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Characterizations of ordered semigroups by $(\in, \in \vee q)$-fuzzy ideals

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Abstract—Let S be an ordered semigroup. In this paper we first introduce the concepts of $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy bi-ideals and $(\in, \in \vee q)$-fuzzy generalized bi-ideals of an ordered semigroup S, and investigate their related properties. Furthermore, we also define the upper and lower parts of fuzzy subsets of an ordered semigroup S, and investigate the properties of $(\in, \in \vee q)$-fuzzy ideals of S. Finally, characterizations of regular ordered semigroups and intra-regular ordered semigroups by means of the lower part of $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals are given.

Keywords—Ordered semigroup; regular ordered semigroup; intra-regular ordered semigroup; $(\in, \in \vee q)$-fuzzy left (right) ideal of an ordered semigroup; $(\in, \in \vee q)$-fuzzy (generalized) bi-ideal of an ordered semigroup.

I. Introduction

The theory of fuzzy sets, proposed by L. A. Zadeh in 1965, has provided a useful mathematical tool for describing the behavior of systems that are too complex to admit precise mathematical analysis by classical methods and tools. Extensive applications of fuzzy set theory have been found in various fields such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. It soon invoked a natural question concerning a possible connection between fuzzy sets and algebraic systems. Kuroki initiated the theory of fuzzy semigroups in his papers [10]. A systematic exposition of fuzzy semigroups by Mordeson et al. appeared in [14], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy Languages and fuzzy finite state machines. In [10-12], Kuroki gave some properties of fuzzy ideals and fuzzy bi-ideals and fuzzy generalized bi-ideals of semigroups (without order) and characterized regular semigroups, intra-regular semigroups, semigroups that are semilattices of left (right) simple semigroups and so on in terms of fuzzy ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals. Following the works by Kuroki [10-12], the concept of (α, β)-fuzzy subgroups was introduced by S.K. Bhakat and P. Das [1,2], based on the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup. In particular, the concept of an $(\in, \in \vee q)$-fuzzy subgroup is a useful generalization of Rosenfeld's fuzzy subgroup [13]. In [3] Davvaz defined $(\in, \in \vee q)$-fuzzy subnearring and ideals of a nearring. In [4] Jun and Song initiated the study of (α, β)-fuzzy interior ideals of a semigroup. Kazancı and Yamak introduced the concept of generalized fuzzy bi-ideals of a semigroup [5] and gave some properties of fuzzy bi-ideals by means of $(\in, \in \vee q)$-fuzzy bi-ideals. Ma and Zhan [6] study $(\in, \in \vee q)$-fuzzy h-bi-ideals and h-quasi-ideals of a hemiring.

Following the terminology given by Zadeh, if S is an ordered semigroup, fuzzy sets in ordered semigroups S were first considered by Kehayopulu and Tsingelis in [18], then they defined " fuzzy" analogous for several notations, which have proven useful in the theory of ordered semigroups. Moreover, they proved that each ordered groupoid can be embedded into an ordered groupoid having the greatest element (poe-groupoid) in terms of fuzzy sets [19]. A theory of fuzzy sets on ordered semigroups has been recently developed (see [8,9,20-23,28-32]). The concept of ordered fuzzy points of an ordered semigroup S was introduced by Xie and Tang [28], and prime fuzzy ideals of an ordered semigroup S were studied in [30]. Authors also introduced the concepts of weakly prime fuzzy ideals, completely prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of an ordered semigroup S, and established the relations among five types ideals. Furthermore, Xie and Tang characterize weakly prime fuzzy ideals, completely semiprime fuzzy ideals and weakly completely prime fuzzy ideals of S by their level ideals [28].

The concept of regular rings was introduced by von Neumann [24]. As we know, regular rings play an important role in the abstract algebra. In 1956, Kov*á*cs characterized a regular rings as rings satisfying the condition $A \cap B = AB$ for every right ideal A and every left ideal B [25]. Since then, Is*é*ki studied the same properties for semigroups, and proved that a commutative semigroup S is regular if and only if every ideal of S is idempotent [26]. In 1961, Calais proved that a semigroup S is regular if and only if the right and the left ideals of S are idempotent and for every right ideal A and every left ideal B of S, the product AB is a quasi-ideal of S [27]. Then Kehayopulu extended those results to ordered semigroups [15, 16]. Recently, Kehayopulu also extended those similar fuzzy results to ordered semigroups, and characterized the regular, the left regular and the right regular ordered semigroups by means of fuzzy left ideals, fuzzy right ideals, fuzzy semiprime subsets and fuzzy quasi-ideals (see [20-22]). In [29], the authors also characterized regular ordered semigroups and intra-regular ordered semigroups in terms of fuzzy left ideals, fuzzy right ideals, fuzzy (generalized) bi-ideals and fuzzy quasi-ideals. Shabir et al. [7] characterized regular ordered semigroups by (α, β)-fuzzy ideals.

As a further study, the concepts of $(\in, \in \vee q)$-fuzzy ideals,

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$(\in, \in \vee q)$-fuzzy bi-ideals and $(\in, \in \vee q)$-fuzzy generalized bi-ideals, which are generalization of fuzzy ideals, fuzzy bi-ideals and fuzzy generalized bi-ideals in an ordered semigroup are introduced, and related properties are investigated. Furthermore, we also define the upper and lower parts of fuzzy subsets of an ordered semigroup S, and investigate the properties of $(\in, \in \vee q)$-fuzzy ideals of S. Then regular ordered semigroups are characterized by means of the lower part of $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals. Finally, we give a main theorem which characterize the regular ordered semigroups and intra-regular ordered semigroups in terms of the lower part of $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy bi-ideals. The paper illustrates that one can pass from the theory of semigroups or ordered semigroups to the theory of "fuzzy" ordered semigroups. As an application of the results of this paper, the corresponding results of semigroup (without order) are also obtained.

II. Preliminaries and some notations

Throughout this paper unless stated otherwise S stands for an ordered semigroup, that is, a semigroup S with an order relation " $\leq$ " such that $a \leq b$ implies $xa \leq xb$ and $ax \leq bx$ for any $x \in S$ (for example, see [17]). A function f from S to the real closed interval [0,1] is a *fuzzy subset* of S. The ordered semigroup S itself is a fuzzy subset of S such that $S(x) \equiv 1$ for all $x \in S$ (the fuzzy subset S is also denoted by 1 in [11]). Let f and g be two fuzzy subsets of S. Then the inclusion relation $f \subseteq g$ is defined by $f(x) \leq g(x)$ for all $x \in S$, and $f \cap g$, $f \cup g$ are defined by

$$(f \cap g)(x) = \min\{f(x), g(x)\} = f(x) \wedge g(x),$$

$$(f \cup g)(x) = \max\{f(x), g(x)\} = f(x) \vee g(x)$$

for all $x \in S$, respectively. The set of all fuzzy subsets of S is denoted by $F(S)$. One can easily show that $(F(S), \subseteq, \cap, \cup)$ forms a complete lattice with the maximum element S and the minimum element 0, which is a mapping from S into $[0,1]$ defined by

$$0: \ S \rightarrow [0,1], \ x \mapsto 0(x) := 0, \ \forall x \in S.$$

Let $(S, \cdot, \leq)$ be an ordered semigroup. For $x \in S$, we define $A_x := \{(y,z) \in S \times S | \ x \leq yz\}$. The product $f \circ g$ of f and g is defined by

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{(y,z)\in A_x} \min\{f(y), g(z)\}, & \text{if } A_x \neq \emptyset \ , \\ 0 & , \ \text{if } A_x = \emptyset. \end{cases}$$

for all $x \in S$. It is well known (see Theorem of [19]) that this operation "$\circ$" is associative and $(F(S), \circ, \subseteq)$ is a *poe*-semigroup.

Let S be an ordered semigroup. For $H \subseteq S$, we define

$$(H] := \{t \in S \mid t \leq h \ \text{ for some } \ h \in H\}.$$

For $H = \{a\}$, we write $(a]$ instead of $(\{a\}]$. For two subsets A, B of S, we have: (1) $A \subseteq (A]$; (2) If $A \subseteq B$, then $(A] \subseteq (B]$; (3) $(A](B] \subseteq (AB]$; (4) $((A]] = (A]$; (5) $((A](B]] = (AB]$ (cf. [17]).

By a *subsemigroup* of S we mean a nonempty subset A of S such that $A^2 \subseteq A$. A nonempty subset A of an ordered semigroup S is called a *left* (resp. *right*) *ideal* of S if

(1) $SA \subseteq A$ (resp. $AS \subseteq A$) and

(2) If $a \in A$ and $S \ni b \leq a$, then $b \in A$.

If A is both a left and a right ideal of S, then it is called an *(two-sided) ideal* of S [17]. We denote by $I(a)$ the (two-sided) ideal of S generated by a $(a \in S)$. Then $I(a) = (a \cup Sa \cup aS \cup SaS]$ [17]. A subsemigroup B of an ordered semigroup S is called a *bi-ideal* of S if

(1) $BSB \subseteq B$ and

(2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$ [20].

A non-empty subset B of an ordered semigroup S is called a *generalized bi-ideal* of S if

(1) $BSB \subseteq B$.

(2) If $a \in B$ and $S \ni b \leq a$, then $b \in B$ [29].

Let A be a nonempty subset of S. We denote by f_A the characteristic mapping of A, that is the mapping of S into $[0,1]$ defined by

$$f_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Then f_A is a fuzzy subset of S. Clearly, for two nonempty subsets A and B of S we have $A \subseteq B$ if and only if $f_A \subseteq f_B$.

Let S be an ordered semigroup. A fuzzy subset f of S is called a *fuzzy susemigroup* of S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$ [18]. A fuzzy subset f of S is called a *fuzzy left* (resp. *right*) *ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$) $\forall x, y \in S$. Equivalently, $S \circ f \subseteq f$ (resp. $f \circ S \subseteq f$).

A fuzzy subset f of S is called a *fuzzy ideal* of S if it is both a fuzzy left and a fuzzy right ideal of S [18, 29]. A fuzzy subsemigroup f of S is called a *fuzzy bi-ideal* of S if

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $f(xyz) \geq \min\{f(x), f(z)\} = f(x) \wedge f(z), \forall x, y, z \in S$ [29].

Lemma 2.1 ([28]): Let f be a fuzzy subset of an ordered semigroup S. Then f is a strongly convex fuzzy subset of S if and only if $x \leq y \Rightarrow f(x) \geq f(y)$, for all $x, y \in S$.

Definition 2.2 ([28]): Let S be an ordered semigroup, $a \in S$ and $\lambda \in [0,1]$. An *ordered fuzzy point* a_λ of S is defined by the rule that

$$a_\lambda(x) = \begin{cases} \lambda, & \text{if } x \in (a], \\ 0, & \text{if } x \notin (a]. \end{cases}$$

It is accepted that a_λ is a mapping from S into $[0,1]$, then an ordered fuzzy point of S is a fuzzy subset of S. For any fuzzy subset f of S, we also denote $a_\lambda \subseteq f$ by $a_\lambda \in f$ in sequel. Moreover, for an ordered fuzzy point a_λ and a strongly convex fuzzy subset f of S we have $a_\lambda \in f$ if and only if $f(a) \geq \lambda$ [28].

Definition 2.3: An ordered fuzzy point a_λ of an ordered semigroup S is said to be *quasi-coincident* with a fuzzy subset f of S, written as $a_\lambda q f$ if $f(a) + \lambda > 1$.

If $a_\lambda \in f$ or $a_\lambda q f$, then we write $a_\lambda \in \vee q f$. To say that the symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

Definition 2.4: Let f be any fuzzy subset of an ordered semigroup S. The set

$$[f]_t := \{x \in S |\ x_t \in \vee q f\}, \text{ where } t \in (0,1]$$

is called an $(\in, \in \vee q)$-*level subset* of f.

Lemma 2.6 ([28]): Let a_λ, b_μ $(\lambda \neq 0, \mu \neq 0)$ be ordered fuzzy points of S, and f, g fuzzy subsets of S. Then the following statements are true:

(1) *$a_\lambda \circ b_\mu = (ab)_{\lambda\wedge\mu}$ for all ordered fuzzy points a_λ and b_μ of S. In particular, $a_\lambda \circ a_\lambda = (a^2)_\lambda$.*

(2) *If f, g are fuzzy ideals of S, then $f \circ g$, $f \cup g$ are fuzzy ideals of S.*

(3) *If $f \subseteq g$, and $h \in F(S)$, then $f \circ h \subseteq g \circ h$, $h \circ f \subseteq h \circ g$.*

The reader is referred to [28, 33, 34] for notation and terminology not defined in this paper.

III. $(\in, \in \vee q)$-FUZZY IDEALS OF ORDERED SEMIGROUPS

Definition 3.1 : A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee q)$-*fuzzy subsemigroup* of S if $x_t \in f$ and $y_r \in f$ imply $(xy)_{t\wedge r} \in \vee q f$ for all $t, r \in (0,1]$ and $x, y \in S$.

Definition 3.2: A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee q)$-*fuzzy left* (resp. *right*) *ideal* of S, if for all $t \in (0,1]$ and $x, y \in S$, the following conditions hold:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, and

(2) $y_t \in f, x \in S \Rightarrow (xy)_t \in \vee q f$ (resp. $(yx)_t \in \vee q f$).

A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee q)$-*fuzzy ideal* if it is both an $(\in \vee q)$-fuzzy left ideal and an $(\in, \in \vee q)$-fuzzy right ideal of S.

Theorem 3.3: Let f be a strongly convex fuzzy subset of an ordered semigroup S. Then f is an $(\in \vee q)$-fuzzy subsemigroup of S if and only if

$$f(xy) \geq \min\{f(x), f(y), 0.5\} = f(x) \wedge f(y) \wedge 0.5$$

for all $x, y \in S$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy subsemigroup of S. On the contrary assume that $f(xy) < \min\{f(x), f(y), 0.5\}$ for some $x, y \in S$. Choose $t \in (0,1]$ such that $f(xy) < t < \min\{f(x), f(y), 0.5\}$. Then, since f is strongly convex, $x_t \in f$ and $y_t \in f$, but $(xy)_t \overline{\in \vee q} f$, which is a contradiction. Hence $f(xy) \geq \min\{f(x), f(y), 0.5\}$.

Conversely, assume that $f(xy) \geq \min\{f(x), f(y), 0.5\}$ for all $x, y \in S$. Let $x_t \in f$ and $y_r \in f$ for all $t, r \in (0,1]$. Then $f(x) \geq t$ and $f(y) \geq r$. So $f(xy) \geq \min\{f(x), f(y), 0.5\} \geq \min\{t, r, 0.5\}$. Now if $\min\{t, r\} \leq 0.5$, then $f(xy) \geq \min\{t, r\}$. Since f is a strongly convex fuzzy subset of S, we have $(xy)_{t\wedge r} \in f$. If $\min\{t, r\} > 0.5$, then $f(xy) \geq 0.5$. So $f(xy) + \min\{t, r\} > 0.5 + 0.5 = 1$, which implies that $(xy)_{t\wedge r} q f$. Hence $(xy)_{t\wedge r} \in \vee q f$. Therefore, f is an $(\in \vee q)$-fuzzy subsemigroup of S.

Theorem 3.4: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy left ideal of S if and only if f satisfies that

(1) *$x \leq y \Rightarrow f(x) \geq f(y)$, and*

(2) *$f(xy) \geq \min\{f(y), 0.5\} = f(y) \wedge 0.5$ for all $x, y \in S$.*

Proof. Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S and $x, y \in S$. Then $f(xy) \geq \min\{f(y), 0.5\}$. Indeed: If $f(xy) < \min\{f(y), 0.5\}$, then there exists $t \in (0,1]$ such that $f(xy) < t < \min\{f(y), 0.5\}$. Then $y_t \in f$ and $(xy)_t \overline{\in \vee q} f$, which is a contradiction. Hence $f(xy) \geq \min\{f(y), 0.5\}$.

Conversely, assume that the conditions (1) and (2) hold. Let $y_t \in f$ Then $f(y) \geq t$. Now $f(xy) \geq \min\{f(y), 0.5\} \geq \min\{t, 0.5\}$. If $t \leq 0.5$, then $f(xy) \geq t$. So $(xy)_t \in f$. If $t > 0.5$, then $f(xy) \geq 0.5$. So $f(xy) + t \geq 0.5 + 0.5 = 1$. Thus $(xy)_t q f$. Hence $(xy)_t \in \vee q f$, and therefore f is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Similar to Theorem 3.4, we have the following two theorems. The proofs are similar to that of Theorem 3.4.

Theorem 3.5: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy right ideal of S if and only if f satisfies that

(1) *$x \leq y \Rightarrow f(x) \geq f(y)$, and*

(2) *$f(xy) \geq \min\{f(x), 0.5\} = f(x) \wedge 0.5$ for all $x, y \in S$.*

Theorem 3.6: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy ideal of S if and only if f satisfies that

(1) *$x \leq y \Rightarrow f(x) \geq f(y)$, and*

(2) *$f(xy) \geq \min\{f(y), 0.5\} = f(y) \wedge 0.5$ and $f(xy) \geq \min\{f(x), 0.5\} = f(x) \wedge 0.5$ for all $x, y \in S$.*

Clearly, every fuzzy ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy ideal of S. But the converse is not necessarily true.

Example 3.7: We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

·	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a), (b,b), (b,a), (c,c), (c,a), (c,b), (d,d)\}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.6, f(b) = 0.7, f(c) = 0.8, f(d) = 0.3$. Then we can easily show that f is an $(\in, \in \vee q)$-fuzzy ideal of S by Theorem 3.6. But f is not a fuzzy ideal of S, since $f(bb) = f(a) = 0.6 < 0.7 = f(b)$.

Theorem 3.8: Let S be an ordered semigroup and f a fuzzy subset of S such that $f(x) \leq 0.5$ for all $x \in S$. Then f is an $(\in, \in \vee q)$-fuzzy ideal of S if and only if the set

$$f_t := \{x \in S \mid f(x) \geq t\}$$

is an ideal of S for all $t \in (0, 0.5]$.

Proof. Assume that f is an $(\in, \in \vee q)$-fuzzy ideal of S. Let $x \in f_t$ for $t \in (0, 0.5]$. Then $f(x) \geq t$. It follows from Theorem 3.6 that

$$f(xy) \geq \min\{f(x), 0.5\} \geq \min\{t, 0.5\} = t$$

and

$$f(yx) \geq \min\{f(x), 0.5\} \geq \min\{t, 0.5\} = t$$

for all $y \in S$. Thus $xy \in f_t$ and $yx \in f_t$. Furthermore, let $x \in f_t, S \ni y \leq x$. Then $y \in f_t$. Indeed: Since $x \in f_t, f(x) \geq t$, and f is an $(\in, \in \vee q)$-fuzzy ideal of S, we have $f(y) \geq f(x) \geq t$, so $y \in f_t$. Therefore, f_t is an ideal of S.

Conversely, for any $x, y \in S$, let $t = \min\{f(x), 0.5\}$. Then

$x \in f_t$. Since f_t is an ideal of S, we have $xy \in f_t$, and so $f(xy) \geq t = \min\{f(x), 0.5\}$, for all $y \in S$. Moreover, if $x \leq y$, then $f(x) \geq f(y)$. Indeed: Let $\lambda = f(y)$. Then $y \in f_\lambda$. Since f_λ is an ideal of S, we have $x \in f_\lambda$. Then $f(x) \geq \lambda = f(y)$. Therefore, f is an $(\in, \in \vee q)$-fuzzy right ideal of S. In a similar way we can show that f is also an $(\in, \in \vee q)$-fuzzy right ideal of S, and so f is an $(\in, \in \vee q)$-fuzzy ideal of S.

Theorem 3.9: Let S be an ordered semigroup and f a strongly convex fuzzy subset of S. Then f is an $(\in \vee q)$-fuzzy ideal of S if and only if the $(\in, \in \vee q)$-level subset $[f]_t$ of f is an ideal of S for all $t \in (0, 1]$.

Proof. Assume that f is an $(\in, \in \vee q)$-fuzzy ideal of S. Let $x \in [f]_t$ for $t \in (0, 1]$. Then $x_t \in \vee q f$, that is $f(x) \geq t$ or $f(x) + t > 1$. Since f is an $(\in, \in \vee q)$-fuzzy ideal of S, we have $f(xy) \geq \min\{f(x), 0.5\}$ and $f(yx) \geq \min\{f(x), 0.5\}$ for all $y \in S$.

Case 1. $f(x) \geq t$. If $t > 0.5$, then $f(xy) \geq \min\{f(x), 0.5\} \geq \min\{t, 0.5\} = 0.5$, and thus $(xy)_t q f$. If $t \leq 0.5$, then $f(xy) \geq \min\{f(x), 0.5\} \geq \min\{t, 0.5\} = t$, and so $(xy)_t \in f$.

Case 2. $f(x) + t > 1$. If $t > 0.5$, then $f(xy) \geq \min\{f(x), 0.5\} \geq \min\{1-t, 0.5\} = 1-t$, i,e., $f(xy)+t > 1$, and thus $(xy)_t q f$. If $t \leq 0.5$, then $f(xy) \geq \min\{f(x), 0.5\} \geq \min\{1-t, 0.5\} = 0.5 \geq t$, and so $(xy)_t \in f$.

Thus, in both cases, we have $(xy)_t \in \vee q f$, and so $xy \in [f]_t$. In the same way we can show that $yx \in [f]_t$. Furthermore, let $x \in [f]_t, S \ni y \leq x$. Then $y \in [f]_t$. Indeed: Since $x \in [f]_t$, we have $x_t \in \vee q f$, that is $f(x) \geq t$ or $f(x) + t > 1$. Then, by Theorem 3.6,we have $f(y) \geq f(x) \geq t$ or $f(y) \geq f(x) \geq 1 - t$, that is, $y_t \in \vee q f$, and so $y \in [f]_t$. Therefore, $[f]_t$ is an ideal of S.

Conversely, let f is a strongly convex fuzzy subset of S and $t \in (0, 1]$ be such that $[f]_t$ is an ideal of S. Assume that $f(xy) < \min\{f(x), 0.5\}$ for some $x, y \in S$. Then we can take $t \in (0, 0.5]$ such that $f(xy) < t < \min\{f(x), 0.5\}$. Then, since f is strongly convex, $x \in [f]_t$, which implies that $xy \in [f]_t$. Hence $f(xy) \geq t$ or $f(xy)+t > 1$, which is impossible. Therefore, $f(xy) \geq \min\{f(x), 0.5\}$ for all $x, y \in S$. In a similar way we can show that $f(xy) \geq \min\{f(y), 0.5\}$ for all $x, y \in S$. Using Lemma 2.1 and Theorem 3.6, we conclude that f is an $(\in, \in \vee q)$-fuzzy ideal of S.

Theorem 3.10: If f is an $(\in, \in \vee q)$-fuzzy left ideal and g an $(\in, \in \vee q)$-fuzzy right ideal of an ordered semigroup S, then $f \circ g$ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Proof. Let $x, y \in S$. Then $(f \circ g)(xy) \geq \min\{(f \circ g)(y), 0.5\}$. Indeed: Since $f \circ g$ is a fuzzy subset of S, we have $(f \circ g)(a) \geq 0$ for all $a \in S$. If $A_y = \emptyset$, then $(f \circ g)(y) = 0$, and so we have

$$(f \circ g)(xy) \geq 0 = \min\{(f \circ g)(y), 0.5\}.$$

If $A_y \neq \emptyset$, then there exist $p, q \in S$ such that $(p, q) \in A_y$, and thus $(xp, q) \in A_{xy}$. Since f is an $(\in, \in \vee q)$-fuzzy left ideal of S, we have, by Theorem 3.4, $f(xp) \geq \min\{f(p), 0.5\}$. Thus

$$\begin{aligned}
& \min\{(f \circ g)(y), 0.5\} \\
= \ & \min\{\bigvee_{(p,q) \in A_y} [\min\{f(p), g(q)\}], 0.5\} \\
= \ & \bigvee_{(p,q) \in A_y} [\min\{f(p), g(q), 0.5\}] \\
= \ & \bigvee_{(p,q) \in A_y} [\min\{f(p) \wedge 0.5, g(q)\}] \\
\leq \ & \bigvee_{(p,q) \in A_y} [\min\{f(xp), g(q)\}] \\
\leq \ & \bigvee_{(a,b) \in A_{xy}} [\min\{f(a), g(b)\}] = (f \circ g)(xy).
\end{aligned}$$

Similarly, we can show that $(f \circ g)(xy) \geq \min\{(f \circ g)(x), 0.5\}$. Furthermore, if $x \leq y$, then $(f \circ g)(x) \geq (f \circ g)(y)$. Indeed: If $A_y = \emptyset$, then $(f \circ g)(y) = 0$. Since $f \circ g$ is a fuzzy subset of S, we have $(f \circ g)(x) \geq 0 = (f \circ g)(y)$. If $A_y \neq \emptyset$, then, since $x \leq y$, we have $A_y \subseteq A_x$. Thus we have

$$\begin{aligned}
(f \circ g)(y) &= \bigvee_{(p,q) \in A_y} [\min\{f(p), g(q)\}] \\
&\leq \bigvee_{(p,q) \in A_x} [\min\{f(p), g(q)\}] = (f \circ g)(x).
\end{aligned}$$

Therefore, $f \circ g$ is an $(\in, \in \vee q)$-fuzzy ideal of S.

Theorem 3.11: Let $\{f_i \mid i \in I\}$ be a family of $(\in, \in \vee q)$-fuzzy right ideals of an ordered semigroup S. Then $f := \bigcap_{i \in I} f_i$ is an $(\in, \in \vee q)$-fuzzy right ideal of S, where $(\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} (f_i(x))$.

Proof. Let $x, y \in S$. Then $f_i(xy) \geq f_i(x) \wedge 0.5$ for all $i \in I$, since each f_i is an $(\in, \in \vee q)$-fuzzy right ideal of S. Thus we have

$$\begin{aligned}
f(xy) &= (\bigcap_{i \in I} f_i)(xy) = \bigwedge_{i \in I} (f_i(xy)) \geq \bigwedge_{i \in I} (f_i(x) \wedge 0.5) \\
&= [\bigwedge_{i \in I} f_i(x)] \wedge 0.5 = [(\bigcap_{i \in I} f_i)(x)] \wedge 0.5 \\
&= f(x) \wedge 0.5.
\end{aligned}$$

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed: Since each $f_i (i \in I)$ is an $(\in, \in \vee q)$-fuzzy right ideal of S, so $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus

$$\begin{aligned}
f(x) &= (\bigcap_{i \in I} f_i)(x) = \bigwedge_{i \in I} (f_i(x)) \\
&\geq \bigwedge_{i \in I} (f_i(y)) = (\bigcap_{i \in I} f_i)(y) = f(y).
\end{aligned}$$

Hence f is an $(\in, \in \vee q)$-fuzzy right ideals of S.

Similarly, we can prove that intersection of $(\in, \in \vee q)$-fuzzy left ideals of an ordered semigroup S is an $(\in : \in \vee q)$-fuzzy left ideal of S. Thus intersection of $(\in, \in \vee q)$-fuzzy ideals of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy ideal of S.

Theorem 3.12: Let $\{f_i \mid i \in I\}$ be a family of $(\in, \in \vee q)$-fuzzy right ideals of an ordered semigroup S. Then $f := \bigcup_{i \in I} f_i$ is an $(\in \vee q)$-fuzzy right ideal of S, where $(\bigcup_{i \in I} f_i)(x) = \bigvee_{i \in I} (f_i(x))$.

Proof. Let $x, y \in S$. Then $f_i(xy) \geq f_i(x) \wedge 0.5$ for all $i \in I$, since each f_i is an $(\in, \in \vee q)$-fuzzy right ideal of S. Thus we have

$$\begin{aligned} f(xy) &= (\bigcup_{i\in I} f_i)(xy) = \bigvee_{i\in I}(f_i(xy)) \geq \bigvee_{i\in I}(f_i(x) \wedge 0.5) \\ &= [\bigvee_{i\in I} f_i(x)] \wedge 0.5 = [(\bigcup_{i\in I} f_i)(x)] \wedge 0.5 \\ &= f(x) \wedge 0.5. \end{aligned}$$

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed: Since each $f_i (i \in I)$ is an $(\in, \in \vee q)$-fuzzy right ideal of S, so $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus

$$\begin{aligned} f(x) &= (\bigcup_{i\in I} f_i)(x) = \bigvee_{i\in I}(f_i(x)) \\ &\geq \bigvee_{i\in I}(f_i(y)) = (\bigcup_{i\in I} f_i)(y) = f(y). \end{aligned}$$

Therefore, f is an $(\in, \in \vee q)$-fuzzy right ideals of S.

Similarly, we can show that union of $(\in, \in \vee q)$-fuzzy left ideals of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy left ideal of S. Thus union of $(\in, \in \vee q)$-fuzzy ideals of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy ideal of S.

IV. $(\in, \in \vee q)$-FUZZY BI-IDEALS AND $(\in, \in \vee q)$-FUZZY GENERALIZED BI-IDEALS OF ORDERED SEMIGROUPS

Definition 4.1: A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee q)$-*fuzzy bi-ideal* of S, if for all $t, r \in (0, 1]$ and $x, y, z \in S$, the following conditions hold:

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $x_t \in f, y_r \in f \Rightarrow (xy)_{\min\{t,r\}} \in \vee qf$.

(3) $x_t \in f, z_r \in f$ and $y \in S \Rightarrow (xyz)_{\min\{t,r\}} \in \vee qf$.

Clearly, every $(\in, \in \vee q)$-fuzzy bi-ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy subsemigroup of S.

Theorem 4.2: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S if and only if f satisfies that

(1) $x \leq y \Rightarrow f(x) \geq f(y)$.

(2) $f(xy) \geq \min\{f(x), f(y), 0.5\} = f(x) \wedge f(y) \wedge 0.5$ *for all* $x, y \in S$.

(3) $f(xyz) \geq \min\{f(x), f(z), 0.5\} = f(x) \wedge f(z) \wedge 0.5$ *for all* $x, y, z \in S$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy bi-ideal of S and $x, y, z \in S$. Then, by Theorem 3.3 and Definition 4.1, the conditions (1) and (2) hold. Furthermore, $f(xyz) \geq \min\{f(x), f(z), 0.5\}$. Indeed: If $f(xyz) < \min\{f(x), f(z), 0.5\}$, then there exists $t \in (0, 1]$ such that $f(xyz) < t < \min\{f(x), f(z), 0.5\}$. Then $x_t \in f, z_t \in f$ and $(xyz)_t \overline{\in \vee q} f$, which is a contradiction. Hence $f(xyz) \geq \min\{f(x), f(z), 0.5\}$.

Conversely, assume that the conditions (1), (2) and (3) hold. Let $x, y \in S$ and $t, r \in (0, 1]$ be such that $x_t \in f, y_r \in f$. Then, by Definition 3.1 and Theorem 3.3, $(xy)_{\min\{t,r\}} \in \vee qf$. Furthermore, let $x, y, z \in S$ and $t, r \in (0, 1]$ be such that $x_t \in f, z_r \in f$. Then $f(x) \geq t, f(z) \geq r$ and

$$f(xyz) \geq \min\{f(x), f(z), 0.5\} \geq \min\{t, r, 0.5\}.$$

If $\min\{t, r\} > 0.5$, then $f(xyz) \geq 0.5$, which implies that $f(xyz) + \min\{t, r\} > 1$. If $\min\{t, r\} \leq 0.5$, then $f(xyz) \geq \min\{t, r\}$. Therefore, $(xyz)_{\min\{t,r\}} \in \vee qf$.

One can easily observe that every fuzzy bi-ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, but not conversely. We can illustrate it by the following example:

Example 4.3: We consider the ordered semigroup $S := \{a, b, c, d, e\}$ defined by the following multiplication "$\cdot$" and the order "$\leq$":

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	c	e
d	a	a	c	d	e
e	a	a	c	c	e

$$\begin{aligned} \leq := \{&(a, a), (a, c), (a, e), (b, b), (b, c), (b, e), (c, c), (c, e), \\ &(d, c), (d, e), (d, d), (e, e)\}. \end{aligned}$$

Let f be a fuzzy subset of S such that $f(a) = f(b) = f(d) = 0.8, f(c) = 0.7, f(e) = 0.6$. Then we can easily show that f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. But f is not a fuzzy bi-ideal of S, since $f(dcd) = f(c) = 0.7 < 0.8 = \min\{f(d), f(d)\}$.

Theorem 4.4: Let $\{f_i \mid i \in I\}$ be a family of $(\in, \in \vee q)$-fuzzy bi-ideals of an ordered semigroup S. Then $f := \bigcap_{i\in I} f_i$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, where $(\bigcap_{i\in I} f_i)(x) = \bigwedge_{i\in I}(f_i(x))$.

Proof. Let $x, y, z \in S$. Then, since each f_i $(i \in I)$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, we have

$$\begin{aligned} f(xy) &= (\bigcap_{i\in I} f_i)(xy) = \bigwedge_{i\in I}(f_i(xy)) \\ &\geq \bigwedge_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) \\ &= [\bigwedge_{i\in I}(f_i(x) \wedge f_i(y))] \wedge 0.5 \\ &= [(\bigcap_{i\in I} f_i)(x) \wedge (\bigcap_{i\in I} f_i)(y)] \wedge 0.5 \\ &= f(x) \wedge f(y) \wedge 0.5, \end{aligned}$$

and

$$\begin{aligned} f(xyz) &= (\bigcap_{i\in I} f_i)(xyz) = \bigwedge_{i\in I}(f_i(xyz)) \\ &\geq \bigwedge_{i\in I}(f_i(x) \wedge f_i(z) \wedge 0.5) \\ &= [\bigwedge_{i\in I}(f_i(x) \wedge f_i(z))] \wedge 0.5 \\ &= [(\bigcap_{i\in I} f_i)(x) \wedge (\bigcap_{i\in I} f_i)(z)] \wedge 0.5 \\ &= f(x) \wedge f(z) \wedge 0.5. \end{aligned}$$

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed: Since each $f_i (i \in I)$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, so $f_i(x) \geq f_i(y)$

for all $i \in I$. Thus

$$\begin{aligned} f(x) &= (\bigcap_{i\in I} f_i)(x) = \bigwedge_{i\in I}(f_i(x)) \\ &\geq \bigwedge_{i\in I}(f_i(y)) = (\bigcap_{i\in I} f_i)(y) = f(y). \end{aligned}$$

Therefore, f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S by Theorem 4.2.

If $\{f_i \mid i \in I\}$ is a family of $(\in, \in \vee q)$-fuzzy bi-ideals of an ordered semigroup S, is $\bigcup_{i\in I} f_i$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S? where $(\bigcup_{i\in I} f_i)(x) = \bigvee_{i\in I}(f_i(x))$. The following example gives a negative answer to the above question.

Example 4.5: We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

·	a	b	c	d
a	a	a	a	a
b	a	a	d	a
c	a	a	a	a
d	a	a	a	a

$$\leq := \{(a, a), (a, d), (b, b), (c, c), (d, d)\}.$$

Let f_1 and f_2 be two fuzzy subsets of S such that

$$f_1(a) = 0.4,\ f_1(b) = 0.4,\ f_1(c) = 0,\ f_1(d) = 0;$$

$$f_2(a) = 0.4,\ f_2(b) = 0,\ f_2(c) = 0.4,\ f_1(d) = 0.$$

Then f_1 and f_2 are both $(\in, \in \vee q)$-fuzzy bi-ideals of S, but $f_1 \cup f_2$ is not an $(\in, \in \vee q)$-fuzzy bi-ideal of S, since

$$\begin{aligned} (f_1 \cup f_2)(bc) &= (f_1 \cup f_2)(d) = f_1(d) \vee f_2(d) = 0 \\ &< 0.4 = \min\{0.4, 0.4, 0.5\} \\ &= \min\{(f_1 \cup f_2)(b), (f_1 \cup f_2)(c), 0.5\}, \end{aligned}$$

that is, $f_1 \cup f_2$ is not an $(\in, \in \vee q)$-fuzzy subsemigroup of S.

The following theorem can be obtained if we present a sufficient condition.

Theorem 4.6: Let $\{f_i \mid i \in I\}$ be a family of $(\in, \in \vee q)$-fuzzy bi-ideals of an ordered semigroup S such that $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for all $i, j \in I$. Then $f := \bigcup_{i\in I} f_i$ is an $(\in \vee q)$-fuzzy bi-ideal of S, where $(\bigcup_{i\in I} f_i)(x) = \bigvee_{i\in I}(f_i(x))$.

Proof. (1) For all $x, y \in S$, we have

$$\begin{aligned} f(xy) &= (\bigcup_{i\in I} f_i)(xy) = \bigvee_{i\in I}(f_i(xy)) \\ &\geq \bigvee_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) \\ &\quad \text{(since } f_i \text{ is an } (\in \vee q)\text{-fuzzy bi-ideal of } S\text{)} \\ &= (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(y) \wedge 0.5 \qquad (*) \\ &= f(x) \wedge f(y) \wedge 0.5, \end{aligned}$$

In the following we show that Eq. $(*)$ holds. It is obvious that $\bigvee_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) \leq (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(y) \wedge 0.5$. Assume that $\bigvee_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) \neq (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(y) \wedge 0.5$. Then there exists r such that

$$\bigvee_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) < r < (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(y) \wedge 0.5.$$

Since $f_i \subseteq f_j$ or $f_j \subseteq f_i$ for all $i, j \in I$, there exists k such that $r < f_k(x) \wedge f_k(y) \wedge 0.5$. On the other hand, $f_i(x) \wedge f_i(y) \wedge 0.5 < r$ for all $i \in I$, which is a contradiction. Hence $\bigvee_{i\in I}(f_i(x) \wedge f_i(y) \wedge 0.5) = (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(y) \wedge 0.5$.

(2) For all $x, y, z \in S$, we have

$$\begin{aligned} f(xyz) &= (\bigcup_{i\in I} f_i)(xyz) = \bigvee_{i\in I}(f_i(xyz)) \\ &\geq \bigvee_{i\in I}(f_i(x) \wedge f_i(z) \wedge 0.5) \\ &\quad \text{(since } f_i \text{ is an } (\in \vee q)\text{-fuzzy bi-ideal of } S\text{)} \\ &= (\bigvee_{i\in I} f_i)(x) \wedge (\bigvee_{i\in I} f_i)(z) \wedge 0.5 \\ &\quad \text{(Similar to the proof of (1))} \\ &= f(x) \wedge f(z) \wedge 0.5. \end{aligned}$$

Furthermore, if $x \leq y$, then $f(x) \geq f(y)$. Indeed: Since f_i is an $(\in, \in \vee q)$-fuzzy bi-ideal of S for all $i \in I$, so $f_i(x) \geq f_i(y)$ for all $i \in I$. Thus

$$\begin{aligned} f(x) &= = (\bigcup_{i\in I} f_i)(x) = \bigvee_{i\in I}(f_i(x)) \\ &\geq \bigvee_{i\in I}(f_i(y)) = (\bigcup_{i\in I} f_i)(y) = f(y). \end{aligned}$$

Therefore f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S by Theorem 4.2.

Definition 4.7: A fuzzy subset f of an ordered semigroup S is called an $(\in, \in \vee q)$*-fuzzy generalized bi-ideal* of S, if for all $t, r \in (0, 1]$ and $x, y, z \in S$, the following conditions hold:
(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $x_t \in f, z_r \in f$ and $y \in S \Rightarrow (xyz)_{\min\{t,r\}} \in \vee q f$.

The following Theorem 4.8 can be proved in a similar way as in the proof of Theorem 4.2.

Theorem 4.8: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S if and only if f satisfies that
(1) $x \leq y \Rightarrow f(x) \geq f(y)$.
(2) $f(xyz) \geq \min\{f(x), f(z), 0.5\} = f(x) \wedge f(z) \wedge 0.5$ *for all $x, y, z \in S$.*

One can easily observe that every $(\in, \in \vee q)$-fuzzy bi-ideal of an ordered semigroup S is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S, but not conversely. We can illustrate it by the following example:

Example 4.9: We consider the ordered semigroup $S := \{a, b, c, d\}$ defined by the following multiplication " $\cdot$ " and the order " $\leq$ ":

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

$$\leq := \{(a,a),(a,b),(a,c),(a,d),(b,b),(c,c),(d,d)\}.$$

Let f be a fuzzy subset of S such that $f(a) = 0.6, f(b) = 0, f(c) = 0.3, f(d) = 0$. Then we can easily show that f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. But f is not an $(\in, \in \vee q)$-fuzzy bi-ideal of S, since $f(cc) = f(b) = 0 < 0.3 = \min\{f(c), f(c), 0.5\}$, that is f is not an $(\in, \in \vee q)$-fuzzy subsemigroup of S.

An ordered semigroup $(S, \cdot, \leq)$ is called *regular* if, for each element a of S, there exists an element x in S such that $a \leq axa$ (cf. [16]). **Equivalent definition:**

(1) $A \subseteq (ASA], \forall A \subseteq S$.

(2) $a \in (aSa], \forall a \in S$.

Theorem 4.10: Every $(\in, \in \vee q)$-fuzzy generalized bi-ideal of a regular ordered semigroup S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Let f be any $(\in, \in \vee q)$-fuzzy generalized bi-ideals of S and $a, b \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$. Then

$$f(ab) \geq f((axa)b)) = f(a(xa)b) \geq \min\{f(a), f(b), 0.5\}.$$

Thus f is an $(\in, \in \vee q)$-fuzzy subsemigroup of S, and so it is an $(\in, \in \vee q)$-fuzzy bi-ideal of S by Theorem 4.2.

V. Upper and lower parts of $(\in, \in \vee q)$-fuzzy ideals of ordered semigroups

Definition 5.1: Let f be a fuzzy subset of an ordered semigroup S. Then we define the *upper part* f^+ and the *lower part* f^- as follows:

$$f^+ = f(x) \vee 0.5 \text{ and } f^- = f(x) \wedge 0.5$$

for all $x \in S$.

Clearly, f^+ and f^- are still fuzzy subsets of S. In the following we mainly investigate the properties of lower parts of $(\in, \in \vee q)$-fuzzy ideals of S.

Lemma 5.2: Let f and g be fuzzy subsets of an ordered semigroup S. Then the following statements are true:

(1) $(f^-)^- = f^-, f^- \subseteq f$.

(2) $(f \cap g)^- = f^- \cap g^-$.

(3) $(f \cup g)^- = f^- \cup g^-$.

(4) $(f \circ g)^- = f^- \circ g^-$.

(5) $(f \circ S)^- = f^- \circ S, (S \circ f)^- = S \circ f^-, (f \circ S \circ f)^- = f^- \circ S \circ f^-$ *and* $(S \circ f \circ S)^- = S \circ f^- \circ S$.

Proof. (1) It is obvious by Definition 5.1.

(2) For all $x \in S$, we have

$$\begin{aligned}(f \cap g)^-(x) &= (f \cap g)(x) \wedge 0.5 \\ &= f(x) \wedge g(x) \wedge 0.5 \\ &= (f(x) \wedge 0.5) \wedge (g(x) \wedge 0.5) \\ &= f^-(x) \wedge g^-(x) = (f^- \cap g^-)(x).\end{aligned}$$

It thus implies that $(f \cap g)^- = f^- \cap g^-$.

(3) For all $x \in S$, we have

$$\begin{aligned}(f \cup g)^-(x) &= (f \cup g)(x) \wedge 0.5 \\ &= (f(x) \vee g(x)) \wedge 0.5 \\ &= (f(x) \wedge 0.5) \vee (g(x) \wedge 0.5) \\ &= f^-(x) \vee g^-(x) = (f^- \cup g^-)(x),\end{aligned}$$

which implies that $(f \cup g)^- = f^- \cup g^-$.

(4) Let $x \in S$. If $A_x = \emptyset$, then $(f \circ g)^-(x) = (f \circ g)(x) \wedge 0.5 = 0 \wedge 0.5 = 0 = (f^- \circ g^-)(x)$. If $A_x \neq \emptyset$, then we have

$$\begin{aligned}(f \circ g)^-(x) &= (f \circ g)(x) \wedge 0.5 \\ &= \bigvee_{(y,z) \in A_x} [f(y) \wedge g(z)] \wedge 0.5 \\ &= \bigvee_{(y,z) \in A_x} [(f(y) \wedge 0.5) \wedge (g(z) \wedge 0.5)] \\ &= \bigvee_{(y,z) \in A_x} [f^-(y) \wedge g^-(z)] = (f^- \circ g^-)(x),\end{aligned}$$

from which we deduce that $(f \circ g)^- = f^- \circ g^-$.

(5) The proof is similar to the proof of (4) with a slight modification.

In Example 3.7, we have illustrated that an $(\in, \in \vee q)$-fuzzy ideal of an ordered semigroup S is not necessarily a fuzzy ideal of S. In the following proposition, we show that if f is an $(\in, \in \vee q)$-fuzzy ideal of S, then the lower part f^- of f is a fuzzy ideal of S.

Proposition 5.3: If f is an $(\in, \in \vee q)$-fuzzy ideal of an ordered semigroup S, then the lower part f^- of f is a fuzzy ideal of S.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy ideal of S and $x, y \in S$. Then, $f(xy) \geq f(x) \wedge 0.5$ and we have $f^-(xy) = f(xy) \wedge 0.5 \geq f(x) \wedge 0.5 = f^-(x)$. Moreover, if $x \leq y$, then $f^-(x) \geq f^-(y)$. Indeed: since f is an $(\in, \in \vee q)$-fuzzy ideal of S, we have $f(x) \geq f(y)$, and so $f^-(x) = f(x) \wedge 0.5 \geq f(y) \wedge 0.5 = f^-(y)$. Therefore, f^- is a fuzzy right ideal of S. In a similar way we can show that f^- is also a fuzzy left ideal of S, and so f^- is a fuzzy ideal of S.

Definition 5.4: Let A be a nonempty subset of an ordered semigroup S. Then, the upper part f_A^+ and the lower part f_A^- of the characteristic function f_A of A are defined by

$$f_A^+ : S \to [0,1], x \mapsto f_A^+(x) := \begin{cases} 1, & \text{if } x \in A, \\ 0.5, & \text{if } x \notin A, \end{cases}$$

and

$$f_A^- : S \to [0,1], x \mapsto f_A^-(x) := \begin{cases} 0.5, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

Clearly, $f_A^+(x) = f_A(x) \vee 0.5$ and $f_A^-(x) = f_A(x) \wedge 0.5$ for all $x \in S$.

Lemma 5.5: Let A, B be any nonempty subsets of an ordered semigroup S. Then the following statements are true:

(1) $(f_A \cap f_B)^- = f_{A \cap B}^-$.

(2) $(f_A \cup f_B)^- = f_{A \cup B}^-$.

(3) $(f_A \circ f_B)^- = f_{(AB]}^-$.

(4) $(S\circ f_A)^- = f^-_{(SA]}, (f_A\circ S)^- = f^-_{(AS]}, (f_A\circ S\circ f_A)^- = f^-_{(ASA]}$ *and* $(S\circ f_A\circ S)^- = f^-_{(SAS]}$.

Proof. (1) $(f_A\cap f_B)^-(x) = f^-_{A\cap B}(x)$ for all $x\in S$. Indeed: If $x\in A\cap B$, then $f^-_{A\cap B}(x) = 0.5$. Since $x\in A$ and $x\in B$, we have $f_A(x) = f_B(x) = 1$, so $(f_A\cap f_B)^-(x) = f_A(x)\wedge f_B(x)\wedge 0.5 = 1\wedge 1\wedge 0.5 = 0.5$. If $x\notin A\cap B$, then $f^-_{A\cap B}(x) = 0$. Suppose that $x\notin A$. Then, by Lemma 5.2(2), $(f_A\cap f_B)^-(x) = f^-_A(x)\wedge f^-_B(x)\leq f^-_A(x) = 0$. On the other hand, since $(f_A\cap f_B)^-$ is a fuzzy subset of S, we have $(f_A\cap f_B)^-(x)\geq 0$. Thus we obtain that $(f_A\cap f_B)^-(x) = f^-_{A\cap B}(x)$.

(2) The proof is similar to the proof of (1) with suitable modification by using the definition.

(3) Let $x\in S$. If $x\in(AB]$, then $f^-_{(AB]}(x) = 0.5$, and $x\leq ab$ for some $a\in A$ and $b\in B$. Thus $(a,b)\in A_x$, and

$$\begin{aligned}(f_A\circ f_B)^-(x) &= (f_A\circ f_B)(x)\wedge 0.5\\ &= \bigvee_{(y,z)\in A_x}[f_A(y)\wedge f_B(z)]\wedge 0.5\\ &\geq [f_A(a)\wedge f_B(b)]\wedge 0.5\\ &= 1\wedge 1\wedge 0.5 = 0.5 = f^-_{(AB]}(x).\end{aligned}$$

On the other hand, since $(f_A\circ f_B)(x)\leq 1$ for all $x\in S$, we have $(f_A\circ f_B)^-(x) = (f_A\circ f_B)(x)\wedge 0.5\leq 0.5 = f^-_{(AB]}(x)$. Therefore, $(f_A\circ f_B)^-(x) = 0.5 = f^-_{(AB]}(x)$.
If $x\notin(AB]$, then $f^-_{(AB]}(x) = 0$. We now prove that $(f_A\circ f_B)^-(x) = 0$. Indeed: If $A_x = \emptyset$, then $(f_A\circ f_B)^-(x) = (f_A\circ f_B)(x)\wedge 0.5 = 0\wedge 0.5 = 0$, and $(f_A\circ f_B)^-(x) = f^-_{(AB]}(x)$. If $A_x\neq\emptyset$, then $x\leq yz$ for all $(y,z)\in A_x$. If $y\in A$ and $z\in B$, then $yz\in AB$, so $x\in(AB]$, which is impossible. Thus $y\notin A$ or $z\notin B$. If $y\notin A$, then $f_A(y) = 0$. Since $f_B(z)\geq 0$, we have $f_A(y)\wedge f_B(z) = 0$. If $z\notin B$ then, as in the previous case, we have $f_A(y)\wedge f_B(z) = 0$. Therefore,

$$(f_A\circ f_B)^-(x) = (f_A\circ f_B)(x)\wedge 0.5 = \bigvee_{(y,z)\in A_x}[f_A(y)\wedge f_B(z)]\wedge 0.5 = 0.$$

(4) The proof is similar to the proof of (3) with a slight modification, we omit it.

Lemma 5.6 ([7, Lemma 16]): Let $(S,\cdot)$ be a semigroup and $\emptyset\neq A\subseteq S$. Then A is a left (right) ideal of S if and only if the lower part f^-_A of the characteristic function f_A of A is an $(\in,\in\vee q)$-fuzzy left (right) ideal of S.

Theorem 5.7: Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset\neq A\subseteq S$. Then A is a left (right) ideal of S if and only if the lower part f^-_A of the characteristic function f_A of A is an $(\in,\in\vee q)$-fuzzy left (right) ideal of S.

Proof. ($\Longrightarrow$) Let $x,y\in S, x\leq y$. Then $f^-_A(x)\geq f^-_A(y)$. Indeed: If $y\in A$, then $f^-_A(y) = 0.5$. Since $S\ni x\leq y\in A$, we have $x\in A$, then $f^-_A(x) = 0.5$. Thus $f^-_A(x)\geq f^-_A(y)$. If $y\notin A$, then $f^-_A(y) = 0$. Since $x\in S$, we have $f^-_A(x)\geq 0$. Thus $f^-_A(y) = 0\leq f^-_A(x)$.

($\Longleftarrow$) Let $x\in A, S\ni y\leq x$. Then $y\in A$. Indeed: It is enough to prove that $f^-_A(y) = 0.5$. Since $x\in A, f^-_A(x) = 0.5$. Since f^-_A is an $(\in,\in\vee q)$-fuzzy left (right) ideal of S and $y\leq x$, we have $f^-_A(y)\geq f^-_A(x) = 0.5$. Since $y\in S$, we have $f^-_A(y)\leq 0.5$.

The rest of the proof is a consequence of Lemma 5.6.

Theorem 5.8: Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset\neq A\subseteq S$. Then A is an ideal of S if and only if the lower part f^-_A of the characteristic function f_A of A is an $(\in,\in\vee q)$-fuzzy ideal of S.

Proof. By Theorem 5.7, it is obvious.

Theorem 5.9: Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset\neq A\subseteq S$. Then A is a bi-ideal of S if and only if the lower part f^-_A of the characteristic function f_A of A is an $(\in,\in\vee q)$-fuzzy bi-ideal of S.

Proof. ($\Longrightarrow$) Let A be a bi-ideal of S and $x,y\in S$. Then $f^-_A(xy)\geq f^-_A(x)\wedge f^-_A(y)\wedge 0.5$. Indeed: If $x\in A$ and $y\in A$, then, since A is a bi-ideal of S, $xy\in A^2\subseteq A$, and we have $f^-_A(xy) = 0.5\geq f^-_A(x)\wedge f^-_A(y)\wedge 0.5$. If $x\notin A$ or $y\notin A$, then $f^-_A(x)\wedge f^-_A(y)\wedge 0.5 = 0$. Since $xy\in S$, we have $f^-_A(xy)\geq 0$. Thus $f^-_A(xy)\geq f^-_A(x)\wedge f^-_A(y)\wedge 0.5$. Now, let $x,y,z\in S$. Then, in the above way, we have $f^-_A(xyz)\geq f^-_A(x)\wedge f^-_A(z)\wedge 0.5$.

($\Longleftarrow$) Assume that f^-_A is an $(\in,\in\vee q)$-fuzzy bi-ideal of S. Then for every $x,y\in A$, we have

$$f^-_A(xy)\geq f^-_A(x)\wedge f^-_A(y)\wedge 0.5 = 0.5\wedge 0.5\wedge 0.5 = 0.5.$$

Also, $f^-_A(xy)\leq 0.5$, and so $f^-_A(xy) = 0.5$, i.e., $xy\in A$. Similarly, for every $x,z\in A, y\in S$, we can prove that $xyz\in A$.

The rest of the proof is similar to the proof process of Theorem 5.7, we omit it.

Theorem 5.10: Let $(S,\cdot,\leq)$ be an ordered semigroup and $\emptyset\neq A\subseteq S$. Then A is a generalized bi-ideal of S if and only if the lower part f^-_A of the characteristic function f_A of A is an $(\in,\in\vee q)$-fuzzy generalized bi-ideal of S.

Proof. The proof is similar to that of Theorem 5.9.

Theorem 5.11: Let $(S,\cdot,\leq)$ be an ordered semigroup and f a strongly convex fuzzy subset of S. Then f is an $(\in,\in\vee q)$-fuzzy subsemigroup of S if and only if $f^-\circ f^-\subseteq f^-$.

Proof. Let f be an $(\in,\in\vee q)$-fuzzy subsemigroup of S and $x\in S$. If $A_x = \emptyset$, then $(f^-\circ f^-)(x) = 0\leq f^-(x)$. If $A_x\neq\emptyset$, then, by Lemma 5.2(4), we have

$$\begin{aligned}(f^-\circ f^-)(x) &= (f\circ f)^-(x) = (f\circ f)(x)\wedge 0.5\\ &= \bigvee_{(y,z)\in A_x}[f(y)\wedge f(z)]\wedge 0.5\\ &= \bigvee_{(y,z)\in A_x}[f(y)\wedge f(z)\wedge 0.5]\wedge 0.5\\ &\leq \bigvee_{(y,z)\in A_x} f(yz)\wedge 0.5\\ &\leq \bigvee_{(y,z)\in A_x} f(x)\wedge 0.5 \text{ (since } x\leq yz)\\ &= f(x)\wedge 0.5 = f^-(x).\end{aligned}$$

It thus implies that $f^-\circ f^-\subseteq f^-$.

Conversely, if $f^-\circ f^-\subseteq f^-$, then for all $x,y\in S$

$$\begin{aligned}f(xy) &\geq f^-(xy)\geq (f^-\circ f^-)(xy)\\ &= \bigvee_{(p,q)\in A_{xy}}[f^-(p)\wedge f^-(q)]\geq f^-(x)\wedge f^-(y)\\ &= (f(x)\wedge 0.5)\wedge(f(y)\wedge 0.5)\end{aligned}$$

$$= f(x) \wedge f(y) \wedge 0.5.$$

Therefore, f is an $(\in, \in \vee q)$-fuzzy subsemigroup of S by Theorem 3.3.

Lemma 5.12: Let $(S, \cdot, \leq)$ be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in \vee q)$-fuzzy left ideal of S if and only if f satisfies that

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

(2) $S \circ f^- \subseteq f^-$.

Proof. ($\Longrightarrow$) Let f be an $(\in, \in \vee q)$-fuzzy left ideal of S and $x \in S$. Then $(S \circ f^-)(x) \leq f^-(x)$. Indeed: If $A_x = \emptyset$, then $(S \circ f^-)(x) = 0 \leq f^-(x)$. If $A_x \neq \emptyset$, then, by Theorem 3.4 and Lemma 5.2(5),

$$\begin{aligned}
(S \circ f^-)(x) &= (S \circ f)^-(x) = (S \circ f)(x) \wedge 0.5 \\
&= \bigvee_{(y,z) \in A_x} [S(y) \wedge f(z)] \wedge 0.5 \\
&= \bigvee_{(y,z) \in A_x} [1 \wedge f(z)] \wedge 0.5 \\
&= \bigvee_{(y,z) \in A_x} [f(z) \wedge 0.5] \wedge 0.5 \\
&\leq \bigvee_{(y,z) \in A_x} f(yz) \wedge 0.5 \\
&\leq \bigvee_{(y,z) \in A_x} f(x) \wedge 0.5 \text{ (since } x \leq yz) \\
&= f(x) \wedge 0.5 = f^-(x).
\end{aligned}$$

($\Longleftarrow$) Assume that (2) holds. Then $f(xy) \geq f(x) \wedge 0.5$ for all $x, y \in S$. Indeed: Let $x, y \in S$. Put $a = xy$. Then, by $S \circ f^- \subseteq f^-$,

$$\begin{aligned}
f(xy) &= f(a) \geq f^-(a) \geq (S \circ f^-)(a) \\
&= \bigvee_{(b,c) \in A_a} [S(b) \wedge f^-(c)] \\
&\geq S(x) \wedge f^-(y) = 1 \wedge f^-(y) \\
&= f^-(y) = f(y) \wedge 0.5.
\end{aligned}$$

By hypothesis (1) and Theorem 3.4, f is an $(\in, \in \vee q)$-fuzzy left ideal of S.

Similar to Lemma 5.12, we have the following two lemmas. The proofs are similar to that of Lemma 5.12.

Lemma 5.13: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy right ideal of S if and only if f satisfies that

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

(2) $f^- \circ S \subseteq f^-$.

Lemma 5.14: Let S be an ordered semigroup and f a fuzzy subset of S. Then f is an $(\in, \in \vee q)$-fuzzy ideal of S if and only if f satisfies that

(1) $x \leq y \Rightarrow f(x) \geq f(y)$, *for all* $x, y \in S$.

(2) $S \circ f^- \subseteq f^-$, $f^- \circ S \subseteq f^-$.

Proposition 5.15: If f is an $(\in \vee q)$-fuzzy generalized bi-ideal of an ordered semigroup S, then $f^- \circ S \circ f^- \subseteq f^-$.

Proof. Let f be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S. Then $(f^- \circ S \circ f^-)(a) \leq f^-(a)$ for all $a \in S$. Indeed: Since f^- is a fuzzy subset of S, we have $f^-(a) \geq 0$ for all $a \in S$. If $(f^- \circ S \circ f^-)(a) = 0$, then $(f^- \circ S \circ f^-)(a) \leq f^-(a)$. If $(f^- \circ S \circ f^-)(a) \neq 0$, then there exist $x, y, p, q \in S$ such that $(x, y) \in A_a$ and $(p, q) \in A_x$, i.e., $a \leq xy$ and $x \leq pq$. Since f is an $(\in, \in \vee q)$-fuzzy generalized bi-ideal of S, we have $f(pqy) \geq f(p) \wedge f(y) \wedge 0.5$. Thus

$$\begin{aligned}
&(f^- \circ S \circ f^-)(a) \\
= &\bigvee_{(x,y) \in A_a} [(f^- \circ S)(x) \wedge f^-(y)] \\
= &\bigvee_{(x,y) \in A_a} \{ \bigvee_{(p,q) \in A_x} [f^-(p) \wedge S(q)] \wedge f^-(y)\} \\
= &\bigvee_{(x,y) \in A_a} \bigvee_{(p,q) \in A_x} [(f(p) \wedge 0.5) \wedge S(q) \wedge (f(y) \wedge 0.5)] \\
= &\bigvee_{(x,y) \in A_a} \bigvee_{(p,q) \in A_x} [f(p) \wedge f(y) \wedge 0.5] \wedge 0.5 \\
\leq &\bigvee_{(x,y) \in A_a} \bigvee_{(p,q) \in A_x} f(pqy) \wedge 0.5 \\
\leq &\bigvee_{(x,y) \in A_a} f(xy) \wedge 0.5 \text{ (Since } xy \leq pqy) \\
\leq &f(a) \wedge 0.5 = f^-(a) \text{ (Since } a \leq xy),
\end{aligned}$$

which means that $f^- \circ S \circ f^- \subseteq f^-$.

In a similar way as in the previous proposition, by Lemma 5.11, we can show the following proposition:

Proposition 5.16: If f is an $(\in \vee q)$-fuzzy bi-ideal of an ordered semigroup S, then $f^- \circ f^- \subseteq f^-$ and $f^- \circ S \circ f^- \subseteq f^-$.

Lemma 5.17: Let f be a fuzzy subset of an ordered semigroup S and $f^- \circ S \circ f^- \subseteq f^-$. Then $f(xyz) \geq f(x) \wedge f(z) \wedge 0.5$ for all $x, y, z \in S$.

Proof. For all $x, y, z \in S$, put $a = xyz$. Since $f^- \circ S \circ f^- \subseteq f^-$, we have

$$\begin{aligned}
f(xyz) &= f(a) \geq f^-(a) \geq (f^- \circ S \circ f^-)(a) \\
&= \bigvee_{(p,q) \in A_a} [(f^- \circ S)(p) \wedge f^-(q)] \\
&\geq (f^- \circ S)(xy) \wedge f^-(z) \\
&= \bigvee_{(u,v) \in A_{xy}} [f^-(u) \wedge S(v)] \wedge f^-(z) \\
&\geq f^-(x) \wedge S(y) \wedge f^-(z) \\
&= (f(x) \wedge 0.5) \wedge 1 \wedge (f(z) \wedge 0.5) \\
&= f(x) \wedge f(z) \wedge 0.5.
\end{aligned}$$

Proposition 5.18: Let f and g be two $(\in, \in \vee q)$-fuzzy generalized bi-ideals of an ordered semigroup S. Then $f^- \circ g^-$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S.

Proof. Let f, g be two $(\in, \in \vee q)$-fuzzy generalized bi-ideals of S. Then, by Proposition 5.15, $f^- \circ S \circ f^- \subseteq f^-$ and $g^- \circ S \circ g^- \subseteq g^-$. Thus

$$(f^- \circ g^-) \circ (f^- \circ g^-) \subseteq f^- \circ (g^- \circ S \circ g^-) \subseteq f^- \circ g^-.$$

Clearly, $f^- \circ g^-$ is a strongly convex fuzzy subset of S. Therefore, by Lemma 5.11, $f^- \circ g^-$ is an $(\in, \in \vee q)$-fuzzy subsemigroup of S. Furthermore, by Lemma 2.5 we have

$$\begin{aligned} & (f^- \circ g^-) \circ S \circ (f^- \circ g^-) \\ = \; & f^- \circ g^- \circ (S \circ f^-) \circ g^- \\ \subseteq \; & f^- \circ (g^- \circ S \circ g^-) \subseteq f^- \circ g^-. \end{aligned}$$

By Lemma 5.17, $(f^- \circ g^-)(xyz) \geq (f^- \circ g^-)(x) \wedge (f^- \circ g^-)(z) \wedge 0.5$ for all $x, y, z \in S$. Furthermore, if $x \leq y$, then $(f^- \circ g^-)(x) \geq (f^- \circ g^-)(y)$. Indeed: If $A_y = \emptyset$, then $(f^- \circ g^-)(y) = 0$. Since $f^- \circ g^-$ is a fuzzy subset of S, we have $(f^- \circ g^-)(x) \geq 0 = (f^- \circ g^-)(y)$. If $A_y \neq \emptyset$, then, since $x \leq y$, we have $A_y \subseteq A_x$. Thus we have

$$\begin{aligned} (f^- \circ g^-)(y) &= \bigvee_{(u,v)\in A_y} [f^-(u) \wedge g^-(v)] \\ &\leq \bigvee_{(u,v)\in A_x} [f^-(u) \wedge g^-(v)] \\ &= (f^- \circ g^-)(x). \end{aligned}$$

Therefore, $f^- \circ g^-$ is an $(\in, \in \vee q)$-fuzzy bi-ideal of S by Theorem 4.2.

VI. Upper and lower parts of $(\in, \in \vee q)$-fuzzy ideals of ordered semigroups

The characterization of regular semigroups or ordered semigroups in terms of left ideals, right ideals and (generalized) bi-ideals is well known (for example, see [15-17]). In [7], regular semigroups are characterized by the properties of $(\in, \in \vee q)$-fuzzy left (resp. right, quasi-, generalized bi-) ideals. In [29], the authors characterized regular ordered semigroups and intra-regular ordered semigroups by the properties of fuzzy left ideals, fuzzy right ideals and fuzzy (generalized) bi-ideals. In this section we give the similar results by means of the lower parts of $(\in, \in \vee q)$-fuzzy ideals of ordered semigroups.

Lemma 6.1 ([29]): An ordered semigroup S is regular if and only if $B = (BSB]$ for any bi-ideal B of S.

Now we give characterizations of regular ordered semigroups by $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals.

Theorem 6.2: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is regular.

(2) $f^- = f^- \circ S \circ f^-$ for any $(\in, \in \vee q)$-fuzzy bi-ideal f of S.

Proof. $(1) \Rightarrow (2)$. Let f be any $(\in, \in \vee q)$-fuzzy bi-ideal of S and $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa$, and $(ax, a) \in A_a$. Thus

$$\begin{aligned} (f^- \circ S \circ f^-)(a) &= \bigvee_{(y,z)\in A_a} [(f^- \circ S)(y) \wedge f^-(z)] \\ &\geq (f^- \circ S)(ax) \wedge f^-(a) \\ &= \bigvee_{(p,q)\in A_{ax}} [f^-(p) \wedge S(q)] \wedge f^-(a) \\ &\geq [f^-(a) \wedge S(x)] \wedge f^-(a) \\ &= f^-(a) \wedge 1 \wedge f^-(a) = f^-(a), \end{aligned}$$

and so we have $f^- \circ S \circ f^- \supseteq f^-$. Since f is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, by Proposition 5.16, we have $f^- \circ S \circ f^- \subseteq f^-$. Thus $f^- \circ S \circ f^- = f^-$.

$(2) \Rightarrow (1)$. Let B be any bi-ideal of S. Then, by Lemma 5.9, f_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, and for each $a \in B$, by hypothesis and Lemma 5.2(1), we have

$$\begin{aligned} & \bigvee_{(y,z)\in A_a} [(f_B^- \circ S)(y) \wedge f_B^-(z)] \\ = \; & [(f_B^- \circ S) \circ f_B^-](a) = [((f_B^-)^- \circ S) \circ (f_B^-)^-](a) \\ = \; & (f_B^-)^-(a) = f_B^-(a) = 0.5, \end{aligned}$$

which implies that $A_a \neq \emptyset$, and there exist $b, c \in S$ such that $a \leq bc$, $(f_B^- \circ S)(b) = 0.5$ and $f_B^-(c) = 0.5$. Then $c \in B$, and

$$\bigvee_{(p,q)\in A_b} [f_B^-(p) \wedge S(q)] = (f_B^- \circ S)(b) = 0.5,$$

which implies that $A_b \neq \emptyset$, and there exist $u, v \in S$ such that $b \leq uv$, $f_B^-(u) = 0.5$ and $S(v) = 1$. Then $u \in B$, and

$$a \leq bc \leq uvc \in BSB,$$

and so $B \subseteq (BSB]$. On the other hand, since B is a bi-ideal of S, we have $(BSB] \subseteq (B] = B$. Therefore, $B = (BSB]$. By Lemma 6.1, S is regular.

In a similar way, the following theorem can be proved.

Theorem 6.3: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is regular.

(2) $f^- = f^- \circ S \circ f^-$ for any $(\in, \in \vee q)$-fuzzy generalized bi-ideal f of S.

Theorem 6.4: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) S is regular.

(2) $f^- \circ g^- \circ f^- = f^- \cap g^-$ for any $(\in, \in \vee q)$-fuzzy bi-ideal f and any $(\in, \in \vee q)$-fuzzy ideal g of S.

Proof. $(1) \Rightarrow (2)$. Let f, g be an $(\in, \in \vee q)$-fuzzy bi-ideal and an $(\in, \in \vee q)$-fuzzy ideal of S, respectively. Then, by Lemma 2.5 and Proposition 5.16, we have $f^- \circ g^- \circ f^- \subseteq f^- \circ S \circ f^- \subseteq f^-$. Since g is an $(\in, \in \vee q)$-fuzzy ideal of S, by Proposition 5.3 we have

$$f^- \circ g^- \circ f^- \subseteq S \circ g^- \circ S \subseteq S \circ g^- \subseteq g^-.$$

Thus $f^- \circ g^- \circ f^- \subseteq f^- \cap g^-$. On the other hand, let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \leq axa \leq (axa)xa$, so $(a, xaxa) \in A_a$. Since g is an $(\in, \in \vee q)$-fuzzy ideal of S, we have $g(xax) \geq g(ax) \wedge 0.5 \geq g(a) \wedge 0.5$, and so $g^-(xax) = g(xax) \wedge 0.5 \geq g(a) \wedge 0.5 \wedge 0.5 = g(a) \wedge 0.5 = g^-(a)$. Thus

$$\begin{aligned} & (f^- \circ g^- \circ f^-)(a) \\ = \; & \bigvee_{(y,z)\in A_a} [f^-(y) \wedge (g^- \circ f^-)(z)] \\ \geq \; & f^-(a) \wedge (g^- \circ f^-)(xaxa)\} \\ = \; & f^-(a) \wedge \{ \bigvee_{(p,q)\in A_{xaxa}} [g^-(p) \wedge f^-(q)]\} \\ \geq \; & f^-(a) \wedge [g^-(xax) \wedge f^-(a)] \\ \geq \; & f^-(a) \wedge g^-(a) \wedge f^-(a) \\ = \; & f^-(a) \wedge g^-(a) = (f^- \cap g^-)(a), \end{aligned}$$

which means that $f^- \circ g^- \circ f^- \supseteq f^- \cap g^-$. Therefore $f^- \circ g^- \circ f^- = f^- \cap g^-$.

(2) $\Rightarrow$ (1). Since S itself is an $(\in, \in \vee q)$-fuzzy ideal of S, by hypothesis we have

$$f^- = f^- \cap S = f^- \circ S \circ f^-.$$

By Theorem 6.2, S is regular.

In a similar way we can show the following two theorems:

Theorem 6.5: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is regular.*

(2) *$f^- \circ g^- \circ f^- = f^- \cap g^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy ideal g of S.*

Theorem 6.6: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is regular.*

(2) *$f^- \circ g^- = f^- \cap g^-$ for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

Lemma 6.7 ([29]): Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is regular.*

(2) *$B \cap L \subseteq (BL]$ for every bi-ideal B and every left ideal L of S.*

(3) *$R \cap B \cap L \subseteq (RBL]$ for every bi-ideal B, every right ideal R and every left ideal L of S.*

Now we give characterizations of regular ordered semigroups by $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals.

Theorem 6.8: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is regular.*

(2) *$f^- \cap g^- \subseteq f^- \circ g^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

(3) *$f^- \cap g^- \subseteq f^- \circ g^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

(4) *$h^- \cap f^- \cap g^- \subseteq h^- \circ f^- \circ g^-$ for every $(\in, \in \vee q)$-fuzzy generalized bi-ideal f, every $(\in, \in \vee q)$-fuzzy left ideal g and every $(\in, \in \vee q)$-fuzzy right ideal h of S.*

(5) *$h^- \cap f^- \cap g^- \subseteq h^- \circ f^- \circ g^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f, every $(\in, \in \vee q)$-fuzzy left ideal g and every $(\in, \in \vee q)$-fuzzy right ideal h of S.*

Proof. (1) $\Rightarrow$ (2). Let f and g be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal and an $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. If $a \in S$, then, since S is regular, there exists $x \in S$ such that $a \le axa$. Since $(a, xa) \in A_a$, we have

$$\begin{aligned} & (f^- \circ g^-)(a) \\ = & \bigvee_{(y,z)\in A_a} [f^-(y) \wedge g^-(z)] \ge f^-(a) \wedge g^-(xa) \\ = & f^-(a) \wedge g(xa) \wedge 0.5 \ge f^-(a) \wedge (g(a) \wedge 0.5) \wedge 0.5 \\ & \text{(since } g \text{ is an } (\in, \in \vee q)\text{-fuzzy left ideal of } S) \\ = & f^-(a) \wedge (g(a) \wedge 0.5) = f^-(a) \wedge g^-(a) = (f^- \cap g^-)(a), \end{aligned}$$

which means that $f^- \cap g^- \subseteq f^- \circ g^-$.

(2) $\Rightarrow$ (3). Clearly.

(3) $\Rightarrow$ (1). Let B and L be a bi-ideal and a left ideal of S, respectively. Let $a \in B \cap L$. Then, by Theorems 5.7, 5.9, f_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of S and f_L^- is an $(\in, \in \vee q)$-fuzzy left ideal of S. Thus, by hypothesis and Lemma 5.2(1), we have

$$\begin{aligned} (f_B^- \circ f_L^-)(a) &= [(f_B^-)^- \circ (f_L^-)^-](a) \\ &\ge [(f_B^-)^- \cap (f_L^-)^-](a) = (f_B^- \cap f_L^-)(a) \\ &= f_B^-(a) \wedge f_L^-(a) = 0.5 \end{aligned}$$

for all $a \in B \cap L$, and $A_a \neq \emptyset$. By Definition 5.1, $(f_B^- \circ f_L^-)(a) \le 0.5$ for all $a \in S$. Then

$$\bigvee_{(y,z)\in A_a} [f_B^-(y) \wedge f_L^-(z)] = (f_B^- \circ f_L^-)(a) = 0.5,$$

which implies that there exist $b, c \in S$ such that $a \le bc$, $f_B^-(b) = 0.5$ and $f_L^-(c) = 0.5$. Then $a \le bc \in BL$, and so $B \cap L \subseteq (BL]$. By Lemma 6.7, S is regular.

(1) $\Rightarrow$ (4). Let f, g and h be an $(\in, \in \vee q)$-fuzzy generalized bi-ideal, an $(\in, \in \vee q)$-fuzzy left ideal and an $(\in, \in \vee q)$-fuzzy right ideal of S, respectively. Let $a \in S$. Then, since S is regular, there exists $x \in S$ such that $a \le axa$. Thus $(ax, a), (a, xa) \in A_a$, and

$$\begin{aligned} & (h^- \circ f^- \circ g^-)(a) \\ = & \bigvee_{(y,z)\in A_a} [h^-(y), (f^- \circ g^-)(z)] \\ \ge & h^-(ax) \wedge (f^- \circ g^-)(a) \\ = & (h(ax) \wedge 0.5) \wedge \{ \bigvee_{(p,q)\in A_a} [f^-(p) \wedge g^-(q)]\} \\ \ge & (h(a) \wedge 0.5) \wedge [f^-(a) \wedge g^-(xa)] \\ = & h^-(a) \wedge [f^-(a) \wedge (g(xa) \wedge 0.5)] \\ \ge & h^-(a) \wedge f^-(a) \wedge (g(a) \wedge 0.5) \\ = & h^-(a) \wedge f^-(a) \wedge g^-(a) = (h^- \cap f^- \cap g^-)(a). \end{aligned}$$

Therefore, $h^- \cap f^- \cap g^- \subseteq h^- \circ f^- \circ g^-$.

(4) $\Rightarrow$ (5). Clearly.

(5) $\Rightarrow$ (1). The proof is similar to that of (3) $\Rightarrow$ (1) with suitable modification.

An ordered semigroup $(S, \cdot, \le)$ is called *intra-regular* if, for each element a of S, there exist $x, y \in S$ such that $a \le xa^2y$.

Equivalent definition: $a \in (Sa^2S], \forall a \in S$ (cf. [15]).

Lemma 6.9 ([29]): Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is intra-regular.*

(2) *$R \cap L \subseteq (LR]$ for every left ideal L and every right ideal R of S.*

Now we give a characterization of an intra-regular ordered semigroup by $(\in, \in \vee q)$-fuzzy left ideals and $(\in \vee q)$-fuzzy right ideals.

Theorem 6.10: Let S be an ordered semigroup. Then the following statements are equivalent:

(1) *S is intra-regular.*

(2) *$f^- \cap g^- \subseteq g^- \circ f^-$ for every $(\in, \in \vee q)$-fuzzy left ideal g and every $(\in, \in \vee q)$-fuzzy right ideal f of S.*

Proof. (1) $\Rightarrow$ (2). Suppose that S is an intra-regular ordered semigroup, f and g are an $(\in, \in \vee q)$-fuzzy right ideal and an $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Let $a \in S$. Then there exist $x, y \in S$ such that $a \leq xa^2y$, i.e., $(xa, ay) \in A_a$. Thus

$$\begin{aligned}(g^- \circ f^-)(a) &= \bigvee_{(y,z)\in A_a} [g^-(y) \wedge f^-(z)] \\ &\geq g^-(xa) \wedge f^-(ay) \\ &= [g(xa) \wedge 0.5] \wedge [f(ay) \wedge 0.5] \\ &\geq [g(a) \wedge 0.5] \wedge [f(a) \wedge 0.5] \\ &= g^-(a) \wedge f^-(a) = (g^- \cap f^-)(a) \\ &= (f^- \cap g^-)(a),\end{aligned}$$

which means that $f^- \cap g^- \subseteq g^- \circ f^-$.

$(2) \Rightarrow (1)$. Assume that (2) holds. Let R and L be any right ideal and left ideal of S, respectively, and $a \in R \cap L$. Then , by Theorem 5.7, f_R^- and f_L^- be an $(\in, \in \vee q)$-fuzzy right ideal and an $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. Thus , by hypothesis and Lemma 5.2(1),

$$\begin{aligned}(f_L^- \circ f_R^-)(a) &= [(f_L^-)^- \circ (f_R^-)^-](a) \geq [(f_L^-)^- \cap (f_R^-)^-](a) \\ &= (f_R^- \cap f_L^-)(a) = f_R^-(a) \wedge f_L^-(a) = 0.5,\end{aligned}$$

and so $A_a \neq \emptyset$. It is easy to show that $(f_L^- \circ f_R^-)(a) \leq 0.5$ for all $a \in S$. Thus we have

$$\bigvee_{(y,z)\in A_a} [f_L^-(y) \wedge f_R^-(z)] = (f_L^- \circ f_R^-)(a) = 0.5,$$

which implies that there exist $b, c \in S$ such that $a \leq bc$, $f_L^-(b) = 0.5$ and $f_R^-(c) = 0.5$. Then $a \leq bc \in LR$, and so $R \cap L \subseteq (LR]$. It thus follows, by Lemma 6.9, that S is intra-regular.

Theorem 6.11: An ordered semigroup S is intra-regular if and only if

$$(\forall a \in S)\ f^-(a) = f^-(a^2)$$

for every $(\in, \in \vee q)$-fuzzy ideal f of S.

Proof. $(\Longrightarrow)$ Let f be an $(\in, \in \vee q)$-fuzzy ideal of S and $a \in S$. Then, by hypothesis, there exist $x, y \in S$ such that $a \leq xa^2y$, and

$$\begin{aligned}f^-(a) &= f(a) \wedge 0.5 \geq f(xa^2y) \wedge 0.5 \\ &\geq (f(a^2y) \wedge 0.5) \wedge 0.5 = f(a^2y) \wedge 0.5 \\ &\geq (f(a^2) \wedge 0.5) \wedge 0.5 = f(a^2) \wedge 0.5 \\ &\geq (f(a) \wedge 0.5) \wedge 0.5 = f(a) \wedge 0.5 = f^-(a),\end{aligned}$$

which implies that $f^-(a) = f(a^2) \wedge 0.5 = f^-(a^2)$.

$(\Longleftarrow)$ By Theorem 5.7, $f_{I(a^2)}^-$ is an $(\in, \in \vee q)$-fuzzy ideal of S. By hypothesis, $f_{I(a^2)}^-(a) = f_{I(a^2)}^-(a^2) = 0.5$, so $a \in I(a^2) = (a^2 \cup Sa^2 \cup a^2S \cup Sa^2S]$. Thus $a \leq t$ for some $t \in a^2 \cup Sa^2 \cup a^2S \cup Sa^2S$. If $t = a^2$, then $a \leq a^2 \leq a^4 \in Sa^2S$, that is $a \in (Sa^2S]$. If $t = xa^2$ for some $x \in S$, then $a \leq xa^2 \leq x(xa^2)a = x^2a^2a \in Sa^2S$. If $t = a^2y$ for some $y \in S$, then $a \leq a^2y \leq a(a^2y)y = aa^2y^2 \in Sa^2S$. If $t \in Sa^2S$, then $a \in (Sa^2S]$. Thus S is intra-regular.

Theorem 6.12: Let S be an intra-regular ordered semigroup. Then, for any $(\in, \in \vee q)$-fuzzy ideal f of S, we have

$$(\forall a, b \in S)\ f^-(ab) = f^-(ba).$$

Proof. Let f be any $(\in, \in \vee q)$-fuzzy ideal of S and $a, b \in S$. Then, by Theorem 6.11 and Proposition 5.3, we have

$$\begin{aligned}f^-(ab) &= f^-((ab)^2) = f^-(a(ba)b) \\ &\geq f^-(ba) = f^-((ba)^2) \\ &= f^-(b(ab)a) \geq f^-(ab).\end{aligned}$$

Thus we obtain that $f^-(ab) = f^-(ba)$.

Lemma 6.13 ([29]): An ordered semigroup S is regular and intra-regular if and only if $B = (B^2]$ for every bi-ideal B of S.

Now we shall give some characterizations of an ordered semigroup which is both regular and intra-regular by $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy bi-ideals.

Theorem 6.14: Let S be an ordered semigroup. Then the following conditions are equivalent:

(1) *S is regular and intra-regular.*

(2) *$f^- \circ f^- = f^-$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f of S.*

(3) *$f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-)$ for any $(\in, \in \vee q)$-fuzzy bi-ideals f and g of S.*

(4) *$f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-)$ for every $(\in, \in \vee q)$-fuzzy bi-ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

(5) *$f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-)$ for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy bi-ideal g of S.*

(6) *$f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-)$ for every $(\in, \in \vee q)$-fuzzy right ideal f and every $(\in, \in \vee q)$-fuzzy left ideal g of S.*

Proof. $(1) \Rightarrow (3)$. Let f and g be $(\in, \in \vee q)$-fuzzy bi-ideals of S and $a \in S$. Then, since S is both regular and intra-regular, there exists $x \in S$ such that $a \leq axa (\leq axaxa)$, and there exist $y, z \in S$ such that $a \leq ya^2z$. Thus

$$a \leq axa \leq axaxa \leq ax(ya^2z)xa = (axya)(azxa),$$

i.e., $(axya, azxa) \in A_a$. Since f and g are both $(\in \vee q)$-fuzzy bi-ideals of S, we have

$$\begin{aligned}f^-(axya) &= f(axya) \wedge 0.5 \\ &\geq (f(a) \wedge f(a) \wedge 0.5) \wedge 0.5 \\ &= f(a) \wedge 0.5 = f^-(a)\end{aligned}$$

and

$$\begin{aligned}g^-(axya) &= g(axya) \wedge 0.5 \\ &\geq (g(a) \wedge g(a) \wedge 0.5) \wedge 0.5 \\ &= g(a) \wedge 0.5 = g^-(a).\end{aligned}$$

Then

$$\begin{aligned}(f^- \circ g^-)(a) &= \bigvee_{(p,q)\in A_a} [f^-(p) \wedge g^-(q)] \\ &\geq f^-(axya) \wedge g^-(azxa) \\ &\geq f^-(a) \wedge g^-(a) = (f^- \cap g^-)(a),\end{aligned}$$

which means that $f^- \cap g^- \subseteq f^- \circ g^-$. In the same way, we can show that $f^- \cap g^- \subseteq g^- \circ f^-$. Hence $f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-)$.

Since every $(\in, \in \vee q)$-fuzzy left (right) ideal of S is an $(\in, \in \vee q)$-fuzzy bi-ideal of S, and so $(3) \Rightarrow (4), (4) \Rightarrow (6), (3) \Rightarrow (2), (3) \Rightarrow (5)$ and $(5) \Rightarrow (6)$ are clear.

$(6) \Rightarrow (1)$. Let f and g be an $(\in, \in \vee q)$-fuzzy right ideal and an $(\in, \in \vee q)$-fuzzy left ideal of S, respectively. By hypothesis, $f^- \cap g^- \subseteq (f^- \circ g^-) \cap (g^- \circ f^-) \subseteq g^- \circ f^-$. By Theorem 6.10, S is intra-regular. On the other hand,

$$\begin{aligned} f^- \cap g^- &\subseteq (f^- \circ g^-) \cap (g^- \circ f^-) \subseteq f^- \circ g^-. \\ f^- \circ g^- &\subseteq f^- \circ S \subseteq f^- (\text{ By Proposition 5.3}). \\ f^- \circ g^- &\subseteq S \circ g^- \subseteq g^- (\text{ By Proposition 5.3}). \end{aligned}$$

Thus $f^- \circ g^- \subseteq f^- \cap g^-$, and $f^- \circ g^- = f^- \cap g^-$. By Theorem 6.6, S is regular.

$(2) \Rightarrow (1)$. Let B be a bi-ideal of S and $a \in B$. By Theorem 5.9, f_B^- is an $(\in, \in \vee q)$-fuzzy bi-ideal of S. Then , by hypothesis, $(f_B^- \circ f_B^-)(a) = f_B^-(a) = 0.5$, which implies that $A_a \neq \emptyset$, and

$$\bigvee_{(y,z) \in A_a} [f_B^-(y) \wedge f_B^-(z)] = (f_B^- \circ f_B^-)(a) = 0.5.$$

Then there exist $b, c \in S$ such that $a \leq bc$, and $f_B^-(b) = f_B^-(c) = 0.5$. Then $a \leq bc \in BB = B^2$, that is $a \in (B^2]$, and so $B \subseteq (B^2]$. On the other hand, since B is a bi-ideal of S, we have $(B^2] \subseteq (B] = B$. Hence we obtain that $B = (B^2]$. By Lemma 6.13, S is regular and intra-regular.

VII. Conclusion

In study the structure of an ordered semigroup S, we notice that the fuzzy ideals (left, right ideals) of S with special properties always play an important role. In this paper we have introduced the concepts of $(\in, \in \vee q)$-fuzzy ideals, $(\in, \in \vee q)$-fuzzy bi-ideals and $(\in, \in \vee q)$-fuzzy generalized bi-ideals of an ordered semigroup, and investigated their related properties. Furthermore, we have given some characterizations of regular ordered semigroups and intra-regular ordered semigroups by $(\in, \in \vee q)$-fuzzy left ideals, $(\in, \in \vee q)$-fuzzy right ideals and $(\in, \in \vee q)$-fuzzy (generalized) bi-ideals. We hope that the research along this direction can be continued, and in fact, some results in this paper have already constituted a platform for further discussion concerning the future development of ordered semigroups. Hopefully, some new results in these topics can be obtained in the forthcoming paper.

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