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An asymptotic formula for pricing an American exchange option

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Abstract—In this paper, the American exchange option (AEO) valuation problem is modelled as a free boundary problem. The critical stock price for an AEO is satisfied an integral equation implicitly. When the remaining time is large enough, an asymptotic formula is provided for pricing an AEO. The numerical results reveal that our asymptotic pricing formula is robust and accurate for the long-term AEO.

Keywords—integral equation, asymptotic solution, free boundary problem, American exchange option

I. Introduction

An American option gives holder a right to exercise prior the maturity date. Several approaches are provided to model the American option valuation problem. Merton [16] first assumed that there is a trigger to early-exercise the American option, which is called critical stock price (CSP), and model the American option valuation model as a free boundary problem. Now finding the explicit representation of the CSP becomes a challenge to solve the free boundary problem. A considerable number of researchers, such as [5], [8], and [11] are devoted to discover the asymptotic expressions of the CSP when the remaining time nears to zero. On the other hand, MacMillan [14], and Barone-Adesi and Whaley [2] used numerical methods such as the fixed-point method or Newton's method to obtain the CSP and provided other approximate solutions of this free boundary problem. These approaches provide a formula to valuate the long-term American option. However, one disadvantage of these approaches is time-consumption.

An exchange option is an option which gives the holder the right to exchange form asset two to asset one. The non-dividend European exchange option (EEO) is first provided by Margrabe [15]. He showed that the EEO's price function has a linear homogeneous property, and applied Euler's theorem and Itô formula to this price function. Thus, he obtained that the price function of the EEO satisfies a two dimensional parabolic equation and provided an explicit expression of the valuation formula by solving this equation. McDonald and Siegel [17] extended this analytic valuation formula to the dividend-paying EEO.

Following the line of the American option valuation model, an AEO valuation model is also modeled as a two dimensional parabolic free boundary problem. Now some researches are concentrated on finding a valuation formula of an AEO. Margrabe [15] showed that a rational investor does not exercise the non-dividend AEO early. This implies that the price of such an AEO is equal to its European counterpart. Broadie and Detemple [2] characterized the optimal exercise regions for American options on multiple assets. They showed that there is a value, which is only depended on time, such that immediate exercise is optimal when the ratio of the two assets is larger than the value. Carr [4] generalized the Geske-Johnson [9] approach to valuate an AEO. The value of a general pseudo-AEO is represented as a formula of the standard multi-normal distribution. The valuation formula becomes cumbersome for a large number of exercise points. Liu and Liu [13] who extended the method of Evans et al. [8] provided an asymptotic pricing formula as time nears to the maturity date. Such a valuation formula a the short-term AEO can be easily implemented but cannot be applied to a mid- or a long-term AEO. So far the robust and fast valuation formula for the long-term AEO has not been provided yet.

In this paper, we first provide the initial region of the free boundary and develop a free boundary problem. Following the spirit of MacMillan [14] and Barone-Adesi and Whaley [2], we generalize their method to provide an asymptotic pricing formula for an AEO. The CSP is implicit in an integral equation and its value can be found by the numerical root finding method such as fixed point method or Newton's method in previous researches. When the remaining time is large enough, an asymptotic solution is derived for this integral equation. The numerical results compared with the binomial approach reveal that our asymptotic pricing formula of an AEO is accurate for a long term AEO.

The rest of this paper is organized as follows: In section 2, we propose a free boundary problem for an AEO and provide the properties of the free boundary. An asymptotic formula for the free boundary problem is provided in section 3. The numerical results are displayed in section 4. Finally, we provide a brief conclusion.

II. Free boundary problem

Let

$$\frac{dS_i}{S_i} = (r - q_i)dt + \sigma_i dw_i,\ i = 1, 2,$$

be the price dynamics of both two assets where S_i, q_i and σ_i are the price, the continuous rate of dividends and the volatility of the asset i. The correlation coefficient between two Wiener processes dw_1 and dw_2 is ρdt. The final payoff of an exchange option is given as

$$V(S_1, S_2, T) = \max(S_1 - S_2, 0), \tag{1}$$

where T and $\tau = T - t$ denote respectively the maturity date and the remaining time and $V(S_1, S_2, \tau)$ denotes the value of

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a EEO at τ. We have known that the value of a EEO satisfies the following equation

$$V_t + \mathcal{L}V = 0,\ t < T \tag{2}$$

with terminal condition (1), where the operator $\mathcal{L}$ is defined as

$$\begin{aligned}\mathcal{L}V = \ & \tfrac{1}{2}\sigma_1^2 S_1^2 V_{S_1S_1} + \rho\sigma_1\sigma_2 S_1 S_2 V_{S_1S_2} \\ & + \tfrac{1}{2}\sigma_2^2 S_2^2 V_{S_2S_2} - q_1 S_1 V_{S_1} - q_2 S_2 V_{S_2}.\end{aligned}$$

The solution of (2) with (1) has been known as

$$\begin{aligned}V(S_1, S_2, \tau) = \ & S_1 e^{-q_1\tau} N(d_1(S_1, S_2, \tau)) \\ & - S_2 e^{-q_2\tau} N(d_2(S_1, S_2, \tau)),\end{aligned} \tag{3}$$

where

$$\begin{aligned}& d_1(S_1, S_2, \tau) = \frac{\log(\frac{S_1}{S_2})}{\sigma\sqrt{\tau}} + \frac{q_2 - q_1 + \frac{\sigma^2}{2}}{\sigma}\sqrt{\tau}, \\ & d_2(S_1, S_2, \tau) = d_1 - \sigma\sqrt{\tau}, \\ & \sigma^2 = \sigma_1^2 - 2\rho\sigma_1\sigma_2 + \sigma_2^2.\end{aligned}$$

The hedge ratios of a EEO, which is she first derivative of (3) are given as follows

$$V_{S_1}(S_1, S_2, \tau) = \ e^{-q_1\tau} N(d_1(S_1, S_2, \tau)), \tag{4}$$
$$V_{S_2}(S_1, S_2, \tau) = \ -e^{-q_2\tau} N(d_2(S_1, S_2, \tau)). \tag{5}$$

Note that (4) and (5) will be applied to valuate the early exercise premium for an AEO in the next section.

An AEO gives holder a right to exercise the option before the maturity date. Based on the no-arbitrage condition, the value of an AEO is always greater than its immediate exercising value; that is

$$P(S_1, S_2, \tau) > \max\{S_1 - S_2, 0\}, \tag{6}$$

where P denotes the value of an AEO.

In order to model the phenomenon of early exercise, we structure an optimal early exercise region. When the pair of the stock prices (S_1, S_2) is in the optimal early exercise region, early exercising becomes the only optimal strategy for the holder of an AEO. Hence, we have $P(S_1, S_2, \tau) = \max\{S_1 - S_2, 0\}$ in the optimal early-exercising region. This implies that $\mathcal{L}P = 0$ only holds outside of the optimal early-exercising region.

Let $\mathbf{Q} = (0, \infty) \times (0, \infty)$. When τ is given, the optimal early exercise region of an AEO is characterized as [3]

$$\mathbf{S}(\tau) = \{(S_1, S_2) \in \mathbf{Q} | S_2 \le L(\tau)S_1, \text{for some } L(\tau)\},$$

where $L(\tau)$ is a function of τ and called a CSP.

Let $\mathbf{C}(\tau) = \mathbf{Q} - \mathbf{S}(\tau)$ for all τ. We have

$$P(S_1, S_2, \tau) > \max\{S_1 - S_2, 0\},\ (S_1, S_2) \in \mathbf{C}(\tau),\ 0 < \tau < T.$$

The pricing formula of an AEO is formulated as follows:

$$P_\tau = \mathcal{L}P,\ (S_1, S_2) \in \mathbf{C}(\tau),\ 0 \le \tau \le T, \tag{7}$$
$$P(S_1, S_2, 0) = (S_1 - S_2)^+,\ \tau = 0, \tag{8}$$
$$\begin{aligned}& P(S_1, L(\tau)S_1, t) = S_1 - L(\tau)S_1, \\ & \qquad 0 < S_1 < \infty,\ 0 < \tau \le T.\end{aligned} \tag{9}$$

The AEO valuation problem is now modelled as a free boundary problem. The challenge of solving the FBP is to find out a mathematical representation of the optimal early region.

Since the exchange option can be considered either as a call on asset one with strike price S_2 or a put on asset two with strike price S_1, the high contact condition (or tangency condition) of a call and a put can be applied to an AEO. Hence we have the following conditions:

$$P_{S_1}(S_1, L(\tau)S_1, \tau) = 1,\ 0 < S_1 < \infty,\ 0 < \tau \le T, \tag{10}$$
$$P_{S_2}(S_1, L(\tau)S_1, \tau) = -1,\ 0 < S_1 < \infty,\ 0 < \tau \le T. \tag{11}$$

By the definition of an AEO, the value of an AEO satisfies the following condition:

$$P(0, S_2, \tau) = 0,\ 0 < S_2 < \infty,\ 0 < \tau \le T, \tag{12}$$
$$P(S_1, 0, \tau) = S_1,\ 0 < S_1 < \infty,\ 0 < \tau \le T. \tag{13}$$
$$\lim_{S_2 \to \infty} P(S_1, S_2, \tau) = 0,\ 0 < S_1 < \infty,\ 0 < \tau \le T. \tag{14}$$
$$\lim_{S_1 \to \infty} P(S_1, S_2, \tau) = S_1,\ 0 < S_2 < \infty,\ 0 < \tau \le T. \tag{15}$$

We call the free boundary problem (7) with conditions (8)-(15) as problem A.

Let $\Omega = \{(S_1, S_2) \in \mathbf{Q} | q_1 S_1 - q_2 S_2 > 0,\ S_1 > S_2\}$, then Ω is an open subset of $\mathbf{Q}$ such that $\mathcal{L}\psi(S_1, S_2) > 0$ for $(S_1, S_2) \in \Omega$ and $\mathcal{L}\psi(S_1, S_2) < 0$ for $(S_1, S_2) \in \mathbf{Q} - \Omega$, where $\psi(S_1, S_2) = \max\{S_1 - S_2, 0\}$.

III. An asymptotic formula

The value of an AEO is always greater than the value of a EEO for the right of exercise prior the expiry when the underlying assets pay dividends. We define the early exercise premium by

$$E(S_1, S_2, \tau) = P(S_1, S_2, \tau) - V(S_1, S_2, \tau), \tag{16}$$

or $E = P - V$ in abbreviated form.

Within the continuation, (7) holds for both P and V; and since the differential equation is linear, (7) also holds for $E(S_1, S_2, \tau)$:

$$E_\tau = \mathcal{L}E,\ (S_1, S_2) \in \mathbf{C}(\tau),\ 0 \le \tau \le T, \tag{17}$$

Since P and V both satisfy the boundary condition (8), we choose $E(S_1, S_2, 0) = 0$ which guarantees that $P(S_1, S_2, \tau)$ written in form (16) satisfies the boundary condition (1). Substituting (9), (10) and (11) into (16), we have that the early exercise premium satisfies the following condition:

$$V(S_1, L(\tau)S_1, \tau) + E(S_1, L(\tau)S_1, \tau) = S_1 - L(\tau)S_1, \tag{18}$$
$$V_{S_1}(S_1, L(\tau)S_1, \tau) + E_{S_1}(S_1, L(\tau)S_1, \tau) = 1, \tag{19}$$
$$V_{S_2}(S_1, L(\tau)S_1, \tau) + E_{S_2}(L(S_1, \tau)S_1, \tau) = -1. \tag{20}$$

So, the pricing problem of an AEO remains to solve (17) with zero initial value and conditions (18)-(20).

Following the method of MacMillan [14], we replace the variable τ by an as yet unspecified function of τ, $h(\tau)$, and rewrite $E(S_1, S_2, \tau)$ in the form:

$$E(S_1, S_2, \tau) = h(\tau) g(S_1, S_2, h(\tau)). \tag{21}$$

Imposing (21) into (17), we have that

$$\frac{h'(\tau)}{h(\tau)}g + h'(\tau)g_h = \mathcal{L}g, \tag{22}$$

where $g_h = \frac{\partial g}{\partial h}$ and $h'(\tau) = \frac{dh}{d\tau}$.

We choose $h(\tau)$ as

$$h(\tau) = 1 - e^{-\tau}$$

so that the left hand side of (22) becomes

$$(1-h)(\frac{g}{h} + g_h). \tag{23}$$

Since $1-h$ approaches to 0 as τ is large enough, we can drop the term $(1-h)(\frac{g}{h}+g_h)$ to produce a useful approximation for large τ with the error controlled by the term $1-h$. The approximate equation for g now becomes

$$\mathcal{L}g = 0$$

as τ is large enough. Thus, the rest of the AEO's pricing problem becomes to solve the following equation:

$$\mathcal{L}g = 0, (S_1, S_2) \in \mathbf{C}(\tau),\ 0 \le \tau \le T, \infty \tag{24}$$

with the following conditions

$$P(S_1, L(\tau)S_1, \tau) + h(\tau)g(S_1, L(\tau)S_1, h(\tau)) = S_1 - L(\tau)S_1, \tag{25}$$

$$P_{S_1}(S_1, L(\tau)S_1, \tau) + h(\tau)g_{S_1}(S_1, L(\tau)S_1, h(\tau)) = 1, \tag{26}$$

$$P_{S_2}(S_1, L(\tau)S_1, \tau) + h(\tau)g_{S_2}(S_1, L(\tau)S_1, h(\tau)) = -1. \tag{27}$$

When h in g is treated as a parameter, a solution for $g(S_1, S_2)$ is given by

$$g(S_1, S_2) = cS_1^{\alpha}S_2^{\beta}. \tag{28}$$

Imposing (28) into (27), we have

$$c\beta L^{\beta-1}S_1^{\alpha+\beta-1}h = e^{-q_2\tau}N(\bar{d}_2) - 1, \tag{29}$$

where

$$\bar{d}_i(S_1, LS_1, \tau) = \frac{\log(L)}{\sigma\sqrt{\tau}} + \frac{q_2 - q_1 + (-1)^{i-1}\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau},\ i = 1, 2,$$

abbreviated as $\bar{d}_i$. We rearrange (29) and obtain the representation of c in (28), that is

$$c = \frac{e^{-q_2\tau}N(\bar{d}_2) - 1}{\beta S_1^{\alpha+\beta-1}L^{\beta-1}h}. \tag{30}$$

Substituting (30) into (28), we obtain

$$g(S_1, S_2, \tau) = \frac{e^{-q_2\tau}N(\bar{d}_2) - 1}{\beta h L^{\beta-1}}S_1^{1-\beta}S_2^{\beta}. \tag{31}$$

In order to select the value of L and β, we substitute (31) into (26) and keep a tedious manipulation. The relation between L and β are represented as the follows

$$\frac{(1-\beta)(e^{-q_2\tau}N(\bar{d}_2) - 1)}{\beta}L = 1 - e^{-q_1\tau}N(\bar{d}_1). \tag{32}$$

Now we rearrange (32) and obtain

$$L = \frac{\beta(1 - e^{-q_1\tau}N(\bar{d}_1))}{(1-\beta)(e^{-q_2\tau}N(\bar{d}_2) - 1)}. \tag{33}$$

Note that $\bar{d}_1$ is still a function of L.

Finally, we remain to select a suitable value of β satisfied equation (24) as τ is large enough. As τ is large enough, we substitute (31) into (24) . Hence, we obtain that

$$A\beta^2 + B\beta + C = 0,$$

where

$$A = \sigma^2,\ B = \sigma^2 + q_1 - q_2,\ C = -q_1.$$

We obtain the value β as follows:

$$\beta = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}. \tag{34}$$

The early exercise premium E of an AEO is in terms of the S_1, S_2 and L. The CSP, $L(\tau)$, is implicit in (33) which is an integral equation. There is a difficulty to find out the explicit analytic form, but some numerical methods are provided to deal with such an integral equation. MacMillan (1986) used fixed-point iteration method and find out a successive approximations convergent to the exactly value of the root.

In the next section, we rewrite the normal distribution function in (33) as a (complementary) error function and use the asymptotic form of such function to provide an asymptotic representation of the critical function as τ tends to infinity.

Before substituting the asymptotic forms into (33), we first discuss the value of $\bar{d}_i(L, \tau)$, as τ is large enough. We have known that $L(\tau)$ lies between 1 and L_∞, where $L_\infty < \infty$ denotes the CSP of the perpetual AEO. This implies that

$$\frac{-\log L}{\sigma\sqrt{\tau}} \sim 0,\ \text{as } \tau \gg 1.$$

Hence, the function

$$\bar{d}_i(S_1, LS_1, \tau) = \frac{-\log(L)}{\sigma\sqrt{\tau}} + \frac{q_2 - q_1 + (-1)^{i-1}\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau},$$

$i = 1, 2$ is dominated by the last term

$$\frac{q_2 - q_1 + (-1)^{i-1}\frac{\sigma^2}{2}}{\sigma}\sqrt{\tau} \equiv a_i(\tau),$$

as τ is large enough.

Since we consider the situation of the long maturity date, the value of $\bar{d}_i$, $i = 1, 2$ are dominated by a_i. So another approximation method is provided by substituting $\bar{d}_i$ with a_i. Thus, the approximation formula of the critical stock price is written as follows:

$$L \sim \frac{\beta(1 - e^{-q_1\tau}N(a_1))}{(1-\beta)(e^{-q_2\tau}N(a_2) - 1)}. \tag{35}$$

However, (35) needs to calculate integrals which is time consuming.

The quadratic asymptotic formula for an AEO have now been derived. In the next section, we present some numerical results intended to show the accuracy of this techniques.

IV. Numerical Results

The one asset binomial option pricing model is first introduced by Sharpe [20] and in detail by Cox, Ross and Rubinstein [7]. Rubinstein [19] extended the one asset binomial model to an exchange option. However, the binomial model is even less efficient for two variables than for one variable. The convergence of the price of a EEO or an AEO is a function of the number of time steps. We choose a binomial model with 500 time steps as our benchmark for the true value (TV). To compare with 200 time steps, our benchmark accurate to 3 decimal digit.

In order to examine the accuracy of our asymptotic solution, we fix S_2=1, $\rho = \sigma_1 = \sigma_2 = 0.5$. This implies $\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2 = 0.125$. The price of these AEOs are computed by the binomial tree model (BT) and the asymptotic solution (35) (QA). These asymptotic solutions are compared with the true value. The relative error, defined as

$$\mathrm{RE} = |\frac{\text{Asymptotic value} - \text{True value}}{\text{True value}}|$$

are used to measure the accuracy of the price of each option. The mean relative error (MRE) is defined as the average of RE from time 1 to time 5. The computations of these formulas are coded by Matlab software.

Table 1 contains 11 options with setting $(S_1, q_1, q_2) = (1.1, 0.1, 0.3)$ and remaining time $\tau = 1, 1.2, \ldots, 3$. Let RT denote the remaining time and PE denote the price of a EEO. The relative error between QA and TV is denoted as RE. The columns in Table 1 from left to right are T, PE, TV, BT200, QA and RE, respectively. The last row is the MRE of the RE from time 1 to time 3. We find that the MRE is 0.0103. This implies that our formula provides a accurate value for pricing an AEO.

TABLE I
The numerical results of case $(S_1, q_1, q_2) = (1.1, 0.1, 0.3)$

RT	PE	TV	BT	QA	RE
1	0.3268	0.3271	0.3273	0.333	0.0180
1.2	0.3512	0.3518	0.352	0.3573	0.0156
1.4	0.3722	0.3736	0.3736	0.3783	0.0126
1.6	0.3906	0.3928	0.3928	0.3967	0.0099
1.8	0.4065	0.4099	0.4098	0.4126	0.0066
2	0.4203	0.4252	0.4251	0.4266	0.0033
2.2	0.4322	0.4389	0.4389	0.4387	0.0005
2.4	0.4425	0.4513	0.4513	0.4491	0.0049
2.6	0.4512	0.4624	0.4625	0.4581	0.0093
2.8	0.4586	0.4724	0.4726	0.4658	0.0140
3	0.4648	0.4815	0.4817	0.4723	0.0191
MRE					0.0103

V. Conclusion

So far many financial problems are modelled as an AEO. Hence, providing a robust valuation formula for an AEO becomes very important for the realistic financial problem. In this paper, we propose an AEO valuation problem which is a free boundary problem and provide a rigorous verification of some properties for the free boundary. Following the spirit of [2] and [14], we have extended their method to the AEO valuation problem and provided an asymptotic valuation formula for an AEO. However, this formula contains an undetermined function. This undetermined function is called as the critical stock price (CSP) for an AEO. For the CSP of an AEO, we have derived an asymptotic formula as the remaining time is large enough.

In order to apply MacMillan's [14] method to other kinds of American option, one needs to know the exact pricing formula of its European counterpart. The pricing formula of the European spread option has been found by Carmona and Durrleman [6]. Hence, one of our future studies is to provide an asymptotic pricing formula for the American spread option.

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