Applications of High-Order Compact Finite Difference Scheme to Nonlinear Goursat Problems

Mohd Agos Salim Nasir, Ahmad Izani Md. Ismail

Abstract—Several numerical schemes utilizing central difference approximations have been developed to solve the Goursat problem. However, in a recent years compact discretization methods which leads to high-order finite difference schemes have been used since it is capable of achieving better accuracy as well as preserving certain features of the equation e.g. linearity. The basic idea of the new scheme is to find the compact approximations to the derivative terms by differentiating centrally the governing equations. Our primary interest is to study the performance of the new scheme when applied to two Goursat partial differential equations against the traditional finite difference scheme.

Keywords—Goursat problem, partial differential equation, finite difference scheme, compact finite difference

I. INTRODUCTION

MOST realistic models of physical and other phenomena are nonlinear in nature and can be described using nonlinear partial differential equations. These nonlinear partial differential equations, usually relating space and time derivatives, need to be solved in order to gain a fuller and deeper understanding of the problem. Due to the nonlinearity, analytical methods cannot be utilized and numerical methods such as the high-order compact discretizations need to be applied.

The hyperbolic partial differential models known as the Goursat problem have been investigated and the solutions of this problem for both linear and nonlinear problems using the aid of numerical methods have been proposed by many researches like [7], [11], [14], [16] and [21]. Applications of the Goursat problem can be found in many science problems, e.g. inverse acoustic scattering problem [13], wave equations in nonhomogeneous media [15], sine-Gordon equation [18] and electric field problem [20].

Traditional numerical discretization scheme for approximating the Goursat problem usually employ forwarded differentiating for the mix derivatives term [14], [16]. The existing scheme (AM scheme-standard scheme) had been used arithmetic mean averaging of functional values in finite difference approximation. [16] and [14] have been used geometric and harmonic mean averaging of functional values, the so-called GM and HM schemes respectively for the Goursat problem.

Mohd Agos Salim Nasir is with the Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA Malaysia, 40450 Shah Alam, Selangor, Malaysia (phone: 603-55435327; e-mail: masn@tmsk.uitm.edu.my).

Ahmad Izani Md. Ismail is with the School of Mathematical Sciences, School of Mathematical Sciences, 11800 Minden, Penang, Malaysia (phone: 604-6533888; e-mail: izani@cs.usm.my).

However all this schemes have been neglecting the higher order terms of the expansions of Taylor series. Eventhough it will simplify the discretization activities, but the accuracy will be reduced.

In recent years, high order compact finite difference approximations have been applied to solve several differential equations, e.g. convergence for second-order elliptic equations [1], nonlinear dispersive waves problem [2], compressible Navier-Stokes equations [3], financial applications of symbolically [4], dispersive media problem [5], elliptic equations on irregular domains and interface problems and their applications [6], unsteady viscous incompressible flows [8], numerical solution of partial differential equation [9] and initial boundary problems for mixed systems [10].

A class of high-order compact finite difference scheme exhibits higher accuracy at the grid points in that applies a compact stencil. The governing differential equations will be used to approximate leading truncation error terms in the central difference scheme [12]. In this paper we present the compact finite difference scheme for the Goursat problems. With the aid of computational software the scheme was programmed for determining the relative errors of nonlinear Goursat problems.

II. FINITE DIFFERENCE SCHEME AND THE GOURSAT PROBLEM

The Goursat problem is of the form [14]:

$$u_{xy} = f(x, y, u, u_x, u_y)$$
$$u(x,0) = \phi(x), u(0, y) = \psi(y), \phi(0) = \psi(0) \qquad (1)$$
$$0 \le x \le a, 0 \le y \le b$$

The established standard finite difference scheme is,

$$\frac{u_{i+1, j+1} + u_{i, j} - u_{i+1, j} - u_{i, j+1}}{h^2}$$
$$= \frac{1}{4}\left(f_{i+1, j+1} + f_{i, j} + f_{i+1, j} + f_{i, j+1}\right) \qquad (2)$$

Here, the function value at grid location $(i+1/2, j+1/2)$ is approximated by:

$$\frac{1}{4}\left(f_{i+1, j+1} + f_{i, j} + f_{i+1, j} + f_{i, j+1}\right) \qquad (3)$$

The derivation of the left hand side of (2) is instructive and is as follows:

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:6, No:8, 2012

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{u(x+\Delta x, y)-u(x,y)}{\Delta x}\right)$$

$$= \frac{1}{\Delta x}\frac{\partial}{\partial y}\left(u(x+\Delta x, y)-u(x,y)\right) \qquad (4)$$

$$= \frac{1}{\Delta x}\left[\begin{array}{c}\frac{(u(x+\Delta x, y+\Delta y)-u(x+\Delta x, y))}{\Delta y}\\ -\frac{(u(x, y+\Delta y)-u(x,y))}{\Delta y}\end{array}\right]$$

$$= \frac{1}{h^2}\left[\begin{array}{c}u(x+\Delta x, y+\Delta y)-u(x+\Delta x, y))-\\ u(x, y+\Delta y)+u(x,y)\end{array}\right]$$

where $h = \Delta x = \Delta y$.

Studies of the use of finite difference schemes utilizing alternatives to the arithmetic mean have been reported by [16] and [14]. However for linear problem these alternative schemes will result in non-linear scheme whereas the standard scheme utilizing the arithmetic mean preserves linearity.

III. THE GOURSAT PROBLEM AND THE HIGH-ORDER COMPACT METHOD

We assumed the unknown function u to be continuously differentiable and have the required partial derivatives on the rectangular domain.

By manipulating the expansions of Taylor series in two independent variables [17]. It can be written that

$$u(x+\Delta x, y+\Delta y)-u(x+\Delta x, y-\Delta y)-u(x-\Delta x, y+\Delta y)+$$

$$u(x-\Delta x, y-\Delta y) = 4(\Delta x)(\Delta y)u_{xy}+\frac{2}{3}(\Delta x)^3(\Delta y)u_{xxxy}+ \qquad (5)$$

$$\frac{2}{3}(\Delta x)(\Delta y)^3 u_{xyyy}+O\left[(\Delta x+\Delta y)^6\right]$$

Thus for the mixed central derivatives of u_{xy}, the expression is as follows.

$$\delta_x \delta_y u = \frac{\left[\begin{array}{c}u(x+\Delta x, y+\Delta y)-u(x+\Delta x, y-\Delta y)-\\ u(x-\Delta x, y+\Delta y)+u(x-\Delta x, y-\Delta y)\end{array}\right]}{4(\Delta x)(\Delta y)}-$$

$$\frac{2}{12}(\Delta x)^2(\delta_x^3\delta_y u + error)-\frac{2}{12}(\Delta y)^2(\delta_x\delta_y^3 u + error)+ \qquad (6)$$

$$O\left[\frac{(\Delta x+\Delta y)^6}{4(\Delta x)(\Delta y)}\right]$$

Substituting (6) into (1) leads to

$$\frac{\left[\begin{array}{c}u(x+\Delta x, y+\Delta y)-u(x+\Delta x, y-\Delta y)-\\ u(x-\Delta x, y+\Delta y)+u(x-\Delta x, y-\Delta y)\end{array}\right]}{4(\Delta x)(\Delta y)}-$$

$$\frac{2}{12}(\Delta x)^2(\delta_x^3\delta_y u + error)-\frac{2}{12}(\Delta y)^2(\delta_x\delta_y^3 u + error)+ \qquad (7)$$

$$O\left[\frac{(\Delta x+\Delta y)^6}{4(\Delta x)(\Delta y)}\right] = f(x, y, u, u_x, u_y)$$

We rewrite the (7) in indexing form as follows:

$$\frac{u_{i+1,j+1}-u_{i+1,j-1}-u_{i-1,j+1}+u_{i-1,j-1}}{4h^2}-\phi_{i,j}=f_{i,j} \qquad (8)$$

where $\Delta x = \Delta y = h$ and $\phi_{i,j}$ is the truncation error evaluated at node (i,j) and given as

$$\phi_{i,j} = \frac{2}{12}h^2(\delta_x^3\delta_y u_{i,j}+error)+$$

$$\frac{2}{12}h^2(\delta_x\delta_y^3 u_{i,j}+error)+O(h^4) \qquad (9)$$

The relations for $\delta_x^3\delta_y u_{i,j}$ and $\delta_x\delta_y^3 u_{i,j}$ can be constructed by differentiating (1) centrally to get

$$\delta_x^3\delta_y u_{i,j} = \delta_x^2 f_{i,j} \qquad (10)$$

In a like manner

$$\delta_x\delta_y^3 u_{i,j} = \delta_y^2 f_{i,j} \qquad (11)$$

The central difference scheme for Goursat problem (1) is obtained by neglecting $\phi_{i,j}$ in (8). The truncation error at grid point (i,j) is $O(h^2)$, corresponding to the leading term in (9). The basic idea behind the high-order compact finite difference approach is to find compact approximations to the derivatives in (9) by differentiating centrally the governing equation. This relations (10) and (11) substituted into (9) to get the new trunction error expression,

$$\phi_{i,j} = \frac{2}{12}h^2(\delta_x^2 f_{i,j}+error)+$$

$$\frac{2}{12}h^2(\delta_y^2 f_{i,j}+error)+O(h^4) \qquad (12)$$

Successively higher-order compact finite difference schemes could be devised by repeating the above procedure to obtain a scheme that is of some arbitrary high-order [12].

IV. NUMERICAL EXPERIMENTS

We consider the nonlinear Goursat problems below:

Problem 1:

$$u_{xy} = e^{2u}$$

$$u(x,0) = \frac{x}{2} - \ln(1 + e^x)$$

$$u(0,y) = \frac{y}{2} - \ln(1 + e^y)$$ (13)

$$0 \le x \le 4, \, 0 \le y \le 4$$

The analytical solution of problem (13) is [14]:

$$u(x,y) = \frac{x+y}{2} - \ln(e^x + e^y)$$

It is of interest to note that our R.H.S function of (13) can be written is as follows:

$$f(u) = e^{2u}$$ (14)

Differentiating (14) over x, evaluated at node (i,j) and omit the errors that is relatively small gives

$$\delta_x f_{i,j} = 2\delta_x u_{i,j} e^{2u_{i,j}}$$ (15)
$$\Rightarrow \delta_x^2 f_{i,j} = 2\delta_x^2 u_{i,j} e^{2u_{i,j}} + 4(\delta_x u_{i,j})^2 e^{2u_{i,j}}$$

In a like manner

$$\delta_y f_{i,j} = 2\delta_y u_{i,j} e^{2u_{i,j}}$$ (16)
$$\Rightarrow \delta_y^2 f_{i,j} = 2\delta_y^2 u_{i,j} e^{2u_{i,j}} + 4(\delta_y u_{i,j})^2 e^{2u_{i,j}}$$

Substituting terms (15) and (16) into (12), gives the discretization of $u_{xy} = u$ i.e.

$$\phi_{i,j} = \frac{h^2 e^{2u_{i,j}}}{3} \left\{ \delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} + 2\left[(\delta_x u_{i,j})^2 + (\delta_y u_{i,j})^2 \right] \right\}$$ (17)

Manipulating the Taylor series expansions and evaluating at node (i,j), we get

$$\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} = \frac{\left(\begin{array}{c} u_{i+1,j+1} + u_{i+1,j-1} + \\ u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j} \end{array} \right)}{2h^2}$$ (18)

$$\delta_x u_{i,j} = \frac{u_{i+1,j+1} + u_{i+1,j-1} - u_{i-1,j+1} - u_{i-1,j-1}}{4h}$$ (19)

$$\delta_y u_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} + u_{i-1,j+1} - u_{i-1,j-1}}{4h}$$ (20)

Hence, (17) becomes

$$\phi_{i,j} = \frac{h^2 e^{2u_{i,j}}}{3} \left\{ \frac{\left(\begin{array}{c} u_{i+1,j+1} + u_{i+1,j-1} + \\ u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j} \end{array} \right)}{2h^2} + 2 \left[\left(\frac{\begin{array}{c} u_{i+1,j+1} + u_{i+1,j-1} - \\ u_{i-1,j+1} - u_{i-1,j-1} \end{array}}{4h} \right)^2 + \left(\frac{\begin{array}{c} u_{i+1,j+1} - u_{i+1,j-1} + \\ u_{i-1,j+1} - u_{i-1,j-1} \end{array}}{4h} \right)^2 \right] \right\}$$ (21)

The new implicit scheme can be written as

$$\frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2} - \frac{h^2 e^{2u_{i,j}}}{3} \left\{ \frac{\left(\begin{array}{c} u_{i+1,j+1} + u_{i+1,j-1} + \\ u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j} \end{array} \right)}{2h^2} + 2 \left[\left(\frac{\begin{array}{c} u_{i+1,j+1} + u_{i+1,j-1} - \\ u_{i-1,j+1} - u_{i-1,j-1} \end{array}}{4h} \right)^2 + \left(\frac{\begin{array}{c} u_{i+1,j+1} - u_{i+1,j-1} + \\ u_{i-1,j+1} - u_{i-1,j-1} \end{array}}{4h} \right)^2 \right] \right\} - e^{2u_{i,j}} = 0$$ (22)

Now we consider another Goursat problem:
Problem 2:

$$u_{xy} + g(u) = r1$$

$$\text{on } \overline{R} = \{(x,y) \mid 0 \le x \le 0.5, 0 \le y \le 0.5\},$$

$$u = x^4 + 1 \qquad 0 \le x \le 0.5, \, y = 0,$$

$$u = 1 + y + \sin y, \qquad x = 0, 0 \le y \le 0.5,$$ (23)

$$\text{where } g = -0.1(u^3 + u),$$

$$r1 = 0.1(\cos y)e^{0.1x} - 0.1(z^3 + z) \text{ and}$$

$$z = (\sin y)e^{0.1x} + y + x^4 + 1$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:6, No:8, 2012

The exact solution of problem (23) is [19]:

$$u(x, y) = (\sin y)e^{0.1x} + y + x^4 + 1$$

We rewrite the (23) as follows

$$
\begin{aligned}
u_{xy} &= f(u, x, y) \\
&= 0.1(\cos y)e^{0.1x} + 0.1(u^3 + u) - \\
&\quad 0.1\left\{ \begin{bmatrix} (\sin y)e^{0.1x} + y + x^4 + 1 \end{bmatrix}^3 + \\ \begin{bmatrix} (\sin y)e^{0.1x} + y + x^4 + 1 \end{bmatrix} \right\}
\end{aligned}
\tag{24}
$$

By using the same procedure, for the problem (24), the new implicit scheme can be constructed as

$$
\frac{u_{i+1,j+1} - u_{i+1,j-1} - u_{i-1,j+1} + u_{i-1,j-1}}{4h^2} - \phi_{i,j} - f_{i,j} = 0
\tag{25}
$$

where

$$
\begin{aligned}
f_{i,j} &= 0.1(\cos j)e^{0.1i} + 0.1((u_{i,j})^3 + u_{i,j}) - \\
&\quad 0.1\left\{ \begin{bmatrix} (\sin j)e^{0.1i} + j + i^4 + 1 \end{bmatrix}^3 + \\ \begin{bmatrix} (\sin j)e^{0.1i} + j + i^4 + 1 \end{bmatrix} \right\}
\end{aligned}
\tag{26}
$$

and

$$
\phi_{i,j} = \frac{h^2(\delta_x^2 f_{i,j} + \delta_y^2 f_{i,j})}{6}
\tag{27}
$$

Here,

$$
\begin{aligned}
&\delta_x^2 f_{i,j} + \delta_y^2 f_{i,j} = \\
&0.1^3(\cos j)e^{0.1i} - 0.1(\cos j)e^{0.1i} \\
&+ 0.1\left\{ \begin{matrix} 6u_{i,j}[(\delta_x u_{i,j})^2 + (\delta_y u_{i,j})^2] + \\ (1 + 3(u_{i,j})^2)(\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j}) \end{matrix} \right\} \\
&- 0.1\left\{ \begin{matrix} 6\big[(\sin j)e^{0.1i} + j + i^4 + 1\big]\big[0.1(\sin j)e^{0.1i} + 4i^3\big] \\ \big[0.1(\sin j)e^{0.1i} + 4i^3\big] + 3\big[(\sin j)e^{0.1i} + j + i^4 + 1\big]^2 \\ \big[0.1^2(\sin j)e^{0.1i} + 12i^2\big] + \big[0.1^2(\sin j)e^{0.1i} + 12i^2\big] \end{matrix} \right\} \\
&- 0.1\left\{ \begin{matrix} 6\big[(\sin j)e^{0.1i} + j + i^4 + 1\big]\big[(\cos j)e^{0.1i} + 1\big] \\ \big[(\cos j)e^{0.1i} + 1\big] + 3\big[(\sin j)e^{0.1i} + j + i^4 + 1\big]^2 \\ \big[-(\sin j)e^{0.1i}\big] - \big[(\sin j)e^{0.1i}\big] \end{matrix} \right\}
\end{aligned}
\tag{28}
$$

with

$$
\delta_x^2 u_{i,j} + \delta_y^2 u_{i,j} = \frac{\begin{pmatrix} u_{i+1,j+1} + u_{i+1,j-1} + \\ u_{i-1,j+1} + u_{i-1,j-1} - 4u_{i,j} \end{pmatrix}}{2h^2}
\tag{29}
$$

$$
\delta_x u_{i,j} = \frac{\begin{pmatrix} u_{i+1,j+1} + u_{i+1,j-1} - \\ u_{i-1,j+1} - u_{i-1,j-1} \end{pmatrix}}{4h}
\tag{30}
$$

$$
\delta_y u_{i,j} = \frac{u_{i+1,j+1} - u_{i+1,j-1} + u_{i-1,j+1} - u_{i-1,j-1}}{4h}
\tag{31}
$$

The programming development for the application of schemes (22) and (25)-(31) to find the relative errors of problems (13) and (23) respectively, need the usage of computing machine and technical coding software. The results are summarized and presented as below:

It is clear that, for the nonlinear Goursat problems (13) and (23) and for the grid sizes investigated, the high-order compact finite difference scheme is more accurate than the standard scheme.

V. CONCLUDING REMARKS

High-order compact schemes have been applied by several researchers to solve partial differential equations. Numerical experiments on the two nonlinear Goursat problems indicate that the high-order scheme performed better than the standard scheme in the literature.

TABLE II
AVERAGE RELATIVE ERRORS OF PROBLEM (23)

Grid size (h)	Standard scheme	Compact scheme
0.0025	2.0022544e-09	2.1175816e-11
0.0050	8.0811486e-09	1.6795880e-10
0.0250	2.1667491e-07	2.0427646e-08
0.0500	9.4208567e-07	1.6287396e-07

ACKNOWLEDGMENT

REFERENCES

[1] J. Zhao, T. Zhang and R.M. Corless. *Convergence of the compact finite difference method for second order elliptic equations.* Applied Mathematics and Computation. 2006, pg. 1454-1469.
[2] J. Li and M.R. Visbal. *High order compact schemes for nonlinear dispersive waves.* Journal of scientific computing. 2006, pg. 1-23.
[3] B.J. Boersma. *A staggered compact finite difference formulation for the compressible Navier-Stokes equations.* Journal of Computational Physics. 2005, pg. 675-690.
[4] J. Zhao, R.M. Corless and M. Davison. *Financial applications of symbolically generated compact finite difference formulae.* Dept. of Applied Mathematics. Western Ontario London University. 2005.
[5] J. Li and Y. Chen. *High order compact schemes for dispersive media.* Electronics Letters. 2004, pg. 14.

[6] Y. Kyei. *Higher order Cartesian grid based finite difference methods for elliptic equations on irregular domains and interface problems and their applications*. PhD dissertation. North Carolina State University. 2004.

[7] M.A.S. Nasir and A.I.M. Ismail. *Numerical solution of a linear Goursat problem: Stability, consistency and convergence.* WSEAS Transactions on Mathematics. 2004.

[8] M. Li and T. Tang. *A compact fourth order finite difference scheme for unsteady viscous incompressible flows*. Journal of Scientific Computing. 2001, pg. 29-45.

[9] M.O. Ahmed. *An exploration of compact finite difference methods for the numerical solution of PDE. PhD dissertation*. Western Ontario London University. 1997.

[10] R. Bodenmann and H.J. Schroll. *Compact difference methods applied to initial boundary value problems for mixed systems*. Numer. Math. 1996, pg. 291-309.

[11] A.M. Wazwaz. *The decomposition method for approximate solution of the Goursat problem*. Applied Mathematics and Computation. 1995, pg. 299-311.

[12] W.F. Spotz. *High order compact finite difference schemes for computational mechanics*. PhD dissertation. Texas University. 1995.

[13] J.R. Mclaughlin, P.L. Polyakov and P.E. Sacks. *Reconstruction of a spherical symmetric speed of sound*. SIAM Journal of Applied Mathematics. 1994, pg. 1203-1223.

[14] A.M. Wazwaz. *On the numerical solution for the Goursat problem*. Applied Mathematics and Computation. 1993, pg. 89-95.

[15] P. Hillion. *A note on the derivation of paraxial equation in nonhomogeneous media*. SIAM Journal of Applied Mathematics. 1992, pg. 337-346.

[16] D.J. Evans and B.B. Sanugi. *Numerical solution of the Goursat problem by a nonlinear trapezoidal formula*. Applied Mathematics Letter. 1988, pg. 221-223.

[17] E.H. Twizell. *Computational Methods For Partial Differential Equations*. Ellis Horwood Limited. 1984.

[18] D.J. Kaup and A.C. Newell. *The Goursat and Cauchy problems for the Sine Gordon equation*. SIAM Journal of Applied Mathematics. 1978, pg. 37-54.

[19] T.Y. Cheung. *Three Nonlinear Initial Value Problems of The Hyperbolic Type*. SIAM J. Numer. Anal. 1977, pg. 484-491.

[20] R.A. Frisch and B.R. Cheo. *On a bounded one dimensional Poisson-Vlasov system*. Society for Industry and Applied Mathematics. 1972.

[21] J.T. Day. *A Runge-Kutta method for the numerical solution of the Goursat problem in hyperbolic partial differential equations*. Computer Journal. 1966, pg. 81-83.

Dr. Mohd Agos Salim Nasir is a senior lecturer in mathematics. He obtained his master from Universiti Sains Malaysia in 2003 and received his PhD in 2010 at the same university. He is also a member of IEEE and SIAM. His research interest is in Numerical Methods and Numerical Analysis.

Dr. Ahmad Izani Md. Ismail is the Dean and a Professor in mathematics at School of Mathematical Sciences, Universiti Sains Malaysia. His research interest is in Numerical Methods and Numerical Analysis.