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Some Separations in Covering Approximation Spaces

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Abstract—Adopting Zakowski's upper approximation operator $\overline{C}$ and lower approximation operator $\underline{C}$, this paper investigates granularity-wise separations in covering approximation spaces. Some characterizations of granularity-wise separations are obtained by means of Pawlak rough sets and some relations among granularity-wise separations are established, which makes it possible to research covering approximation spaces by logical methods and mathematical methods in computer science. Results of this paper give further applications of Pawlak rough set theory in pattern recognition and artificial intelligence.

Keywords—Rough set, covering approximation space, granularity-wise separation.

I. Introduction

Rough set theory, which was proposed by Z. Pawlak in [8], is a useful tool in researches and applications of pattern recognition and artificial intelligence (see [1], [4], [5], [6], [7], [9], [13], [15], [16], for example). In rough set theory, Pawlak approximation spaces are based on equivalence class partitioning of sets. However, in an equivalence class partitioning of a set, we can usually not separate points by equivalence classes. In other words, Pawlak approximation spaces do not satisfy "granularity-wise separations" in general. This leads us to explore the richer rough set theory. In the past years, covering approximation spaces arouse our extensive interest and their usefulness has been demonstrated by many successful applications in pattern recognition and artificial intelligence (see [10], [11], [12], [15], [17], [18], [20], for example). Naturally, the following question is worthy to be considered.

Question 1.1: (1) How to characterize granularity-wise separations in covering approximation spaces?

(2) What relations are there among granularity-wise separations in covering approximation spaces?

As some investigations of the above question, we adopt Zakowski's upper approximation operator $\overline{C}$ and lower approximation operator $\underline{C}$ to characterize granularity-wise separations in a covering approximation space $(U, \mathcal{C})$ and establish some relations among these separations. Results of this paper give further applications of Pawlak rough set theory in pattern recognition and artificial intelligence.

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II. Preliminaries

Definition 2.1 ([20]): Let U, the universe of discourse, be a finite set and $\mathcal{C}$ be a family of nonempty subsets of U.

(1) $\mathcal{C}$ is called a cover of U if $\bigcup\{K : K \in \mathcal{C}\} = U$.

(2) The pair $(U;\mathcal{C})$ is called a covering approximation space if $\mathcal{C}$ is a cover of U.

Definition 2.2 ([8]): Let $(U;\mathcal{C})$ be a covering approximation space.

(1) $\mathcal{C}$ is called a partition on U if $K \bigcap K' = \emptyset$ for all $K, K' \in \mathcal{C}$, where $K \neq K'$.

(2) $(U;\mathcal{C})$ is called a Pawlak approximation space if $\mathcal{C}$ is a partition on U.

Notation 2.3: Let $(U;\mathcal{C})$ be a covering approximation space. Throughout this paper, we use the following notations, where $x \in U$, $X \subset U$ and $\mathcal{F} \subset 2^U$.

(1) $\bigcap \mathcal{F} = \bigcap\{F : F \in \mathcal{F}\}$.

(2) $\bigcup \mathcal{F} = \bigcup\{F : F \in \mathcal{F}\}$.

(3) $\mathcal{C}_x = \{K : x \in K \in \mathcal{C}\}$.

(4) $N(x) = \bigcap\{K : K \in \mathcal{C}_x\} = \bigcap \mathcal{C}_x$.

(5) $D(X) = U - \bigcup\{K : K \in \mathcal{C} \bigwedge K \bigcap X = \emptyset\}$.

(6) $D(x) = D(\{x\}) = U - \bigcup(\mathcal{C} - \mathcal{C}_x)$.

Now we give the following granularity-wise separations in covering approximation spaces. Ideas of these separations come from topology (see [2], for example).

Definition 2.4: Let $(U;\mathcal{C})$ be a covering approximation space. $(U;\mathcal{C})$ is called a G_0- (resp. G_1-, G_2-, G_3-, G_d-, G_r-) covering approximation space if $(U;\mathcal{C})$ satisfies the following G_0- (resp. G_1-, G_2-, G_3-, G_d-, G_r-) separation axiom.

(1) G_0-separation axiom: $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K \in \mathcal{C}(K \bigcap\{x, y\} = \{x\} \bigvee K \bigcap\{x, y\} = \{y\})$.

(2) G_1-separation axiom: $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K_x, K_y \in \mathcal{C}(K_x \bigcap\{x, y\} = \{x\} \bigwedge K_y \bigcap\{x, y\} = \{y\})$.

(3) G_2-separation axiom: $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K_x, K_y \in \mathcal{C}(x \in K_x \bigwedge y \in K_y \bigwedge K_x \bigcap K_y = \emptyset)$.

(4) G_3-separation axiom: $x \in U \bigwedge x \notin X \subset U \Longrightarrow \exists K \in \mathcal{C}(x \in K \bigwedge K \bigcap X = \emptyset)$.

(5) G_d-separation axiom: $x \in U \Longrightarrow \exists K \in \mathcal{C}(\{x\} = K \bigcap D(x))$.

(6) G_r-separation axiom: $x \in K \in \mathcal{C} \Longrightarrow D(x) \subset K$.

For short, G_i-covering approximation spaces are called G_i-spaces, $i = 0, 1, 2, 3, d, r$.

In order to investigate the above separations in covering approximation spaces by means of Pawlak rough sets, we need the following definition (see [11], for example).

Definition 2.5: Let $(U;\mathcal{C})$ be a covering approximation space. For each $X \subset U$, Put

$$\underline{C}(X) = \bigcup\{K : K \in \mathcal{C} \ and \ K \subset X\};$$

$$\overline{C}(X) = U - \underline{C}(U - X).$$

(1) $\overline{C} : 2^U \longrightarrow 2^U$ is called covering upper approximation operator.

(2) $\underline{C} : 2^U \longrightarrow 2^U$ is called covering lower approximation operator.

(3) $\overline{C}(X)$ is called a covering upper approximation of X, which is abbreviated to X^*.

(4) $\underline{C}(X)$ is called a covering lower approximation of X, which is abbreviated to X_*.

(5) X is called a definable set of $(U; \mathcal{C})$ if $X_* = X^*$.

(6) X is called a rough set of $(U; \mathcal{C})$ if $X_* \neq X^*$.

Remark 2.6: Lemma 3.12 of this paper shows that $D(x) = \{x\}^*$. So G_d-separation axiom (resp. G_r-separation axiom) in this paper is equivalent to G_d-separation axiom (resp. G_r-separation axiom) in [3]. In order to avoid approximation operators appearing in definitions of granularity-wise separations, we use $D(x)$ in G_d-separation axiom and G_r-separation axiom of this paper.

III. Characterizations of Separations

Lemma 3.1 ([11], [20]): Let $(U; \mathcal{C})$ be a covering approximation space. Then the following hold.

(1) $U^* = U_* = U$ and $\emptyset^* = \emptyset_* = \emptyset$.

(2) If $X \subset U$, then $X_* \subset X \subset X^*$.

(3) If $X \subset Y \subset U$, then $X_* \subset Y_*$ and $X^* \subset Y^*$.

(4) If $X \subset U$, then $(X_*)_* = X_*$ and $(X^*)^* = X^*$.

(5) If $K \in \mathcal{C}$, then $K_* = K$.

(6) If $X \subset U$, then $(U - X)_* = U - X^*$ and $(U - X)^* = U - X_*$.

Lemma 3.2: Let $(U; \mathcal{C})$ be a covering approximation space, $x, y \in U$ and $x \neq y$. Then the following are equivalent.

(1) $x \notin \{y\}^*$.

(2) $\exists K \in \mathcal{C}(K \bigcap \{x, y\} = \{x\})$.

Proof. (1) $\Longrightarrow$ (2): If $x \notin \{y\}^*$, then $x \notin U - (U - \{y\})_*$ by Lemma 3.1(6), and hence $x \in (U - \{y\})_*$. So there is $K \in \mathcal{C}$ such that $x \in K \subset U - \{y\}$. Thus $y \notin K$, and hence $K \bigcap \{x, y\} = \{x\}$.

(2) $\Longrightarrow$ (1): If there is $K \in \mathcal{C}$ such that $K \bigcap \{x, y\} = \{x\}$, then $y \notin K$, and so $y \in U - K$. By Lemma 3.1(3),(5),(6), $\{y\}^* \subset (U - K)^* = U - K_* = U - K$. Note that $x \in K$, so $x \notin \{y\}^*$.

Corollary 3.3: Let $(U; \mathcal{C})$ be a covering approximation space, $x, y \in U$ and $x \neq y$. Then the following are equivalent.

(1) $x \in \{y\}^*$.

(2) $\forall K \in \mathcal{C}(x \in K \longrightarrow y \in K)$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $x \in \{y\}^*$. Let $K \in \mathcal{C}$. then $K \bigcap \{x, y\} \neq \{x\}$ by Lemma 3.2. Thus $K \bigcap \{x, y\} = \emptyset$ or $K \bigcap \{x, y\} = \{y\}$ or $K \bigcap \{x, y\} = \{x, y\}$. Consequently, if $x \in K$, then $K \bigcap \{x, y\} = \{x, y\}$, and hence $y \in K$.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Then, for each $K \in \mathcal{C}$, $y \in K$ providing $x \in K$, and hence $K \bigcap \{x, y\} \neq \{x\}$. By Lemma 3.2, $x \in \{y\}^*$.

Theorem 3.4: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_0-space.

(2) $x, y \in U \bigwedge x \neq y \Longrightarrow x \notin \{y\}^* \bigvee y \notin \{x\}^*$.

(3) $x, y \in U \bigwedge x \neq y \Longrightarrow \{x\}^* \neq \{y\}^*$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U; \mathcal{C})$ is a G_0-space. Let $x, y \in U$ and $x \neq y$. Without loss of generality, assume that there is $K \in \mathcal{C}$ such that $K \bigcap \{x, y\} = \{x\}$. Then $x \notin \{y\}^*$ by Lemma 3.2.

(2) $\Longrightarrow$ (3): Suppose that (2) holds. Let $x, y \in U$ and $x \neq y$. Without loss of generality, assume that $x \notin \{y\}^*$. Since $x \in \{x\}^*$ by Lemma 3.1(2), $\{x\}^* \neq \{y\}^*$.

(3) $\Longrightarrow$ (1): Suppose that (3) holds. Let $x, y \in U$ and $x \neq y$. Then $\{x\}^* \neq \{y\}^*$. Without loss of generality, assume that there is $z \notin \{y\}^*$ and $z \in \{x\}^*$. By Lemma 1(6), $z \notin \{y\}^* = U - (U - \{y\})_*$, and hence $z \in (U - \{y\})_*$. So there is $K \in \mathcal{C}$ such that $z \in K \subset U - \{y\}$, i.e., $z \in K$ and $y \notin K$. Since $z \in \{x\}^*$, $x \in K$ by Corollary 3.3. Thus $K \bigcap \{x, y\} = \{x\}$. It follows that $(U; \mathcal{C})$ is a G_0-space.

Theorem 3.5: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_1-space.

(2) $x, y \in U \bigwedge x \neq y \Longrightarrow x \notin \{y\}^* \bigwedge y \notin \{x\}^*$.

(3) $x \in U \Longrightarrow \{x\} = \{x\}^*$.

(4) $x \in U \Longrightarrow \{x\} = N(x)$.

Proof. (1) $\Longrightarrow$ (2): It holds by Lemma 3.2.

(2) $\Longrightarrow$ (3): Suppose that (2) holds. Let $x \in U$. If $y \in U$ and $y \neq x$, then $y \notin \{x\}^*$. So $\{x\}^* \subset \{x\}$. On the other hand, $\{x\} \subset \{x\}^*$ by Lemma 1(2). It follows that $\{x\}^* = \{x\}$.

(3) $\Longrightarrow$ (4): Suppose that (3) holds. Let $x \in U$. Assume that $y \in N(x)$, then $y \in K$ for each $K \in \mathcal{C}_x$. If $y \neq x$, then $x \notin \{y\}^* = U - (U - \{y\})_*$, and hence $x \in (U - \{y\})_*$. So there is $K \in \mathcal{C}_x$ such that $K \subset U - \{y\}$, i.e., $y \notin K$. This is a contradiction. It follows that $N(x) \subset \{x\}$. On the other hand, it is clear that $\{x\} \subset N(x)$. So $N(x) = \{x\}$.

(4) $\Longrightarrow$ (1): Suppose that (3) holds. Let $x, y \in U$ and $x \neq y$, then $y \notin N(x)$, and hence there is $K_x \in \mathcal{C}_x \subset \mathcal{C}$ such that $y \notin K_x$. Thus $K_x \bigcap \{x, y\} = \{x\}$. In a similar way, there is $K_y \in \mathcal{C}$ such that $K_y \bigcap \{x, y\} = \{y\}$. Consequently, $(U; \mathcal{C})$ is a G_1-space.

Theorem 3.6: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_2-space.

(2) $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K_x, K_y \in \mathcal{C}(x \in K_x \bigwedge y \in K_y \bigwedge K_x^* \bigcap K_y = \emptyset)$.

(3) $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K \in \mathcal{C}(x \in K \bigwedge y \notin K^*)$.

(4) $x, y \in U \bigwedge x \neq y \Longrightarrow \exists K \in \mathcal{C}(x \in K \subset K^* \subset U - \{y\})$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U; \mathcal{C})$ is a G_2-space. Let $x, y \in U$ and $x \neq y$. Then there are $K_x, K_y \in \mathcal{C}$ such that $x \in K_x$, $y \in K_y$ and $K_x \bigcap K_y = \emptyset$. So $K_x \subset U - K_y$. By Lemma 3.1(3),(5), $K_x^* \subset (U - K_y)^* = U - K_{y*} = U - K_y$. So $K_x^* \bigcap K_y = \emptyset$

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (4): Suppose that (3) holds. Let $x, y \in U$ and $x \neq y$. Then there is $K \in \mathcal{C}$ such that $x \in K$ and $y \notin K^*$. Note that $y \notin K^*$ if and only if $K^* \subset U - \{y\}$. So $x \in K \subset K^* \subset U - \{y\}$.

(4) $\Longrightarrow$ (1): Suppose that (4) holds. Let $x, y \in U$ and $x \neq y$. Then there is $K \in \mathcal{C}$ such that $x \in K \subset K^* \subset U - \{y\}$. So $y \in U - K^* = (U - K)_*$, and hence there is $K' \in \mathcal{C}$ such that $y \in K' \subset U - K$. It is clear that $K \bigcap K' = \emptyset$. It follows that $(U; \mathcal{C})$ is a G_2-space.

Remark 3.7: In Theorem 3.6(2), "$K_x^* \bigcap K_y = \emptyset$" can not be replaced by "$K_x^* \bigcap K_y^* = \emptyset$".

Proof. Let $U = \{a, b, c, d, e\}$, and $\mathcal{C} = \{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}, \{a, b, e\}, \{c, d, e\}\}$. It is not difficult to check that $(U; \mathcal{C})$ is a G_2-space. If $K_a \in \mathcal{C}_a$, $K_b \in \mathcal{C}_b$ and $K_a \bigcap K_b = \emptyset$, then $K_a = \{a, c\}$ and $K_b = \{b, d\}$. Since $K_a^* = \{a, c\}^* = U - (U - \{a, c\})_* = U - \{b, d, e\}_* = U - \{b, d\} = \{a, c, e\}$ and $K_b^* = \{bd\}^* = U - (U - \{b, d\})_* = U - \{a, c, e\}_* = U - \{a, c\} = \{b, d, e\}$, $e \in K_a^* \bigcap K_b^* \neq \emptyset$.

Theorem 3.8: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_3-space.
(2) $x \in U \bigwedge x \notin X \subset U \Longrightarrow x \notin X^*$.
(3) $X \subset U \Longrightarrow X = X^*$.
(4) $X \subset U \Longrightarrow X = X_*$.
(5) $x \in U \Longrightarrow \{x\} = \{x\}_*$.
(6) $x \in U \Longrightarrow \{x\} \in \mathcal{C}$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U; \mathcal{C})$ is a G_3-space. Let $x \in U$ and $x \notin X \subset U$. Then there is $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap X = \emptyset$. So $X \subset U - K$, and hence $X^* \subset (U - K)^* = U - K_* = U - K$. It follows that $x \notin X^*$.

(2) $\Longrightarrow$ (3): Suppose that (2) holds. Let $X \subset U$. Then $x \notin X^*$ if $x \notin X$. This shows that $X^* \subset X$. On the other hand, $X \subset X^*$ by Lemma 3.1(2). Consequently, $X = X^*$.

(3) $\Longrightarrow$ (4): Suppose that (3) holds. Let $X \subset U$. Then $U - X_* = (U - X)^* = U - X$, and so $X_* = U - (U - X_*) = U - (U - X) = X$.

(4) $\Longrightarrow$ (5): It is clear.

(5) $\Longrightarrow$ (6): Suppose that (5) holds. Let $x \in U$. Then $x \in \{x\} = \{x\}_*$. So there is $K \in \mathcal{C}$ such that $x \in K \subset \{x\}$. It follows that $\{x\} = K \in \mathcal{C}$.

(6) $\Longrightarrow$ (1): Suppose that (6) holds. Let $x \in U \bigwedge x \notin X \subset U$. Then $\{x\} \in \mathcal{C}$ and $\{x\} \bigcap X = \emptyset$. So $(U; \mathcal{C})$ is a G_3-space.

Lemma 3.9: Let $(U; \mathcal{C})$ be a covering approximation space, $x \in U$ and $X \subset U$. Then the following are equivalent.

(1) $x \notin X^*$.
(2) $\exists K \in \mathcal{C}(x \in K \bigwedge K \bigcap X^* = \emptyset)$.
(3) $\exists K \in \mathcal{C}(x \in K \bigwedge K \bigcap X = \emptyset)$.

Proof. (1) $\Longrightarrow$ (2): Let $x \notin X^*$, i.e., $x \notin U - (U - X)_*$. Then $x \in (U - X)_*$. So there is $K \in \mathcal{C}$ such that $x \in K \subset (U - X)_* = U - X^*$. Consequently, $K \bigcap X^* = \emptyset$.

(2) $\Longrightarrow$ (3): It holds by Lemma 3.1(2).

(3) $\Longrightarrow$ (1): If there is $K \in \mathcal{C}$ such that $x \in K$ and $K \bigcap X = \emptyset$, then $K \subset U - X$, and hence $x \in K \subset (U - X)_*$. So $x \notin U - (U - X)_* = X^*$.

Corollary 3.10: Let $(U; \mathcal{C})$ be a covering approximation space, $x \in U$ and $X \subset U$. Then the following are equivalent.

(1) $x \in X^*$.
(2) $\forall K \in \mathcal{C} \bigwedge x \in K(K \bigcap X^* \neq \emptyset)$.
(3) $\forall K \in \mathcal{C} \bigwedge x \in K(K \bigcap X \neq \emptyset)$.

The following lemma can be obtained immediately from Definition 2.5.

Lemma 3.11: Let $(U; \mathcal{C})$ be a covering approximation space, $x \in U$ and $X \subset U$. Then the following are equivalent.

(1) $x \in X_*$
(2) $\exists K \in \mathcal{C}(x \in K \subset X)$.
(3) $\exists K \in \mathcal{C}(x \in K \subset X_*)$.

Lemma 3.12: Let $(U; \mathcal{C})$ be a covering approximation space and $x \in U$. Then $\{x\}^* = D(x)$.

Proof. Let $y \in \{x\}^* = U - (U - \{x\})_*$. Then $y \notin (U - \{x\})_*$. Thus, for each $K \in \mathcal{C}$, if $y \in K$, then $K \not\subset U - \{x\}$, and hence $x \in K$, i.e., $K \in \mathcal{C}_x$. So $y \notin K$ for each $K \in \mathcal{C} - \mathcal{C}_x$, and hence $y \notin \bigcup(\mathcal{C} - \mathcal{C}_x)$. Consequently, $y \in U - \bigcup(\mathcal{C} - \mathcal{C}_x) = D(x)$. On the other hand, let $y \in D(x)$. Then we can obtain $y \in \{x\}^*$ by reversing the above proof. This proves that $\{x\}^* = D(x)$.

Theorem 3.13: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_d-space.
(2) $x \in U \Longrightarrow \exists X, Y \subset U(\{x\} = X_* \bigcap Y^*)$.

Proof. (1) $\Longrightarrow$ (2): Let $(U; \mathcal{C})$ be a G_d-space. If $x \in U$, then there is $K \in \mathcal{C}$ such that $\{x\} = K \bigcap D(x)$. Put $X = K$ and $Y = \{x\}$, then $\{x\} = K \bigcap D(x) = K_* \bigcap \{x\}^* = X_* \bigcap Y^*$ by Lemma 3.1(5) and Lemma 3.12.

(2) $\Longrightarrow$ (1): Suppose that (2) holds. Let $x \in U$. Then there are $X, Y \subset U$ such that $\{x\} = X_* \bigcap Y^*$. By Lemma 3.11, there is $K \in \mathcal{C}$ such that $x \in K \subset X_*$, and hence $\{x\} = K \bigcap Y^*$. Note that $x \in Y^*$. By Lemma 3.12 and Lemma 3.1(3)(4), $D(x) = \{x\}^* \subset (Y^*)^* = Y^*$. Thus $\{x\} \subset K \bigcap \{x\}^* = K \bigcap D(x) \subset K \bigcap Y^* = \{x\}$. Consequently, $\{x\} = K \bigcap D(x)$. This proves that $(U; \mathcal{C})$ is a G_d-space.

Theorem 3.14: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_r-space.
(2) $\forall x, y \in U(x \notin \{y\}^* \Longrightarrow y \notin \{x\}^*)$.
(3) $\forall x, y \in U(x \in \{y\}^* \Longrightarrow y \in \{x\}^*)$.

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U; \mathcal{C})$ is a G_r-space. Let $x, y \in U$ and $x \notin \{y\}^*$. Then $x \in U - \{y\}^* = (U - \{y\})_*$. So there is $K \in \mathcal{C}$ such that $x \in K \subset U - \{y\}$. Since $(U; \mathcal{C})$ is an G_r-space, $D(x) \subset K \subset U - \{y\}$. By Lemma 3.12, $\{x\}^* \subset U - \{y\}$, and hence $y \notin \{x\}^*$.

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (1): Suppose that (3) holds. Let $x \in K \in \mathcal{C}$. We only need to prove that $D(x) \subset K$. Let $y \in D(x)$. By Lemma 3.12, $y \in \{x\}^*$, and hence $x \in \{y\}^* = U - (U - \{y\})_*$. So $x \notin (U - \{y\})_*$. It follows that $K \not\subset U - \{y\}$, thus $y \in K$. This proves that $D(x) \subset K$.

IV. Relations among Separations

Theorem 4.1: Let $(U; \mathcal{C})$ be a covering approximation space. Consider the following condition.

(1) $(U; \mathcal{C})$ is a G_3-space.
(2) $(U; \mathcal{C})$ is a G_2-space.
(3) $(U; \mathcal{C})$ is a G_1-space.
(4) $(U; \mathcal{C})$ is a G_d-space.
(5) $(U; \mathcal{C})$ is a G_0-space.
(6) $(U; \mathcal{C})$ is a G_r-space.

Then (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5) and (3) $\Longrightarrow$ (6).

Proof. (1) $\Longrightarrow$ (2): Suppose that $(U; \mathcal{C})$ is a G_3-space. Let $x, y \in U$ and $x \neq y$, then $\{x\}, \{y\} \in \mathcal{C}$ by Theorem 3.8(6), and $\{x\} \bigcap \{y\} = \emptyset$. So $(U; \mathcal{C})$ is a G_2-space.

(2) $\Longrightarrow$ (3): It is clear.

(3) $\Longrightarrow$ (4): Suppose that $(U; \mathcal{C})$ is a G_1-space. For each $x \in U$, $\{x\} = \{x\}^*$ by Theorem 3.5(3). Note that $U = U_*$ by

Lemma 3.1(1). So $\{x\} = \{x\} \bigcap U = \{x\}^* \bigcap U_*$. By Theorem 3.13, $(U; \mathcal{C})$ is a G_d-space.

(4) $\Longrightarrow$ (5): Suppose that $(U; \mathcal{C})$ is a G_d-space. Let $x, y \in U$ and $x \neq y$. Then there is $K \in \mathcal{C}$ such that $\{x\} = K \bigcap D(x) = K \bigcap \{x\}^*$. If $y \notin \{x\}^*$, by Lemma 3.9, there is $K' \in \mathcal{C}$ such that $y \in K'$ and $K' \bigcap \{x\} = \emptyset$, and hence $K' \bigcap \{x, y\} = \{y\}$. If $y \in \{x\}^*$, then $y \notin K$, and hence $K \bigcap \{x, y\} = \{x\}$. Consequently, $(U; \mathcal{C})$ is a G_0-space.

(3) $\Longrightarrow$ (6): Suppose that $(U; \mathcal{C})$ is a G_1-space. For each $x \in U$, If $x \in K \in \mathcal{C}$, then $\{x\} = \{x\}^* = D(x)$ by Theorem 3.5(3) and Lemma 3.12, and hence $D(x) \subset K$. So $(U; \mathcal{C})$ is a G_r-space.

Remark 4.2: None of implications in Theorem 4.1 can be reversed (see the following counterexamples).

Example 4.3: G_2-space $\not\Longrightarrow$ G_3-space.

Proof. Let $U = \{a, b, c, d\}$, and $\mathcal{C} = \{\{a, b\}, \{a, c\}, \{c, d\}, \{b, d\}\}$.

(1) It is not difficult to check that $(U; \mathcal{C})$ is a G_2-space.

(2) $\{a\} \notin \mathcal{C}$, so $(U; \mathcal{C})$ is not a G_3-space by Theorem 3.8(6).

Example 4.4: G_1-space $\not\Longrightarrow$ G_2-space.

Proof. Let $U = \{a, b, c\}$, and $\mathcal{C} = \{\{a, b\}, \{a, c\}, \{b, c\}\}$. Then $\{a\}^* = U - (U - \{a\})_* = U - \{b, c\}_* = U - \{b, c\} = \{a\}$. Similarly, $\{b\}^* = \{b\}$ and $\{c\}^* \{c\}$. $\{a, b\}^* = U - (U - \{a, b\})_* = U - \{c\}_* = U - \emptyset = U$. Similarly, $\{b, c\}^* = \{a, c\}^* = U$.

(1) Since $\{a\}^* = \{a\}$, $\{b\}^* = \{b\}$ and $\{c\}^* = \{c\}$ $(U; \mathcal{C})$ is a G_1-space by Theorem 3.5(3).

(2) For each $K \in \mathcal{C}$, $K^* = U$, $(U; \mathcal{C})$ is not a G_2-space by Theorem 3.6(3).

Example 4.5: G_d-space $\not\Longrightarrow$ G_1-space.

Proof. Let $U = \{a, b\}$, and $\mathcal{C} = \{\{a\}, \{a, b\}\}$. Then $\{a\}_* = \{a\}$, $\{a\}^* = U - (U - \{a\})_* = U - \{b\}_* = U - \emptyset = U$ and $\{b\}^* = U - (U - \{b\})_* = U - \{a\}_* = U - \{a\} = \{b\}$.

(1) Since $\{a\} = \{a\} \bigcap U = \{a\}_* \bigcap U^*$ and $\{b\} = U \bigcap \{b\} = U_* \bigcap \{b\}^*$, $(U; \mathcal{C})$ is a G_d-space by Theorem 3.13.

(2) $\{a\}^* \neq \{a\}$, so $(U; \mathcal{C})$ is not a G_1-space by Theorem 3.5(3).

Example 4.6: G_0-space $\not\Longrightarrow$ G_d-space.

Proof. Let $U = \{a, b, c\}$, and $\mathcal{C} = \{\{a, c\}, \{b, c\}\}$. Then $\{a\}^* = U - (U - \{a\})_* = U - \{b, c\}_* = U - \{b, c\} = \{a\}$, $\{b\}^* = U - (U - \{b\})_* = U - \{a, c\}_* = U - \{a, c\} = \{b\}$ and $\{c\}^* = U - (U - \{c\})_* = U - \{a, b\}_* = U - \emptyset = U$.

(1) Since $\{a\}^* \neq \{b\}^* \neq \{c\}^* \neq \{a\}^*$, $(U; \mathcal{C})$ is a G_0-space by Theorem 3.4(3).

(2) For each $K \in \mathcal{C}$, $K \bigcap D(c) = K \bigcap \{c\}^* = K \bigcap U = K \neq \{c\}$, so $(U; \mathcal{C})$ is not a G_d-space.

Example 4.7: G_r-space $\not\Longrightarrow$ G_0-space.

Proof. Let $U = \{a, b\}$, and $\mathcal{C} = \{\{a, b\}\}$. It is easy to see that $(U; \mathcal{C})$ is a G_r-space and it is not a G_0-space. In addition, $(U; \mathcal{C})$ is also a Pawlak approximation space.

Example 4.8: G_d-space $\not\Longrightarrow$ G_r-space.

Proof. If G_d-space $\Longrightarrow$ G_r-space, then G_d-space $\Longrightarrow$ G_1-space by the following Theorem 4.9. This contradicts Example 4.5.

Theorem 4.9: Let $(U; \mathcal{C})$ be a covering approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_1-space.

(2) $(U; \mathcal{C})$ is a G_0- and G_r-space.

Proof. (1) $\Longrightarrow$ (2): It holds by Theorem 4.1.

(2) $\Longrightarrow$ (1): Suppose that $(U; \mathcal{C})$ is a G_0- and G_r-space. Let $x, y \in U$ and $x \neq y$. By Theorem 3.4(2), $x \notin \{y\}^*$ or $y \notin \{x\}^*$. Without loss of generality, assume that $x \notin \{y\}^*$. Then $y \notin \{x\}^*$ by Theorem 3.14(2). So $(U; \mathcal{C})$ is n G_1-space by Theorem 3.5(2).

Although none of implications in Theorem 4.1 can be reversed, we have the following equivalences on separations in Pawlak approximation spaces.

Theorem 4.10: Let $(U; \mathcal{C})$ be a Pawlak approximation space. Then the following are equivalent.

(1) $(U; \mathcal{C})$ is a G_3-space.

(2) $(U; \mathcal{C})$ is a G_2-space.

(3) $(U; \mathcal{C})$ is a G_1-space.

(4) $(U; \mathcal{C})$ is a G_d-space.

(5) $(U; \mathcal{C})$ is a G_0-space.

(6) $\mathcal{C} = \{\{x\} : x \in U\}$.

Proof. (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) $\Longrightarrow$ (5): They are hold by Theorem 4.1.

(5) $\Longrightarrow$ (6): Suppose $(U; \mathcal{C})$ is a G_0-space. Let $x \in U$. Then there is $K_x \in \mathcal{C}$ such that $x \in K_x$. If $\{x\} \notin \mathcal{C}$, then $K_x \neq \{x\}$, and hence there is $y \in K_x$ and $y \neq x$. Thus $K_x \bigcap \{x, y\} = \{x, y\}$. On the other hand, since $(U; \mathcal{C})$ be a Pawlak approximation space, $\mathcal{C}$ is a partition of U, and hence $K \bigcap \{x, y\} = \emptyset$ for each $K \in \mathcal{C} - \{K_x\}$. This contradicts that $(U; \mathcal{C})$ is a G_0-space. So $\{x\} \in \mathcal{C}$.

(6) $\Longrightarrow$ (1): Suppose that (6) holds. Then $\{x\} \in \mathcal{C}$ for each $x \in U$. By Theorem 3.8(6), $(U; \mathcal{C})$ is a G_3-space.

Remark 4.11: By Theorem 4.10, G_0-Pawlak approximation spaces are G_r-spaces. But G_r-Pawlak approximation space $\not\Longrightarrow$ G_0-space by Example 4.7. So the condition "$(U; \mathcal{C})$ is a G_r-space" is not equivalent to conditions in Theorem 4.10.

V. Conclusion

This paper explore a new property in covering approximation spaces: granularity-wise separation. Adopting Zakowski's covering approximation operators $\overline{C}$ and $\underline{C}$, this paper give some characterization of covering approximation spaces with some granularity-wise separation and establish some relations among these spaces. Results of this paper deepen and enrich rough set theory, which is helpful to understand inherent property of covering approximation spaces completely.

In this paper, our investigations are based on Zakowski's covering approximation operators $\overline{C}$ and $\underline{C}$. Because there are also other useful covering approximation operators [10], [11], [14], [15], [19], [20], it is a natural question how to investigate separations in corresponding covering approximation spaces with these covering approximation operators. This is still worthy to be considered in subsequent research.

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