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Connected Vertex Cover in 2-Connected Planar Graph with Maximum Degree 4 is NP-complete

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***Abstract*—This paper proves that the problem of finding connected vertex cover in a 2-connected planar graph (CVC-2) with maximum degree 4 is NP-complete. The motivation for proving this result is to give a shorter and simpler proof of NP-Completeness of TRA-MLC (the Top Right Access point Minimum-Length Corridor) problem [1], by finding the reduction from CVC-2. TRA-MLC has many applications in laying optical fibre cables for data communication and electrical wiring in floor plans.The problem of finding connected vertex cover in any planar graph (CVC) with maximum degree 4 is NP-complete [2]. We first show that CVC-2 belongs to NP and then we find a polynomial reduction from CVC to CVC-2. Let a graph G_0 and an integer K form an instance of CVC, where G_0 is a planar graph and K is an upper bound on the size of the connected vertex cover in G_0. We construct a 2-connected planar graph, say G, by identifying the blocks and cut vertices of G_0, and then finding the planar representation of all the blocks of G_0, leading to a plane graph G_1. We replace the cut vertices with cycles in such a way that the resultant graph G is a 2-connected planar graph with maximum degree 4. We consider $L = K - 2t + 3\sum_{i=1}^{t} d_i$ where t is the number of cut vertices in G_1 and d_i is the number of blocks for which i^{th} cut vertex is common. We prove that G will have a connected vertex cover with size less than or equal to L if and only if G_0 has a connected vertex cover of size less than or equal to K.

***Keywords*—NP-complete, 2-Connected planar graph, block, cut vertex.

I. Introduction

A brief overview of the relevent definitions of graph theory ([3],[4]) is presented in this section before introducing the problem.

Any graph G is said to be *planar* or embeddable in the plane, if it can be drawn in the plane so that the vertices are distinct points in the plane and its edges intersect only at their end points. Such a drawing of a planar graph G is called a planar embedding of G or a *plane graph*. There are many polynomial time algorithms for finding a planar embedding of planar graph [5]. A subset V_2 of V is said to be a *vertex cut* if the removal of the vertices in V_2 disconnects the graph. A *cut vertex* is a single vertex, removal of which disconnects the graph. A graph G is said to be *2-connected* if and only if any two vertices of G are connected by atleast two internally-disjoint paths. Any 2-connected graph does not have a cut vertex. A *block* of a graph G is a *maximal* connected subgraph of G that has no cut vertex for itself. Any block of a graph G is an isolated vertex or a cut edge, or a maximal 2-connected component with more than 2 vertices. Every pair of blocks will have at the most one vertex in common and that will be a cut vertex. So, any cut vertex will be adjacent to a cut vertex in another block, or any other vertex in a 2-connected component, or a pendent vertex . A Graph G is planar if and only if each of its blocks is planar. There are polynomial time algorithms to identify the blocks in any planar graph and also to find out the cut vertices in a graph [3][6].

A *vertex cover* in a planar graph G is a subset V_1 of V such that every edge of G has atleast one end in V_1 and it is said to be *connected* if the vertices in this subset are all connected in G. A vertex cover V_1 is said to be a *Minimum Vertex Cover* if G has no other vertex cover $\bar{V}$ with $|\bar{V}| < |V_1|$.

The problem of finding a *Minimum vertex cover in a graph* is NP-complete [7]. Garey and Johnson proved many restricted versions of this problem and specifically, the *vertex cover in planar graphs* to be NP-complete [2],[8]. They also proved that the problem of finding *Minimum connected vertex cover in planar graphs with maximum degree 4* (hereafter referred to as CVC) is NP-complete [4] .

In this paper, we attempt to prove that the problem of *connected vertex cover in 2-connected planar graph with maximum degree 4* (hereafter referred to as CVC-2) is NP-complete. Given a 2-connected planar graph with maximum degree 4, the problem is to find a conneted vertex cover with minimum size. The decision version of this problem can be stated as follows:

> **Instance**: A 2-connected planar graph $G(V, E)$ with all the vertices having degree less than or equal to 4 and an integer L.
> **Question**: Does there exist a subset V_2 of V, with $|V_2| = c_2$, such that $c_2 \leq L$, V_2 is connected and it covers all the edges in G.

The motivation behind giving the proof of the complexity of CVC-2 is to give a proof of NP-completeness of Top Right Access point Minimum-Length Corridor (TRA-MLC)problem. In the Minimum-Length Corridor (MLC) problem [1], a rectangular boundary partitioned into rectilinear polygons is given and the problem is to find a *corridor* of *least* total length. A *corridor* is a tree containing a set of line segments lying along the outer rectangular boundary and/or on the boundary of the rectilinear polygons. The corridor must contain at least one point from the boundaries of the outer rectangle and also the rectilinear polygons. An access point of a cooridor is any point on the rectangular boundary. If this access point is constrained to be at the top right corner of the outer rectangular boundary, then this problem is referred to as TRA-MLC. In

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the MLC problem, and in its variants, it is assumed that the rectangular boundary and the partitions are orthogonal.

This problem has many applications in laying optical fibre cables for data communication and electrical wiring in floor plans.There are many other applcations which include signal communication in circuit layout design [1]. We are going to work towards finding a polynomial reduction from CVC-2 to TRA-MLC thereby proving TRA-MLC is NP-complete.

To prove that any problem P to be NP-complete we need to show that

1. $P \in NP$: x is a *yes* instance of P if and only if there exists a *concise certificate* $c(x)$, and it is *verifiable* by a polynomial time algorithm.
2. Some known NP-complete problem P' is *polynomially reducible* to P : For any given instance x of P' , we should be able to construct an instance y of P within polynomial in $|x|$ time, such that x is a *yes* instance of P' if and only if y is a *yes* instance of P.

For more explanation on NP-completeness, reader is referred to [7][9]. In the next section, we give a proof of NP-completeness of CVC-2 by giving a *polynomial reduction* from CVC to CVC-2.

II. The proof

Theorem:*"Connected vertex cover in 2 - connected planar graph with maximum degree 4" is NP-complete.*

Proof: Inorder to prove that CVC-2 is NP-complete, first we need to show that CVC-2 $\in$ NP. For any instance of CVC-2 given by a 2-connected planar graph G with maximum degree 4 and an integer K, assume that a certificate V_2 which is a subset of vertices of G is given. We can find whether the vertices of V_2 are connected and whether they cover all the edges of G in polynomial time. Also we can find, in polynomial time, whether the size of V_2 is less than or equal to K or not. So it is obvious to say that CVC-2 $\in$ NP.

Now, we give a polynomial reduction from CVC to CVC-2. Assume that an instance of the decision version of CVC is given by a connected planar graph $G_0 = (V_0, E_0)$ (as the required vertex cover is connected, the given graph should obviously be connected) , in which the vertex degree is at the most 4 and an integer K, which is the upper bound on the size of the connected vertex cover. We restrict our problem to graphs with more than 2 vertices.

We construct an instance of our problem from G_0. First we find the blocks and cut vertices of G_0, and let the number of cut vertices be t. As G_0 is connected, it does not have isolated vertices as blocks. Let $C = \{c_1, c_2, ...c_t\}$ be the set of cut vertices in G_0. For any cut vertex c_i, let d_i be the number of blocks having c_i as common vertex for $1 \leq i \leq t$. Now, we find planar representation of each block in G_0, by using any polynomial time algorithm, thereby finding a plane graph G_1 of G_0.

Now the construction of the instance begins with G_1. For every integer i from 1 to t, we construct a plane graph G_{i+1} from G_i. Consider the cut vertex c_i which is a common vertex for d_i blocks and let $b_0, b_1, ...b_{d_i-1}$ be the blocks in clockwise order around c_i in G_i. Replace c_i with a cycle consisting of $3d_i$ vertices namely $v_{i(j)}$ for $0 \leq j \leq (3d_i - 1)$. There will be $3d_i$ edges in this cycle and they are $(v_{i(j)}, v_{i(j+1)})$ for $0 \leq j \leq (3d_i - 2)$ and the edge $(v_{i(3d_i-1)}, v_{i(1)})$. For any block b_k ($0 \leq k \leq d_i - 1$) containing c_i, assume that there are p vertices ($p > 1$), $v_1, v_2, ...v_p$ adjacent to c_i in clockwise order around c_i within b_k. We replace these edges, $(c_i, v_1), (c_i, v_2), ...(c_i, v_{p-1})$ with $(v_{i(3k)}, v_1), (v_{i(3k)}, v_2), ...(v_{i(3k)}, v_{p-1})$ and $(v_{i(3k+1)}, v_p)$. Any cut vertex is a common vertex for atleast two blocks. As the degree of any cut vertex in G_1 can not exceed 4, there can be at the most three vertices in a block, which are adjacent to the cut vertex ie. $p \leq 3$. So the degree of the vertex $v_{i(3k)}$, will be at the most 4 and the degree of $v_{i(3k+1)}$ will be at the most 3. If $p = 1$, then the block is a cut edge, ie. c_i is adjacent to only one vertex v_1 in that block which is either a pendent vertex or another cut vertex. In this case we replace the edge (c_i, v_1) with two edges namely $(v_{i(3k)}, v_1)$ and $(v_{i(3k+1)}, v_1)$. Here, the degrees of $v_{i(3k)}$ and $v_{i(3k+1)}$ becomes equal to 3. The degrees of all other vertices in the cycle will be equal to 2.

$Fig.1(a), 1(b), 1(c)$ give an example of this construction. The resultant graph after t steps , $G_{t+1} = G$, is a 2-connected graph as there will be atleast two paths between every pair of vertices. maximum degree in G is 4, as we are replacing only the cut vertices of G_1 with cycles in such a way that the degrees of all the vertices in each cycle does not exceed 4. It is also a planar graph as the individual blocks are planar [1] and we are replacing only cut vertices of G_0 with *cycles*. This construction can obviously be done in Polynomial time.

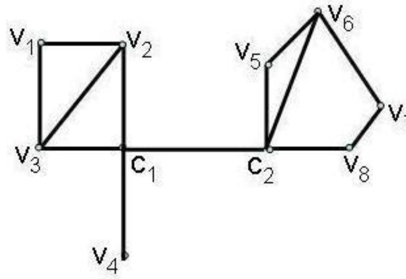

Fig 1(a): Given plane graph G_1

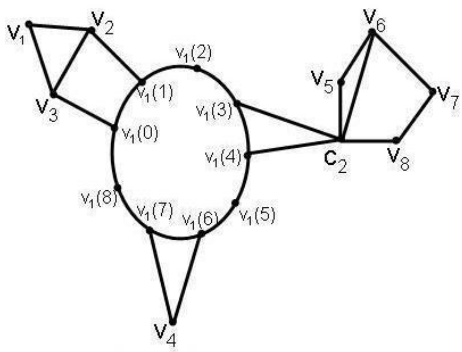

Fig 1(b): G_2 (After first interation)

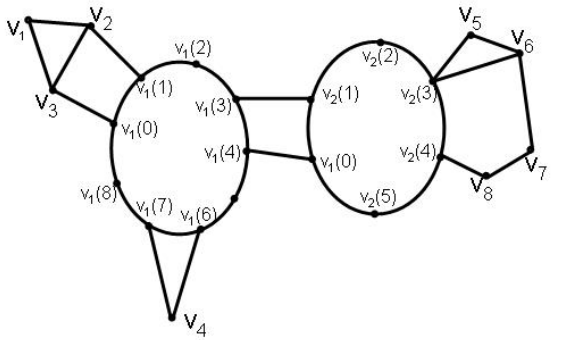

Fig 1(c): G_3 **(After second iteration)**
(2-connected plane graph)

Hereafter, in any cycle of G representing a cut vertex of G_0, let us call the vertices connected to the other vertices of blocks, as *B-type* vertices. All other vertices (in the form $v_{i(3k-1)}$) which have degree 2, will be called as *Connector-type* vertices for further reference.

Let us take an integer $L = K - 2t + 3\sum_{i=1}^{t} d_i$. Now, we show that graph G will have a connected vertex cover of size $c_2 \leq L$ if and only if G_0 has a connected vertex cover of size $c_1 \leq K$.

First assume that G_0 has a subset of vertices V_1 , which is a connected vertex cover of size $c_1 \leq K$. V_1 must contain all the cut vertices of G_0, because the blocks of G_0 are connected only through their common cut vertex. Let us construct a subset V_2 of vertices of G initially starting with $(c_1 - t)$ vertices corresponding to the vertices of the set $(V_1 - C)$ in G_0. From any cycle in G, corresponding to a cut vertex c_i, except $v_{i(2)}$ (a *connector-type* vertex), we add all other $3d_i - 1$ vertices to V_2 and they cover all the edges of the cycle. There are t cycles of this type and the number of vertices added to V_2 will be $(3\sum_{i=1}^{t} d_i) - t$. It can easily be understood that the set V_2 will cover all the edges of G and it is connected. The size of the set V_2, given by c_2, will be $(c1 - t) + (3\sum_{i=1}^{t} d_i) - t$ and we can say that $c_2 \leq K - 2t + (3\sum_{i=1}^{t} d_i)$ because $c_1 \leq K$. So, we proved that G will have a connected vertex cover of size less than or equal to L if G_0 has a connected vertex cover of size less than or equal to K.

Conversely, suppose G has a connected vertex cover V_2 of size $c_2 \leq L$. We have to prove that G_0 will also have a connected vertex cover of size $c_1 \leq K$. First let us consider the cycles in G corresponding to cut vertices in G_0. V_2 should contain $r - 1$ vertices from any of these cycles containing r vertices ie.only one vertex can be absent from these cycles in V_2. If possible , let us assume that two vertices $v_{i(j)}, v_{i(k)}$, in any cycle S_i corresponding to a cut vertex c_i for $1 \leq i \leq t$, are not present in V_2 and also without loss of generality we can assume that $j < k$. If those 2 vertices are adjacent , then the edge between them will not be covered by V_2, so they will not be adjacent. Now let us consider vertices of the cycle S_i as a union of four subsets : $A = \{v_{i(j)}\}$, $B = \{v_{i(k)}\}$, $C = \{v_{i(l)} \,/\, j < l < k\}$ and $D = \{\ v_{i(l)} \,/\, 0 \leq l \leq (3d_i - 1) \wedge v_{i(l)} \notin (A \cup B \cup C)\ \}$. We know that a cut vertex is a common vertex for atleast two blocks in any graph. So, we can assume that the vertices other than the cut vertex in atleast two blocks b_1, b_2, in G_0, are connected to S_i representing the cut vertex c_i. Also let us recollect that the vertices from different blocks in G_0 are connected to each other only through cut vertices and there will not be any other path between them. Now let us consider the cases that can arise.

1. Both $v_{i(j)}, v_{i(k)}$ are of *B-type* :
   As they can not be adjacent, $v_{i(j)}, v_{i(k)}$ will not be connected to the vertices of a *single* block. Without loss of generality, assume that $v_{i(j)}$ is connected to the vertices of a block b_1 and $v_{i(k)}$ to that of b_2. The vertices of b_1, present in V_2, will be connected to the cycle through only one vertex (either $v_{i(j-1)}$ or $v_{i(j+1)}$). In the same way, the vertices of b_2, present in V_2, will be connected to the cycle through only one vertex (either $v_{i(k-1)}$ or $v_{i(k+1)}$). Now , two cases arise.
   (a) Either C or D, consists of only one vertex which is of *connector-type* :
   If either C or D has only one vertex which is of *connector-type*, then it should be in V_2. On either side of this vertex, there will be $v_{i(j)}$ & $v_{i(k)}$ and hence it will not be connected to the vertices of V_2 from b_1 and b_2.
   (b) Atleast two vertices are present in each set $C \,\&\, D$:
   By the way we constructed G, we can say that atleast one block each will be connected to vertices of each set, and the veritices of V_2 from these two blocks are not connected because of the absence of $v_{i(j)}, v_{i(k)}$ in V_2.
2. Both $v_{i(j)}, v_{i(k)}$ are of *connector-type* :
   By the way of construction of G, the vertices of V_2 from atleast one block each will be connected to the vertices of C and D. So, the absence of $v_{i(j)}, v_{i(k)}$ will disconnect V_2.
3. one vertex is of *connector-type*, and another is of *B-type*:
   Atleast one vertex of *B-type*, connected to a block say b_1, will be present in either C or D, say C, as $v_{i(j)}$ and $v_{i(k)}$ are not adjacent. The vertices of V_2 from b_1 are connected to the cycle through this vertex. Atleast one more block will be connected to the other set D, and the vertices of V_2 from that block are not connected to that of b_1 as $v_{i(j)}, v_{i(k)}$ are not present in V_2.

From the above discussion, we can say that any cycle with r vertices , corresponding to the cut vertex in G_0, either $r - 1$ or r vertices should be present in V_2. Even with the absence of one vertex of *connector-type* (in some cases *B-type* vertex can be absent) in V_2, all the edges of the cycle will be covered and the set V_2 will be connected. So for any cycle S_i corresponding to cut vertex c_i in G_0, only $r - 1$ vertices are sufficient to be present in V_2. Let $v_{i(j)}$ in S_i be the single vertex not present in V_2, and if it is not of *connector type*, check whether there is a subgraph K_3, formed by $v_{i(j)}, v_{i(j+1)}, v_k$ or $v_{i(j-1)}, v_{i(j)}, v_k$ and v_k is of degree 2. If yes, v_k must be present in V_2. This implies that v_k is a pendent vertex in G_0. We can replace v_k with $v_{i(j)}$ without affecting the covering. Consider the cycles corresponding to the cut vertices and which are having all the r vertices in V_2. We can take out a *connector-type* vertex, of these cycles, from V_2 without affecting the covering and *reducing* the size of the vertex cover. Now, from each cycle S_i corresponding to c_i for $1 \leq i \leq t$, $d_i - 1$ vertices are present in V_2, implying that $(3 \sum_{i=1}^{t} d_i) - t$ vertices of V_2 will be from the cycles corresponding to cut vertices of G_0. So the number of vertices in V_2, which are outside these cycles will be at the most $K - t$. These $K - t$ vertices will cover the edges in all the blocks , in G_0, probably excepting those incident on cut vertices. If we consider a subset V_1 of V_0 in G_0 containing these $K - t$ vertices along with t cut vertices , it will cover all the edges in G_0 and is connected. The size of V_1 will be at the most K. Hence the proof.

III. Conclusions

As we have already mentioned, The inspiration to prove this result is to next prove that TRA-MLC is NP-complete. A.Gonzalez-Gutierrez & T.F.Gonzalez have proved this problem and many of its varients to be NP-complete [1]. But we are going to work towards a shorter proof and also by using most commonly known graph theory concepts.

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