\,b_2\,\mu^3\\
&+72\,k^3a_0{}^2a_1\,b_1\,\mu^3 - 384\,ka_1\,a_2\,b_2{}^2b_1\,\mu - 384\,ka_0{}^2a_1\,b_1\,b_2\,\mu\\
&-128\,ka_0\,a_1\,b_1{}^3\mu - 192\,ka_1{}^2b_1{}^2b_2\,\mu - 768\,ka_0\,a_1\,b_1\,b_2{}^2\\
&-384\,ka_0\,b_1{}^2b_2\,a_2\,\mu - 192\,ka_0{}^2b_2{}^2a_2\,\mu - 192\,ka_0\,a_1{}^2b_2{}^2\mu\\
&-38\,k^2lb_1{}^2\mu\,b_2 - 40\,k^2lb_1{}^2\mu^2a_0 - 8\,k^2lb_2{}^2\mu\,a_0\\
&-160\,k^2lb_2\,\mu^2a_0{}^2 + 240\,k^3a_0\,b_1{}^2b_2\,\mu + 1104\,k^3a_0\,a_1\,b_1\,b_2\,\mu^2 = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{3}:\ -16\,la_1{}^2\mu\,a_2 - 16\,la_2{}^2\mu\,a_0 + 30\,k^2la_1{}^2b_2\\
&-16\,lb_1\,a_1\,a_2 + 150\,k^2la_2\,b_1{}^2 - 144\,k^3a_0\,a_1{}^2b_2\\
&-288\,k^3a_0{}^2b_2\,a_2 - 456\,k^3a_2{}^2b_1{}^2\mu - 288\,k^3a_0\,b_1{}^2a_2\\
&-32\,k^3a_2{}^3b_2\,\mu^2 - 88\,k^3a_1{}^3b_1\,\mu - 72\,k^3a_0{}^2b_1\,a_1\\
&-96\,k^3a_0{}^2a_1{}^2\mu - 192\,k^3a_0{}^2a_2{}^2\mu^2 - 160\,k^3a_0{}^3a_2\,\mu\\
&+64\,ka_0{}^3a_1{}^2 + 32\,ka_0{}^4a_2 - 288\,k^3a_2{}^2b_2{}^2\\
&-8\,k^3a_1{}^4\mu^2 + 64\,ka_2{}^3b_2{}^2 - 8\,la_2\,a_0{}^2 - 8\,la_2{}^2b_2\\
&-8\,la_1{}^2a_0 - 72\,k^3a_1{}^2b_1{}^2 + 32\,ka_1{}^4b_2\\
&-80\,k^2la_2{}^2\mu\,b_2 + 38\,k^2la_1{}^2\mu^2a_2 + 200\,k^2la_1\,\mu\,a_2\,b_1\\
&+240\,k^2la_2\,b_2\,a_0 + 60\,k^2lb_1\,a_1\,a_0 + 40\,k^2la_1{}^2\mu\,a_0\\
&+8\,k^2la_2{}^2\mu^2a_0 + 160\,k^2la_2\,\mu\,a_0{}^2 - 312\,k^3a_1\,a_2{}^2b_1\,\mu^2\\
&-480\,k^3a_1{}^2b_2\,a_2\,\mu - 768\,k^3a_0\,a_2{}^2b_2\,\mu - 1104\,k^3a_0\,a_1\,a_2\,b_1\,\mu\\
&-240\,k^3a_0\,a_1{}^2a_2\,\mu^2 + 128\,ka_2{}^3b_1{}^2\mu + 192\,ka_1{}^2a_2\,b_1{}^2\\
&+192\,ka_0\,a_2{}^2b_1{}^2 + 192\,ka_0{}^2a_2{}^2b_2 + 128\,ka_0\,a_1{}^3b_1\\
&+128\,ka_0{}^3a_2{}^2\mu + 64\,ka_0\,a_1{}^4\mu + 384\,ka_1{}^2a_2{}^2b_2\,\mu\\
&+384\,ka_1\,a_2{}^2b_1\,b_2 + 256\,ka_1{}^3b_1\,a_2\,\mu + 384\,ka_0\,a_1{}^2a_2\,b_2\\
&+768\,ka_0\,a_2{}^2b_1\,a_1\,\mu + 384\,ka_0{}^2a_1\,a_2\,b_1 + 384\,ka_0{}^2a_1{}^2a_2\,\mu\\
&-720\,k^3a_1\,b_1\,b_2\,a_2 + 256\,ka_0\,a_2{}^3b_2\,\mu = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{-2}:\ -24\,k^3b_2{}^3a_1 - 48\,ka_1\,b_1{}^4 + 32\,k^3b_1{}^3a_2\,\mu^2\\
&+96\,k^3a_1{}^3b_2\,\mu^3 + 4\,lb_1{}^2\mu\,a_1 + 12\,la_1\,b_2{}^2\\
&-96\,ka_0{}^2b_1{}^3 + 4\,lb_1{}^3 + 8\,lb_1\,\mu\,a_2\,b_2\\
&+8\,lb_2\,\mu\,a_0\,a_1 - 360\,k^2lb_2\,\mu^3a_1\,a_2 + 56\,k^2la_2\,b_1\,\mu^2b_2\\
&-17\,k^2la_1\,b_1{}^2\mu^2 + 120\,k^2lb_2{}^2\mu\,a_1 - 364\,k^2lb_2\,\mu^2a_1\,a_0\\
&-60\,k^2la_2\,\mu^3b_1\,a_0 - 45\,k^2lb_1\,\mu^3a_1{}^2 + 360\,k^3a_2{}^2b_1\,b_2\,\mu^3\\
&+768\,k^3a_0\,b_1\,b_2\,a_2\,\mu^2 + 288\,k^3a_0\,a_1\,b_2{}^2\mu + 72\,k^3a_0{}^2a_2\,b_1\,\mu^3\\
&+528\,k^3a_1{}^2b_1\,b_2\,\mu^2 + 72\,k^3a_0\,a_1{}^2b_1\,\mu^3 + 144\,k^3a_1\,b_1{}^2b_2\,\mu\\
&+144\,k^3a_1\,a_2\,b_1{}^2\mu^3 + 576\,k^3a_0\,a_1\,a_2\,b_2\,\mu^3 - 72\,k^3b_1\,b_2{}^2a_2\,\mu\\
&+744\,k^3a_1\,a_2\,b_2{}^2\mu^2 + 456\,k^3a_0{}^2a_1\,b_2\,\mu^2 + 168\,k^3a_0\,a_1\,b_1{}^2\mu^2\\
&-96\,ka_2{}^2b_1\,b_2{}^2\mu - 64\,ka_0\,b_1{}^3a_2\,\mu - 192\,ka_0{}^2b_1\,b_2\,a_2\,\mu\\
&-192\,ka_0\,a_1{}^2b_1\,b_2\,\mu - 192\,ka_0\,a_1\,a_2\,b_2{}^2\mu - 192\,ka_1\,a_2\,b_1{}^2b_2\,\mu\\
&-576\,ka_0\,b_1\,b_2{}^2a_2 - 192\,ka_1\,a_2\,b_2{}^3 - 32\,ka_1{}^3b_2{}^2\mu\\
&-288\,ka_1{}^2b_1\,b_2{}^2 - 192\,kb_1{}^3b_2\,a_2 - 32\,ka_1{}^2b_1{}^3\mu\\
&+4\,lb_1\,\mu\,a_0{}^2 + 24\,lb_1\,a_0\,b_2 - 5\,k^2lb_1{}^3\mu + 24\,k^3a_0\,b_1{}^3\mu\\
&+32\,k^3a_0{}^3b_1\,\mu^2 - 16\,ka_0{}^4b_1\,\mu - 192\,ka_0{}^3b_1\,b_2\\
&-576\,ka_0\,a_1\,b_1{}^2b_2 - 64\,ka_0{}^3a_1\,b_2\,\mu - 96\,ka_0{}^2a_1\,b_1{}^2\mu\\
&-60\,k^2lb_1\,\mu\,b_2\,a_0 - 32\,k^2lb_1\,\mu^2a_0{}^2 - 288\,ka_0{}^2a_1\,b_2{}^2\\
&+216\,k^3a_0{}^2b_2\,b_1\,\mu = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{2}:\ 16\,ka_0{}^4a_1 - 4\,lb_1\,a_1{}^2 + 32\,ka_2{}^2b_1{}^3\\
&-96\,k^3b_1{}^3a_2 + 32\,ka_1{}^3b_1{}^2 - 4\,la_1\,a_0{}^2\\
&-4\,la_1{}^3\mu - 8\,la_1\,a_2\,b_2 + 45\,k^2la_1\,b_1{}^2\\
&+5\,k^2la_1{}^3\mu^2 - 12\,lb_1\,\mu\,a_2{}^2 - 8\,lb_1\,a_0\,a_2\\
&-32\,k^3a_1{}^3b_2\,\mu - 72\,k^3a_0\,b_1{}^2a_1 - 72\,k^3a_0{}^2b_2\,a_1\\
&+24\,k^3a_2{}^3b_1\,\mu^3 - 144\,k^3a_1{}^2b_1\,b_2 - 360\,k^3a_1\,b_2{}^2a_2\\
&-32\,k^3a_0{}^3a_1\,\mu - 24\,k^3a_0\,a_1{}^3\mu^2 + 96\,ka_1\,a_2{}^2b_2{}^2\\
&+64\,ka_0\,a_1{}^3b_2 + 48\,ka_1{}^4b_1\,\mu + 96\,ka_0{}^2a_1{}^2b_1\\
&+64\,ka_0{}^3a_2\,b_1 - 24\,la_1\,\mu\,a_0\,a_2 - 120\,k^2la_2{}^2\mu^2b_1\\
&+17\,k^2la_1{}^2\mu\,b_1 - 56\,k^2la_1\,\mu\,b_2\,a_2 + 360\,k^2lb_1\,b_2\,a_2\\
&+60\,k^2la_1\,b_2\,a_0 + 60\,k^2la_1\,\mu^2a_2\,a_0 + 32\,k^2la_1\,\mu\,a_0{}^2\\
&+72\,k^3a_1\,a_2{}^2b_2\,\mu^2 - 744\,k^3a_2{}^2b_1\,b_2\,\mu - 576\,k^3a_0\,b_1\,b_2\,a_2\\
&-528\,k^3a_1\,a_2\,b_1{}^2\mu - 768\,k^3a_0\,a_1\,a_2\,b_2\,\mu - 288\,k^3a_0\,a_2{}^2b_1\,\mu^2\\
&-144\,k^3a_1{}^2b_1\,a_2\,\mu^2 - 168\,k^3a_0\,a_1{}^2b_1\,\mu - 456\,k^3a_0{}^2a_2\,b_1\,\mu\\
&-216\,k^3a_0{}^2a_1\,a_2\,\mu^2 + 192\,ka_2{}^3b_1\,b_2\,\mu + 576\,ka_0\,a_2{}^2b_2\,a_1\,\mu\\
&+192\,ka_0\,a_2{}^2b_1\,b_2 + 288\,ka_1\,a_2{}^2b_1{}^2\mu + 192\,ka_1{}^3b_2\,a_2\,\mu\\
&+192\,ka_1{}^2a_2\,b_2\,b_1 + 192\,ka_0\,a_1\,a_2\,b_1{}^2 + 576\,ka_0\,a_1{}^2b_1\,a_2\,\mu\\
&+192\,ka_0{}^2a_1\,a_2\,b_2 + 288\,ka_0{}^2a_2{}^2b_1\,\mu + 192\,ka_0{}^3a_1\,a_2\,\mu\\
&+96\,ka_0{}^2a_1{}^3\mu + 364\,k^2lb_1\,a_2\,\mu\,a_0 = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{-1}:\ -32\,kb_1{}^4a_2 + 8\,la_0\,b_1{}^2 - 96\,k^3b_2{}^3a_2\\
&-64\,ka_0{}^3b_1{}^2 + 64\,k^3a_0{}^3b_2\,\mu + 24\,k^3a_0{}^2b_1{}^2\mu\\
&+24\,k^2lb_2{}^2a_0 - 6\,k^2lb_1{}^2b_2 + 8\,la_2\,b_2{}^2\\
&-192\,ka_0\,a_1{}^2b_2{}^2 + 72\,k^3a_1{}^2b_1{}^2\mu^2 + 96\,k^3a_1{}^2b_2{}^2\mu\\
&-128\,ka_0\,a_1\,b_1{}^3 - 192\,ka_1{}^2b_1{}^2b_2 - 192\,ka_0{}^2b_2{}^2a_2\\
&+288\,k^3a_2{}^2b_2{}^2\mu^2 + 24\,k^3a_1{}^3b_1\,\mu^3 - 72\,k^3b_1\,b_2{}^2a_1\\
&-32\,ka_0{}^4b_2 + 8\,lb_2\,a_0{}^2 - 64\,ka_2{}^2b_2{}^3 - 8\,k^3a_1\,b_1{}^3\mu\\
&+72\,k^3a_2{}^2b_1{}^2\mu^3 + 16\,la_1\,b_1\,b_2 + 112\,k^2lb_1\,\mu\,a_1\,b_2\\
&-270\,k^2la_1{}^2b_2\,\mu^2 - 240\,k^2lb_2\,\mu^3a_2{}^2 + 416\,k^2lb_2{}^2\mu\,a_2\\
&+90\,k^2la_2\,b_1{}^2\mu^2 - 240\,k^2la_2\,\mu^2b_2\,a_0 - 60\,k^2lb_1\,\mu^2a_1\,a_0\\
&+288\,k^3a_0\,a_2{}^2b_2\,\mu^3 + 288\,k^3a_1{}^2a_2\,b_2\,\mu^3 + 144\,k^3a_0\,a_1\,a_2\,b_1\,\mu^3\\
&-96\,k^3a_0\,b_2{}^2a_2\,\mu - 240\,k^3b_1{}^2b_2\,a_2\,\mu + 720\,k^3a_1\,a_2\,b_1\,b_2\,\mu^2\\
&+288\,k^3a_0{}^2b_2\,a_2\,\mu^2 + 240\,k^3a_0\,a_1\,b_1\,b_2\,\mu + 432\,k^3a_0\,a_1{}^2b_2\,\mu^2\\
&+72\,k^3a_0{}^2a_1\,b_1\,\mu^2 - 384\,ka_1\,a_2\,b_2{}^2b_1 - 384\,ka_0\,b_1{}^2b_2\,a_2\\
&-384\,ka_0{}^2a_1\,b_1\,b_2 - 64\,k^2lb_2\,\mu\,a_0{}^2 - 4\,k^2lb_1{}^2\mu\,a_0\\
&-120\,k^2lb_1\,\mu^3a_1\,a_2 = 0,
\end{aligned}$$

$$\begin{aligned}
&\left(\tfrac{G'}{G}\right)^{1}:\ -24\,k^3b_1{}^3a_1 - 72\,k^3a_1{}^2b_2{}^2 - 8\,la_1{}^2\mu\,a_0\\
&+240\,k^2la_2\,b_2{}^2 + 8\,k^3a_1{}^3b_1\,\mu^2 - 96\,k^3a_2{}^2b_1{}^2\mu^2\\
&-72\,k^3a_1{}^2b_1{}^2\mu - 8\,la_2\,\mu\,a_0{}^2 - 8\,lb_2\,\mu\,a_2{}^2\\
&-288\,k^3a_2{}^2b_2{}^2\mu - 64\,k^3a_0{}^3a_2\,\mu^2 - 288\,k^3a_0\,b_2{}^2a_2\\
&-288\,k^3b_1{}^2b_2\,a_2 + 96\,k^3a_2{}^3b_2\,\mu^3 + 64\,ka_0{}^3a_1{}^2\mu\\
&+32\,ka_0{}^4a_2\,\mu + 64\,ka_2{}^3b_2{}^2\mu - 24\,k^3a_0{}^2a_1{}^2\mu^2\\
&+32\,ka_1{}^4b_2\,\mu - 16\,lb_1\,\mu\,a_1\,a_2 - 416\,k^2la_2{}^2b_2\,\mu^2\\
&-90\,k^2la_1{}^2\mu\,b_2 - 112\,k^2la_1\,\mu^2a_2\,b_1 + 270\,k^2la_2\,\mu\,b_1{}^2\\
&+120\,k^2lb_1\,b_2\,a_1 + 6\,k^2la_1{}^2\mu^3a_2 + 60\,k^2lb_1\,a_1\,\mu\,a_0\\
&+4\,k^2la_1{}^2\mu^2a_0 - 24\,k^2la_2{}^2\mu^3a_0 + 64\,k^2la_2\,\mu^2a_0{}^2\\
&+72\,k^3a_1\,a_2{}^2b_1\,\mu^3 - 720\,k^3a_1\,a_2\,b_1\,b_2\,\mu + 96\,k^3a_0\,a_2{}^2b_2\,\mu^2\\
&-144\,k^3a_0\,b_1\,b_2\,a_1 - 240\,k^3a_0\,a_1\,a_2\,b_1\,\mu^2 - 288\,k^3a_0{}^2b_2\,a_2\,\mu\\
&-432\,k^3a_0\,b_1{}^2a_2\,\mu - 72\,k^3a_0{}^2a_1\,b_1\,\mu + 384\,ka_1\,a_2{}^2b_1\,b_2\,\mu\\
&+192\,ka_0{}^2a_2{}^2b_2\,\mu + 192\,ka_0\,a_2{}^2b_1{}^2\mu + 384\,ka_0\,a_1{}^2b_2\,a_2\,\mu\\
&+192\,ka_1{}^2b_1{}^2a_2\,\mu + 128\,ka_0\,a_1{}^3b_1\,\mu + 384\,ka_0{}^2a_1\,a_2\,b_1\,\mu\\
&+240\,k^2la_2\,\mu\,b_2\,a_0 + 240\,k^3a_1{}^2b_2\,a_2\,\mu^2 = 0,
\end{aligned}$$

$$\left(\frac{G'}{G}\right)^0: \ -32\,ka_1{}^2b_1{}^3-8\,lb_2\,\mu\,a_1\,a_2-8\,lb_1\,\mu\,a_0\,a_2$$
$$-584\,k^2la_1\,b_2\,\mu^2a_2+8\,la_2\,b_1\,b_2-4\,la_1\,\mu\,a_0{}^2$$
$$+96\,k^2la_1\,b_2{}^2-4\,lb_1\,\mu\,a_1{}^2+8\,la_1\,a_0\,b_2$$
$$+3\,k^2la_1{}^3\mu^3-288\,k^3b_1\,b_2{}^2a_2-136\,k^3b_1{}^3a_2\,\mu$$
$$-72\,k^3b_1{}^2b_2\,a_1-72\,k^3a_0\,b_2{}^2a_1-8\,k^3a_0{}^3a_1\,\mu^2$$
$$+8\,k^3a_0{}^3b_1\,\mu-96\,ka_2{}^2b_2{}^2b_1+32\,ka_2{}^2b_1{}^3\mu$$
$$+32\,ka_1{}^3b_1{}^2\mu-64\,ka_0\,b_1{}^3a_2-96\,ka_0{}^2a_1\,b_1{}^2$$
$$-64\,ka_0{}^3a_1\,b_2+16\,ka_0{}^4a_1\,\mu-3\,k^2lb_1{}^3$$
$$-32\,ka_1{}^3b_2{}^2+4\,lb_1\,a_0{}^2+4\,la_1\,b_1{}^2$$
$$+584\,k^2lb_1\,\mu\,a_2\,b_2-96\,k^2lb_1\,\mu^3a_2{}^2+67\,k^2la_1\,\mu\,b_1{}^2$$
$$-67\,k^2la_1{}^2\mu^2b_1-76\,k^2la_1\,\mu\,b_2\,a_0+76\,k^2lb_1\,a_2\,\mu^2a_0$$
$$+12\,k^2lb_1\,b_2\,a_0-12\,k^2la_1\,\mu^3a_2\,a_0-8\,k^2lb_1\,\mu\,a_0{}^2$$
$$+8\,k^2la_1\,\mu^2a_0{}^2+288\,k^3a_1\,a_2{}^2b_2\,\mu^3+192\,k^3a_2{}^2b_1\,b_2\,\mu^2$$
$$+72\,k^3a_1{}^2a_2\,b_1\,\mu^3-192\,k^3a_1\,a_2\,b_2{}^2\mu+72\,k^3a_0\,a_2{}^2b_1\,\mu^3$$
$$+24\,k^3a_1{}^2b_1\,b_2\,\mu-528\,k^3a_0\,b_1\,b_2\,a_2\,\mu+136\,k^3a_1{}^3b_2\,\mu^2$$
$$+528\,k^3a_0\,a_1\,a_2\,b_2\,\mu^2-24\,k^3a_1\,a_2\,b_1{}^2\mu^2-48\,k^3a_0\,a_1\,b_1{}^2\mu$$
$$+96\,k^3a_0{}^2a_1\,b_2\,\mu-16\,ka_0{}^4b_1-96\,k^3a_0{}^2b_1\,a_2\,\mu^2$$
$$+48\,k^3a_0\,a_1{}^2b_1\,\mu^2+96\,ka_1\,a_2{}^2b_2{}^2\mu+192\,ka_0\,a_2{}^2b_1\,b_2\,\mu$$
$$-192\,ka_1\,a_2\,b_1{}^2b_2+192\,ka_1{}^2b_1\,b_2\,a_2\,\mu-192\,ka_0\,a_1\,a_2\,b_2{}^2$$
$$+192\,ka_0{}^2a_1\,a_2\,b_2\,\mu+192\,ka_0\,a_1\,a_2\,b_1{}^2\mu+64\,ka_0\,a_1{}^3b_2\,\mu$$
$$-192\,ka_0\,a_1{}^2b_1\,b_2-192\,ka_0{}^2b_1\,b_2\,a_2+64\,ka_0{}^3b_1\,a_2\,\mu$$
$$+96\,ka_0{}^2a_1{}^2b_1\,\mu=0, \tag{24}$$

Solving the algebraic equations (24), yields the following cases:

Case (i). $a_0=0,\ a_1=0,\ a_2=3k^2,\ b_1=0,$ $b_2=0,\ k=k,\ l=2k,\ \mu=\frac{1}{2k^2},$

Case (ii). $a_0=2k^2\mu,\ a_1=0,\ a_2=3k^2,\ b_1=0,$ $b_2=0,\ k=k,\ l=0,\ \mu=\mu,$

Case (iii). $a_0=-3,\ a_1=0,\ a_2=3k^2,\ b_1=0,$ $b_2=0,\ k=k,\ l=2k,\ \mu=-\frac{1}{k^2},$

Case(iv). $a_0=0,\ a_1=0,\ a_2=0,\ b_1=0,$ $b_2=\frac{3}{4k^2},\ k=k,\ l=2k,\ \mu=\frac{1}{2k^2},$

Case (v). $a_0=2k^2\mu,\ a_1=0,\ a_2=0,\ b_1=0,$ $b_2=3k^2\mu^2,\ k=k,\ l=0,\ \mu=\mu,$

Case (vi). $a_0=-3,\ a_1=0,\ a_2=0,\ b_1=0,$ $b_2=\frac{3}{k^2},\ k=k,\ l=2k,\ \mu:=-\frac{1}{k^2},$

Case (vii). $a_0:=2k^2\mu,\ a_1=0,\ a_2=3k^2,\ b_1=0,$ $b_2=3k^2\mu^2,\ k=k,\ l=0,\ \mu=\mu.$ (25)

Corresponding to the above seven cases and on substituting the general solution of (19) given by (20) and (21), we obtained traveling wave solutions of the mN equation (5) which have been descibed in following subcases:

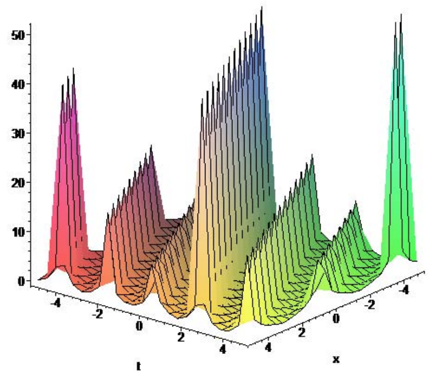

Fig. 1. The periodic solution $u_1(x,t)$ which is a periodic wave for $k=2, A=3, B=1$

Subcase 1. If $\mu>0$, then we obtain
Case (i) gives the following value of $u(x,t)$:

$$u_1(x,t)=1/2\,\sqrt{3}\sqrt{2}\sqrt{\frac{\left(A\cos\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)-B\sin\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\sin\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)+B\cos\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{26}$$

The traveling waves corresponding to $u_1(x,t)$ for $k=2, A=3, B=1$ has been shown in figure 1. Corresponding to Case (ii), we have

$$u_2(x,t)=\sqrt{2\,k^2\mu+3\,\frac{k^2\mu\left(A\cos\left(\sqrt{\mu}kx\right)-B\sin\left(\sqrt{\mu}kx\right)\right)^2}{\left(A\sin\left(\sqrt{\mu}kx\right)+B\cos\left(\sqrt{\mu}kx\right)\right)^2}}. \tag{27}$$

Case (iii) yields the following solution of Eq. (5):

$$u_3(x,t)=\sqrt{-3-3\,\frac{\left(A\cos\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)-B\sin\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\sin\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)+B\cos\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{28}$$

Corresponding to Case (iv), we get

$$u_4(x,t)=1/2\,\sqrt{3}\sqrt{2}\sqrt{\frac{\left(A\sin\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)+B\cos\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\cos\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)-B\sin\left(1/2\,\sqrt{2}\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{29}$$

Case (v) corresponds to the following solution of Eq. (5):

$$u_5(x,t)=\sqrt{2\,k^2\mu+3\,\frac{k^2\mu\left(A\sin\left(\sqrt{\mu}kx\right)+B\cos\left(\sqrt{\mu}kx\right)\right)^2}{\left(A\cos\left(\sqrt{\mu}kx\right)-B\sin\left(\sqrt{\mu}kx\right)\right)^2}}. \tag{30}$$

The mN equation has the following solution with respect to Case (vi):

$$u_6(x,t)=\sqrt{-3-3\,\frac{\left(A\sin\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)+B\cos\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\cos\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)-B\sin\left(\sqrt{-k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{31}$$

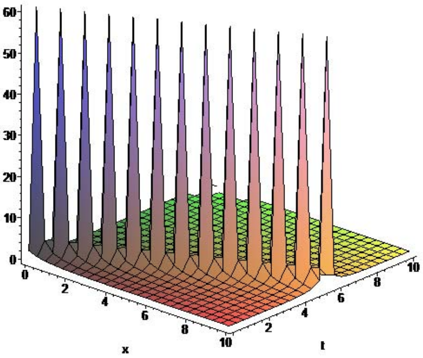

Fig. 2. The traveling solution of the field $u_{10}(x,t)$ for $k=1, A=2, B=3$

Equation (5) possesses the following solution corresponding to Case (vii):

$$u_7(x,t) = \sqrt{2 \cdots - \frac{3k^2\mu\left(A\cos\left(\sqrt{\mu}kx\right)-B\sin\left(\sqrt{\mu}kx\right)\right)^2}{\left(A\sin\left(\sqrt{\mu}kx\right)+B\cos\left(\sqrt{\mu}kx\right)\right)^2} + \frac{3k^2\mu\left(A\sin\left(\sqrt{\mu}kx\right)+B\cos\left(\sqrt{\mu}kx\right)\right)^2}{\left(A\cos\left(\sqrt{\mu}kx\right)-B\sin\left(\sqrt{\mu}kx\right)\right)^2}} \tag{32}$$

Subcase 2. If $\mu < 0$, then we obtain
Case (i) gives the exact traveling wave solution of Eq. (5) as:

$$u_8(x,t) = \sqrt{\cdots 2\, \frac{\left(A\sinh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)+B\cosh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\cosh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)+B\sinh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{33}$$

Equation (5) possesses the following solution corresponding to Case (ii):

$$u_9(x,t) = \sqrt{2\,k^2\mu - 3\,\frac{k^2\mu\left(A\sinh\left(\sqrt{-\mu}kx\right)+B\cosh\left(\sqrt{-\mu}kx\right)\right)^2}{\left(A\cosh\left(\sqrt{-\mu}kx\right)+B\sinh\left(\sqrt{-\mu}kx\right)\right)^2}}. \tag{34}$$

Case (iii) corresponds to the following value of $u(x,t)$:

$$u_{10}(x,t) = \sqrt{\cdots 3 + 3\,\frac{\left(A\sinh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)+B\cosh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\cosh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)+B\sinh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{35}$$

The traveling waves for $u_{10}(x,t)$ for $k=1, A=2, B=3$ are shown in figure 2.

Case (iv) yields the following solution of Eq. (5):

$$u_{11}(x,t) = \sqrt{-3/2\,\frac{\left(A\cosh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)+B\sinh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\sinh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)+B\cosh\left(\sqrt{-1/2\,k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{36}$$

Case (v) gives the following traveling wave solution:

$$u_{12}(x,t) = \sqrt{2\,k^2\mu - 3\,\frac{k^2\mu\left(A\cosh\left(\sqrt{-\mu}kx\right)+B\sinh\left(\sqrt{-\mu}kx\right)\right)^2}{\left(A\sinh\left(\sqrt{-\mu}kx\right)+B\cosh\left(\sqrt{-\mu}kx\right)\right)^2}}. \tag{37}$$

Equation (5) has the following solution corresponds to Case (vi):

$$u_{13}(x,t) = \sqrt{-3 + 3\,\frac{\left(A\cosh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)+B\sinh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}{\left(A\sinh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)+B\cosh\left(\sqrt{k^{-2}}(kx-2\,kt)\right)\right)^2}}. \tag{38}$$

The mN equation has the following solution corresponding to Case (vii):

$$u_{14}(x,t) = \sqrt{2k^2\mu - \frac{3k^2\mu\left(A\sinh\left(\sqrt{-\mu}kx\right)+B\cosh\left(\sqrt{-\mu}kx\right)\right)^2}{\left(A\cosh\left(\sqrt{-\mu}kx\right)+B\sinh\left(\sqrt{-\mu}kx\right)\right)^2} - \frac{3k^2\mu\left(A\cosh\left(\sqrt{-\mu}kx\right)+B\sinh\left(\sqrt{-\mu}kx\right)\right)^2}{\left(A\sinh\left(\sqrt{-\mu}kx\right)+B\cosh\left(\sqrt{-\mu}kx\right)\right)^2}}. \tag{39}$$

V. Conclusion and Discussion

In this article, we have established the traveling wave solutions of the mN Eq. (5) using Lie classical method, Nonclassical method and the modified (G'/G)-expansion method. These traveling wave solutions are expressed in terms of hyperbolic, trigonometric and rational functions involving arbitrary parameters. When these parameters are taken special values, the solitary waves are derived from the traveling waves. It has been shown that the proposed method is direct, concise, basic and effective and easy to calculate, and it is a powerful mathematical tool for obtaining exact traveling wave solutions of nonlinear evolution equations and can be used to solve other NPDEs in mathematical physics. The availability of mathematical computer software like Maple facilitates the tedious algebraic calculations. It is worth to mention here that the correctness of the solutions has been checked with the aid of software Maple.

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