

Application of the Central-Difference with Half-Sweep Gauss-Seidel Method for Solving First Order Linear Fredholm Integro-Differential Equations

E. Aruchunan and J. Sulaiman

Abstract—The objective of this paper is to analyse the application of the Half-Sweep Gauss-Seidel (HSGS) method by using the Half-sweep approximation equation based on central difference (CD) and repeated trapezoidal (RT) formulas to solve linear fredholm integro-differential equations of first order. The formulation and implementation of the Full-Sweep Gauss-Seidel (FSGS) and Half-Sweep Gauss-Seidel (HSGS) methods are also presented. The HSGS method has been shown to rapid compared to the FSGS methods. Some numerical tests were illustrated to show that the HSGS method is superior to the FSGS method.

Keywords—Integro-differential equations, Linear fredholm equations, Finite difference, Quadrature formulas, Half-Sweep iteration.

I. INTRODUCTION

INTEGRO-DIFFERENTIAL equations (IDEs) arise from many branches of science, for example in control theory and financial mathematics [1]. Especially in physics, it arises naturally such as scattering theory, colloidal dispersions, heat transfer in the presence memory effects, quark dynamic [2], etc. IDE is an equation that the unknown function appears under the sign of integration and it also contains the derivatives of the unknown function. Commonly, it can be classified into Fredholm equations or Volterra equations. The upper bound of the region for integral part of Volterra type is variable, while it is a fixed number for that of Fredholm type. However, in this paper we focus on Fredholm integro-differential. Generally, first-order linear Fredholm integro-differential equations can be defined as follows

$$y'(x) = p(x)y(x) + f(x) + \int_a^b K(x,t)y(t) dt, \quad a \leq x \leq b \quad (1)$$
$$y(a) = y_a$$

where the functions , and the kernel are known and is the solution to be determined. In the engineering field, numerical methods for solution of linear Fredholm integro-differential equations (LFIDEs) have been studied by many authors such as Lagrange interpolation method [3], Tau method [4], quadrature-difference method [5], variational method [6], collocation method [7], homotopy perturbation method [8], Euler-Chebyshev method [9] and GMRES method [10].

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LFIDEs are usually difficult to solve analytically so numerical approaches are practiced to obtain an approximation solution for the problem (1). To solve a LFIDE equation numerically, discretization of differential and integral parts to the solution of system of linear algebraic equations is the basic concept used by researchers to solve LFIDE problems. By considering numerical techniques, there are many schemes that can be used to discretize problem (1) independently for linear differential and integral terms. Many researchers have implemented discretization schemes for linear differential term such as finite difference scheme [11]-[12]), Taylor polynomial scheme [13], Chebyshev polynomial method [14], Runge-Kutta scheme [15] and Euler implicit schemes [16] whilst to discretize linear integral term numerically, many discretization schemes can be used for approximation such as quadrature [17]-[20], projection method [21]-[22]) and least squares [23]. The concept of Half-sweep iterative method was introduced by[24] by the employ of Explicit Decoupled Group (EDG) to solve two-dimensional Poisson equations. Then this concept has been discussed in [25]-[30]. This concept is essential to reduce the computational complexities during the iterative process, whereas the implementation of the half-sweep iterations will only consider nearly half of all node points in a solution domain. In this paper, we carried out the application of the half-sweep iteration technique with Gauss-Seidel (GS) iterative methods by using approximation equation based on finite difference and quadrature schemes for solving problem (1). The standard GS iterative method also called as the Full-Sweep Gauss-Seidel iterative method was implemented with half-sweep iterations process whereas it can be indicated as Half-Sweep Gauss-Seidel (HSGS). The organization of the paper is as follows. In section 2, the formulation of the finite difference and quadrature approximation equations for full- and half-sweep cases will be elaborated. In section 3, formulation of the FSGS and HSGS methods will be demonstrated. In section 4, some numerical results will be illustrated to emphasize effectiveness of the methods. Conclusion is in section 5.

II. FORMULATION OF HALF-SWEEP APPROXIMATION EQUATION

Based on Fig. 1, the full- and half-sweep iterative methods will compute approximate values onto only solid node points until the convergence criterion is reached. It seems that the implementation of the half-sweep iterative method just involves by nearly one-half of whole inner points as shown in Figure 1(b) compared with the full-sweep iterative method.

III. FORMULATION OF THE FULL- AND HALF-SWEEP GAUSS-SEIDEL METHODS

In this paper, FSGS and HSGS iterative methods will be applied to solve linear system generated from the discretization of the problem (1) as shown in Eq. (7). Let matrix M be articulated into

$$M = D - L - U \quad (8)$$

where D , L and U are diagonal, strictly lower triangular and strictly upper triangular matrices respectively. Thus, the general scheme for FSGS and HSGS iterative methods can be written as

$$\tilde{y}^{(k+1)} = (D - L)^{-1} \left(U \tilde{y}^{(k)} + f \right). \quad (9)$$

The iterative methods attempt to find a solution to the system of linear equations by repeatedly solving the linear system using approximations to the vector y . Iterations for

FSGS and HSGS methods continue until the solution is within a predetermined acceptable bound on the error. The general algorithms for FSGS and HSGS iterative methods to solve problem (1) would be generally described in Algorithm 1.

Algorithm: FSGS and HSGS methods

- i) Initializing all the parameters. Set $k = 0$.
- ii) $i = p, 2p, \dots, n - 2p, n - p, n$

Calculate

$$y_i^{(k+1)} = \frac{1}{M_{ii}} \left(f_i - h \sum_{j=p, 2p, \dots, i-p} M_{i,j} y_j^{(k+1)} - h \sum_{j=i+p, i+2p, \dots, n} M_{i,j} y_j^{(k)} \right)$$

- iii) Convergence test

- iv) If the error of tolerance $|y_i^{(k+1)} - y_i^{(k)}| < \epsilon = 10^{-10}$ is

satisfied, the value option at that time is $y_i^{(k+1)}$ and the algorithm end.

- v) Else, set $k = k + 1$ and go to step (ii).

IV. ILLUSTRATIVE EXAMPLES

In this section, 3 numerical examples are illustrated to show the accuracy and effectiveness of the proposed methods and all of them were performed by using C language. Three criteria will be considered in comparison for FSGS and HSGS such as number of iterations, execution time and maximum absolute error.

Example 1 [31]

$$y'(x) = 1 - \frac{1}{3}x + \int_0^1 xty(t) dt \quad 0 \leq x \leq 1$$

with the condition

$$y(0) = 0$$

and exact solution of the problem is

$$y(x) = x.$$

Example 2 [31]

$$y'(x) = xe^x + e^x - x + \int_0^1 y(t) dt \quad 0 \leq x \leq 1$$

with the condition

$$y(0) = 0$$

and exact solution of the problem is

$$y(x) = xe^x.$$

Example 3 [32]

$$y'(x) = \sinh x + \frac{1}{8}(1 - e^{-1})x - \frac{1}{8} \int_0^1 xty(t) dt \quad 0 \leq x \leq 1$$

with the condition

$$y(0) = 1$$

and exact solution of the problem is

$$y(x) = \cosh x.$$

Throughout the experiments, the convergence test considered the tolerance error of $\epsilon = 10^{-10}$. The experiments were carried out in different mesh sizes such as 60, 120, 240, 480 and 960. Results of numerical simulations which were obtained from implementations of the FSGS and HSGS iterative methods for Examples 1, 2 and 3 have been recorded in Tables 1, 2 and 3 respectively.

V. CONCLUSION

In this paper, the HSGS iterative method was employed to solve LFIDE for first-order. Based on numerical results observed in Tables 1, 2 and 3, it manifestly shows that the application of the half-sweep iterative concept significantly reduces computational time (refer table 4) with the tolerable precision. In the other hand, the number of iterations also reduced extensively corresponding to the mesh sizes. In all purpose, HSGS iterative method is faster for the computational works compared to FSGS iterative method. This is due to the computational complexity of the HSGS is reduced approximately 50% compared to FSGS method. In future works this concept can also can be used for high order IDEs problems.

TABLE I

COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND
 MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 1

Methods	Number of iteration				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	33174	107988	375982	1394346	5487814
HSGS+CD+RT	10952	33174	107988	375982	1394346
Methods	Execution time (seconds)				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	512.36	17122.44	60347.03	143653.12	5434556.95
HSGS+CD+RT	47.87	563.54	19656.32	61202.98	153655.84
Methods	Maximum Absolute Error				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	2.623E-5	5.853E-6	3.506E-6	1.359E-7	9.858E-7
HSGS+CD+RT	1.057E-4	2.623E-5	5.853E-6	3.506E-6	1.359E-7

TABLE II

COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND
 MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 2

Methods	Number of iteration				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	43268	137637	459828	1653228	6136092
HSGS+CD+RT	14595	43268	137637	459828	1653228
Methods	Execution time (seconds)				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	421.65	5324.21	55324.20	155159.78	1073214.21
HSGS+CD+RT	20.66	795.54	16845.02	64324.17	579548.36
Methods	Maximum Absolute Error				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	2.9883E-4	6.2354E-4	2.3785E-5	4.3312E-5	1.2032E-6
HSGS+CD+RT	1.2228E-3	2.9883E-4	6.2354E-4	2.3785E-5	4.3312E-5

TABLE III

COMPARISON OF A NUMBER OF ITERATIONS, EXECUTION TIME (SECONDS) AND
 MAXIMUM ABSOLUTE ERROR FOR THE ITERATIVE METHODS FOR EXAMPLE 2

Methods	Number of iteration				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	27766	92736	331899	1229548	5988642
HSGS+CD+RT	8737	27766	92736	331899	1229548
Methods	Execution time (seconds)				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	256.65	4651.23	48898.78	145694.01	1002365.64
HSGS+CD+RT	11.86	257.84	4856.35	49584.13	149653.47
Methods	Maximum Absolute Error				
	Mesh size				
	60	120	240	480	960
FSGS+CD+RT	2.572E-5	3.265E-6	4.397E-6	6.320E-7	1.254E-8
HSGS+CD+RT	1.335E-4	2.572E-5	3.265E-6	4.397E-6	6.320E-7

TABLE IV

PERCENTAGES OF REDUCTION FOR EXECUTION TIME FOR HSGS ITERATIVE
 METHODS COMPARED WITH FSGS METHOD

Methods	HSGS+CD+RT
	Execution time
Example 1	57.39%-97.17%
Example 2	45.99%-95.10%
Example 3	65.96%-95.37%

REFERENCES

- [1] E. Yusufoglu, "Numerical solving initial value problem for Fredholm type linear integro-differential equation system", *Journal of the Franklin Institute*, 346, 2009, pp. 636-649.
- [2] H. Jorquera, "Simple algorithm for solving linear integrodifferential equations with variable limits", *Computer physics communications*, 86, 1995, 91-96.
- [3] M. T. Rashed, "Lagrange interpolation to compute the numerical solutions differential and integro-differential equations", *Applied Mathematics and Computation*, 151, 2003, pp. 869-878.
- [4] S.M. Hosseini and S. Shahmorad. "Tau numerical solution of Fredholm integro-differential equations with arbitrary polynomial bases". *Applied Math. Model.*, 27, 2003, pp. 145-154.
- [5] A. I. Fedotov, "Quadrature-difference methods for solving linear and nonlinear singular integro-differential equations", *Nonlinear Analysis*, 71, 12, 2009, pp. e303.
- [6] N.H. Sweilam. "Fourth order integro-differential equations using variational iteration method". *Comput. Math. Appl.*, 54, 2007, pp.1086-1091,
- [7] M. Aguilar and H. Brunner. "Collocation methods for second-order Volterra integro-differential equations". *Applied Numer. Math.*, 4, 1988, pp. 455-470.
- [8] A. Yildirim, "Solution of BVPs for fourth-order integro differential equations by using homotopy perturbation method". *Comput. Math. Appl.*, 32, 2009, pp.1711-1716.
- [9] P.J. Van der Houwen, and B.P. Sommeijer, "Euler-Chebyshe methods for integro-differential equations". *Applied Numer. Math.*, 24: 1997, pp.203-218.
- [10] E. Aruchunan and J. Sulaiman, "Numerical Solution of Second-Order Linear Fredholm Integro-Differential Equation Using Generalized Minimal Residual (GMRES) Method". *American Journal of Applied Sciences, Science Publication*, 7 (6): 2010, pp.780-783.
- [11] K. Styś and T. Styś, "A higher-order finite difference method for solving a system of integro-differential equations". *Journal of Computational and Applied Mathematics*, 126, 2000, pp. 33-46.
- [12] K. S. Jacob, "A Zero-Stable Optimal Order Method for Direct Solution of Second Order Differential Equations", *Journal of Mathematics and Statistics*, 6 (3), 2010, pp.367-371.
- [13] M. Sezer, "A method for the approximate solution of the second order linear differential equations in terms of Taylor polynomials", *Int. J. Math. Educ. Sci. Technol.*, 27(6), 1996, pp. 821-834.
- [14] M. Sezer and M. Kaynak, "Chebyshev polynomial solutions of linear differential equations", *Int. J.Math. Educ. Sci. Technol.* 27(4), 1996, pp.607-618.
- [15] R. Alexander, "Diagonally implicit Runge-Kutta methods for stiff ODES", *SIAM J. Numer. Anal.* 14, 1977, pp. 1006-1021.
- [16] A. Rathinasamy and K. Balachandran, "Mean square stability of semi-implicit Euler method for linear stochastic differential equations with multiple delays and Markovian switching". *Appl Math Comput* 2008,206, pp.968-979.
- [17] C.T.H. Baker, "*The Numerical Treatment of Integral Equations*", Clarendon Press Oxford, 1977.
- [18] A.D. Polyanin and A.V. Manzhirov, "*Handbook of integral equations*", CRC Press LCC, 1998.
- [19] M.A. Abdou, "Fredholm-Volterra integral equation with singular kernel." *Applied Mathematics and Computation* 137, 2003, pp. 231-243.
- [20] D.P. Laurie, "Computation of Gauss-type quadrature formulas". *Journal of Computational and Applied Mathematics*, 127, 2001, pp. 201-217.
- [21] K. Maleknejad and M.T. Kajani, "Solving second kind integral equations by Galerkin methods with hybrid Legendre and Block-Pulse functions". *Appl. Math. Comput.*, 145, 2003, ppt 623-629.
- [22] S.O.Oladejo, T.A. Mojeed, and K.A. Olurode, "The application of cubic spline collocation to the solution of integral equations". *Journal of Applied Sciences Research*, 4(6), 2008, pp.748-753.
- [23] S.A. Ashour, "Numerical solution of integral equations with finite part integrals," *Internat. J. Math. & Math. Sci.* 22 (1), 1999, pp.155-160.
- [24] A. R. Abdullah, "The four point Explicit Decoupled Group (EDG) method: A fast Poisson solver". *International Journal of Computer Mathematics* 38, 1991, pp. 61-70.
- [25] J. Sulaiman, M.K. Hasan and M. Othman, "Red-Black Half-Sweep iterative method using triangle finite element approximation for 2D Poisson equations." In Y. Shi et al. (Eds.), *Computational Science, Lecture Notes in Computer Science (LNCS 4487)*, 2007, pp. 326-333. Springer-Verlag, Berlin.
- [26] E. Aruchunan and J. Sulaiman, "Half-sweep Conjugate Gradient Method for Solving First Order Linear Fredholm Integro-differential Equations." *Australian Journal of Basic and Applied Sciences*, 5(3), 2011, pp.38-43.
- [27] J. Sulaiman, M.K. Hasan and M. Othman, *The Half-Sweep Iterative Alternating Decomposition Explicit (HSIADE) method for diffusion equation*. In J. Zhang, J.-H. He and Y. Fu (Eds.), *Computational and Information Science, Lecture Notes in Computer Science (LNCS 3314)*: 2004, pp.57-63. Springer-Verlag, Berlin.
- [28] J. Sulaiman, M. Othman and M.K. Hasan. *Half-Sweep Algebraic Multigrid (HSAMG) method applied to diffusion equations*. In *Modeling, Simulation and Optimization of Complex Processes: 2008*, pp. 547-556. Springer-Verlag, Berlin.
- [29] M.S. Muthuvalu, and J. Sulaiman, "Half-Sweep Arithmetic Mean method with high-order Newton-Cotes quadrature schemes to solve linear second kind Fredholm equations". *Journal of Fundamental Sciences* 5(1), 2009, pp. 7-16.
- [30] M.S. Muthuvalu and J. Sulaiman, "Half-Sweep Arithmetic mean method with composite trapezoidal scheme for solving linear Fredholm integral equation." *Applied Mathematics and Computation*. 217 (2011) 5442-5448.
- [31] P. Darania and A. Ebadia. "A method for numerical Solution of tintegro-differetial equations, *Applied Mathematics and Computation*, 188, 2007, pp. 657-668.
- [32] B. Raftari, A. Ahmadi, and H. Adibi, "The Use of Finite Difference Method, Homotopy Perturbation Method and Variational Iteration Method for a Special Type of Linear Fredholm Integro-differential Equations", *Australian Journal of Basic and Applied Sciences*, 4(6): 2010, pp.1221-1239,.

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