

Performance of Heterogeneous Autoregressive Models of Realized Volatility: Evidence from U.S. Stock Market

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Abstract—This paper deals with heterogeneous autoregressive models of realized volatility (HAR-RV models) on high-frequency data of stock indices in the USA. Its aim is to capture the behavior of three groups of market participants trading on a daily, weekly and monthly basis and assess their role in predicting the daily realized volatility. The benefits of this work lies mainly in the application of heterogeneous autoregressive models of realized volatility on stock indices in the USA with a special aim to analyze an impact of the global financial crisis on applied models forecasting performance. We use three data sets, the first one from the period before the global financial crisis occurred in the years 2006-2007, the second one from the period when the global financial crisis fully hit the U.S. financial market in 2008-2009 years, and the last period was defined over 2010-2011 years. The model output indicates that estimated realized volatility in the market is very much determined by daily traders and in some cases excludes the impact of those market participants who trade on monthly basis.

Keywords—Global financial crisis, heterogeneous autoregressive model, in-sample forecast, realized volatility, U.S. stock market.

I. INTRODUCTION

VOLATILITY modeling was born 30 years ago and it is still and will remain one of the most active research topics of financial econometrics. The development of econometric models of volatility has gone along with their application in academia and progressive use in the financial industry.

Many recent investigations have considered volatility as an unobservable variable and therefore used a fully specified conditional mean and conditional variance model to estimate and analyze latent volatility. This approach led to all kinds of ARCH class and stochastic volatility models. One can see some weaknesses of that kind of models. The estimation procedure is often not trivial especially in stochastic volatility models and they are not able to replicate empirical features of financial data as mentioned in [13].

However, with computer sciences development, it has become possible to collect and store a large volume of financial data, which can be recorded at various time intervals.

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The smallest unit is a tick. In liquid markets, thousands of ticks are generated every trading day. The use of high frequency data thus gives us new scope for modeling as we are not restricted to one observation per day only.

In order to simplify volatility estimation there would be used some observable proxy for the latent volatility. Therefore, there was suggested model of an observable volatility with the use of high-frequency data in [1]. It was called realized volatility. Within the context of Heterogeneous Market Hypothesis suggested in [15] and newly defined realized volatility, there was defined a new model of three volatility components, each created by different type of market participants, see [9].

This model was called Heterogeneous Autoregressive model of Realized Volatility (HAR-RV). This is a simple AR-type model, which can capture volatility persistence and therefore induced many succeeding extensions.

First, it has been investigated behavior of residuals estimated by HAR-RV models and concludes that residuals exhibit non-Gaussianity and volatility clustering, see [10] for details. It was therefore proposed to specify a GARCH process to account for volatility clustering in the squared residuals.

Subsequently, the discussion turns intentionally to the presence of potential jumps and their impact on predicting future realized volatility. Afterwards, a lot of studies on the role of jumps arise, for details see [2], [4], [5] and [11] for a review. An overall list of references can be found in a literature survey in [8].

In times of low market volatility it is relatively straight forward to measure volatility and understand volatility dynamics. At other times, financial markets are affected by severe disruptions which may be largely isolated events like the market crash of 1987 or the global financial crisis in 2008-2009 years. During such periods, apparent spikes in volatility and large movements in asset prices complicate estimation of volatility and volatility dynamics. These crises dramatically influenced the market volatility and diversification opportunities for investors.

This paper applies newly developed HAR-RV models on high frequency data of the U.S. stock market represented by one of the main U.S. stock index S&P 500 in years 2006-2012, thus also focusing on period of financial turmoil. It tries to capture behavior of three different market agents in time of high uncertainty. Moreover, financial crisis is characterized by many unexpected jumps in volatility; therefore forecasting

performance of the model estimated also remains at the forefront of our concerns.

II. THEORETICAL BACKGROUND

The aim of this chapter is to introduce Heterogeneous Market Hypothesis which is a key to understand a structure of HAR-RV models. Subsequently, quadratic variation theory and realized volatility measurement will be explained.

A. Heterogeneous Market Hypothesis

Heterogeneous Market Hypothesis presented by [15] recognizes the presence of heterogeneity of the traders. This specific view on financial markets can also be related with the Fractal Market Hypothesis of [16] and the Interacting Agent View of [14]. This view on the multi-component structure stems from the heterogeneous nature of the information arrivals rather than from the heterogeneity of the agents.

According to the Heterogeneous Market Hypothesis one can explain the empirical observation of a strong positive correlation between volatility and market presence. In heterogeneous markets, different actors are likely to settle for different prices and decide to execute their transactions in different market situations, hence they create volatility. The heterogeneity of the agents may be caused by various reasons: differences in degree of information, prior belief, temporal horizons, geographical location, institutional constraints, and risk profile and so on. Thus, each such participant has different reaction times to news, related to its time horizon and characteristic dealing frequency.

In this paper we focus on the heterogeneity which originates from the difference in investment time horizons. Financial market is usually composed by participants having a large spectrum of dealing frequency. The basic idea of Heterogeneous Market Hypothesis is that participants with different time horizons perceive, react and cause different types of volatility components. All these agents have different reaction paths and thus create different types of volatility. In simply way it can be explained as follows, we can define three primary volatility components: the short-term with daily or higher dealing frequency, the medium-term typically made of portfolio manager who rebalance their positions weekly, and the long-term with one or more months dealing frequency.

B. Realized Volatility Measurement

Quadratic variation is one type of variation which is frequently used in analysis of stochastic processes. The mathematics behind the quadratic variation is based on the research work of [6]. The quadratic variation theory comes from [1], there was build the concept of realized volatility in this theory. In their study they summarize the theory and explain the links to realized volatility modeling, thus creating a standard setting in this area.

In literature, three different computations of realized volatility exist. The first one appears in [9]:

$$RV_t^d = \left(\frac{1}{M} \sum_{j=0}^{M-1} r_{t-j\delta}^2 \right)^{\frac{1}{2}}, \quad (1)$$

where $\delta = 1d/M$ is the number of observations during one day and $r_{t-j\delta} = p(t-j\delta) - p(t-(j+1)\delta)$ defines the intra-day return of the price process for the sampling frequency δ . Under these conditions, the realized volatility becomes an unbiased volatility estimator. Notice that the definition of realized volatility involves two time parameters: first, the sampling frequency δ , which depends on the format of data used, and second, the aggregation period $1d$.

The second type of realized volatility is used in [8]:

$$RV_t^d = \left(\sum_{j=0}^{M-1} r_{t-j\delta}^2 \right)^{\frac{1}{2}}. \quad (2)$$

In our paper we also use the last version of realized volatility defined in [10]:

$$RV_t^d = \left(\sum_{j=0}^{M-1} r_{t-j\delta}^2 \right), \quad (3)$$

where r means logarithmic return of the variable close after overnight cleaning.

It is also relevant to explain the connection between quadratic variation theory and realized volatility models. First, the theory suggests that realized volatility can be estimated in probability for time intervals approaching zero. Second, it means that realized volatility is the major factor in determining conditional return covariance and finally, under the condition of purely continuous processes, suggests that returns are approximately normally distributed with integrated volatility having the highest impact on the shape of the distribution.

III. MODEL FORMULATION

This chapter deals with heterogeneous autoregressive models of realized volatility called HAR-RV. The first version of model was proposed by [9] and inspired by Heterogeneous Market Hypothesis suggested by [15]. Some extensions of the HAR-RV model were proposed in [8], [10], [11] and [2].

However, the technical derivation of the HAR-RV model is based on quadratic variation theory, which suggests a way of approximation of quadratic variation called realized volatility.

A. HAR-RV Model

The initial HAR-RV model suggested in [8] considers the following stochastic volatility process:

$$dp(t) = \mu(t)dt + \sigma(t)dW(t), \quad (4)$$

where $p(t)$ is a logarithmic price process, $\mu(t)$ is continuous finite variation process, $dW(t)$ is Brownian motion and $\sigma(t)$ is stochastic process independent of $dW(t)$. Integrated volatility related to day t as integral of the stochastic volatility process over a whole day $1d$:

$$\sigma_t^d = \left(\int_{t-1d}^t \sigma^2(\omega) d\omega \right)^{\frac{1}{2}}. \quad (5)$$

The theory behind (5) was first formulated in [3] and later generalized for a class of finite semi-martingales in [1]. It means that the integrated volatility of Brownian motion can be approximated by a sum of intra-day squared returns, therefore allows us to build up an error free estimate of the current volatility, i.e. realized volatility, as defined in (1), (2) or (3).

In the following we will also consider latent integrated volatility and realized volatility viewed over different time horizons longer than one day. These multi-period volatilities will simply be normalized sums of the one-period volatilities. For instance, in our notation, a weekly realized volatility at time t will be given by:

$$RV_t^w = \frac{1}{5} (RV_{t-1d}^d + RV_{t-2d}^d + RV_{t-3d}^d + RV_{t-4d}^d + RV_{t-5d}^d), \quad (6)$$

where $5d$ indicate a time interval of one week which takes the last 5 working days into consideration. Monthly volatility estimator is defined in the same way using the last 22 observations. According (6), daily volatility was defined and is used as input variable into weekly and monthly volatility estimators. In particular we will thus make use of weekly and monthly integrated and realized volatility.

To model an unobserved partial volatility process $\tilde{\sigma}_t^{(i)}$ at each level of time scale or the cascade, it is assumed to be a function of the past realized volatility experienced at the same time scale and the expectation of the next period values of the longer term partial volatilities.

Such cascade model can be simply defined as:

$$\sigma_{t+1wd}^d = \beta_0 + \beta^d RV_t^d + \beta^w RV_t^w + \beta^m RV_t^m + \tilde{\omega}_{t+1d}^d, \quad (7)$$

where RV_t^d , RV_t^w , and RV_t^m are respectively daily, weekly and monthly observed realized volatilities, while $\tilde{\omega}_{t+1d}^d$ means volatility innovation, which is contemporaneously and serially independent.

In addition, a presumption of the daily volatility is equal to the realized daily volatility plus an innovation term is added:

$$\sigma_{t+1wd}^d = RV_{t+1d}^d + \omega_{t+1d}^d. \quad (8)$$

The innovation term ω_{t+1d}^d is included in (8) the equation because in fact not all assumptions are achieved and a measurement errors can occur mainly due to properties of empirical data. By substituting (8) into (7), we get:

$$RV_{t+1d}^d = \beta_0 + \beta^d RV_t^d + \beta^w RV_t^w + \beta^m RV_t^m + \omega_{t+1d}^d, \quad (9)$$

where $\omega_{t+1d}^d = \tilde{\omega}_{t+1d}^d - \omega_{t+1d}^d$ is considered as a disturbance term in the regression.

To summarize initial HAR-RV model, (9) represents an AR-type model of realized volatility whose inputs present different market trading agents and which can be estimated using historical data from stock markets.

B. HAR-RV-GARCH Model

According to [8] it is commonly assumed that the residuals are Gaussian and independently identically distributed. However, volatility clustering in the residuals of the HAR-RV model is often observed in practical applications. In other words, the empirical results point to the *iid* and Gaussianity violation and also confirm volatility clustering. The presence of time-varying conditional distribution in realized volatility models.

To account for the observed volatility clustering in realized volatility, in [10] it was extended the HAR-RV model by explicitly modeling the volatility of realized volatility. Their work presents a new version of HAR-RV model in which they give residuals more flexibility through more flexible normal inverse Gaussian distribution.

This new version of model tries to account for these properties that not only distort the HAR-RV model estimation, but also diminish its forecasting accuracy. They also specify a GARCH process to account for volatility clustering in the squared residuals.

In accordance with literature dealing with HAR-RV models, see for instance [10], we can suggest to perform a GARCH-LM test to justify the following extension of the HAR-RV model by the GARCH (p, q) component:

$$RV_{t+1d}^d = \beta_0 + \beta^d RV_t^d + \beta^w RV_t^w + \beta^m RV_t^m + h_t u_t, \quad (10)$$

$$h_t^2 = \varpi + \sum_{i=1}^p \alpha_i h_{t-i}^2 + \sum_{j=1}^q \beta_j u_{t-j}^2, \quad (11)$$

$$u_t | \Omega_{t-1} \approx (0, 1), \quad (12)$$

where Ω_{t-1} denotes σ -field generated by all the information available up to time $t-1$ and $u_t = \sqrt{h_t} \varepsilon_t$. In this version of HAR-RV model, the error term $h_t u_t$ follows a conditional density with time varying variance. This model is estimated under the name HAR-RV-GARCH.

In [10] there were tested the performance of four HAR-RV family models: the standard HAR-RV model, its extension for

GARCH component, another extension for residuals distribution and finally both extensions together. The authors found out that using HAR-RV-GARCH model the prediction ability improves in every case.

IV. DATA

In this chapter the data used for empirical analysis will be described, construction of variables explained, and their empirical properties and features depicted. In accordance with the theory infinitely small time intervals should ensure a consistent and unbiased estimator of daily integrated volatility. Nevertheless, empirical data often differ largely from the theoretical arbitrage free continuous time process especially due to the presence of market microstructure effects which is present in every typical stock market.

According to [9] one can suggest that under small time intervals, the unbiasedness of volatility estimator is induced by a systematic error. For instance, for a FX market this error is positive and ranges from 30% at 1-minute intervals to 80% for tick-by-tick data and depends on liquidity of the currency. On the other hand, the shorter time intervals mean the higher number of observations per day, and therefore more effective the stochastic error of measurement.

A. S&P 500 Characterization

We estimate the HAR-RV models on U.S. market index S&P 500. The S&P 500, or the Standard & Poor 500, is a stock market index based on the common stock prices of 500 top publicly traded American companies. It differs from other stock market indices like the Dow Jones Industrial Average and the Nasdaq Composite because it tracks a different number of stocks and weights the stocks differently. It is one of the most commonly followed indices and many consider it the best representation of the market and a bellwether for the U.S. economy. It is a free-float capitalization-weighted index.

The index is maintained by Standard & Poor's; the components of the S&P 500 are selected by committee. This is similar to the Dow 30, but different from others such as the Russell 1000, which are strictly rules-based.

The index has traditionally been market-value weighted; that is, movements in the prices of stocks with higher market capitalizations have a greater effect on the index than companies with smaller market caps. Standard & Poor's now calculates the market caps relevant to the index using only the number of shares available for public trading.

In order to keep the S&P 500 Index comparable across time, the index needs to take into account corporate actions such as stock splits, share issuance, dividends and restructuring events such as mergers or spinoffs. Additionally, in order to keep the Index reflective of American stocks, the constituent stocks need to be changed from time to time. The index is updated every 15 seconds during trading sessions.

To prevent the value of the Index from changing merely as a result of corporate financial actions, all such actions affecting the market value of the Index require a Divisor adjustment. Also, when a company is dropped and replaced by another with a different market capitalization, the divisor

needs to be adjusted in such a way that the value of the S&P 500 Index remains constant.

B. S&P 500 Empirical Features Description

S&P 500 series cover a period from January 1, 2006 till December 30, 2011, thus focusing also on the period of the global financial crisis. In total, there are 1565 trading days, respectively. In addition, for every trading day we have the information on its open and close price as well as the highest and the lowest, recorded at 1-minute frequency.

The graphs below show index price or values and corresponding returns of the S&P 500 covering the whole period. It can be seen a period of high price and return volatility. Also, volatility clustering is apparent.

As it has been empirically confirmed, stock markets may face such instability sometime, for instance see [12] or [17]. Following the spread of bad news about U.S financial crisis the U.S. equity markets have seen a more than 50 per cent decline of the S&P 500 index, see Fig. 1. This happened primarily due to the withdrawal by portfolio investors between September and December 2008 and its psychological impact on other investors.

It can be seen that from Fig. 2 that return fluctuates around mean value that is close to zero. Volatility is low for certain time periods and high for other periods. The movements are in the positive and negative territory and larger fluctuations tend to cluster together separated by periods of relative calm. Thus Fig. 2 show volatility clustering where large returns tend to be followed by small returns leading to continuous periods of volatility and stability. Volatility clustering implies a strong autocorrelation in squared return.

Since the volatility was highest in 2008 when the value of S&P 500 reached the minimum in investigated period, we divided the basic period 2006-2011 into three testing period. First, pre-crisis period, was defined from 2006 to the end of 2007, the second, crisis period, started at the beginning of 2008 and finished by the end of 2009; the last, post-crisis period, was defined from 2010 to the end of 2011.

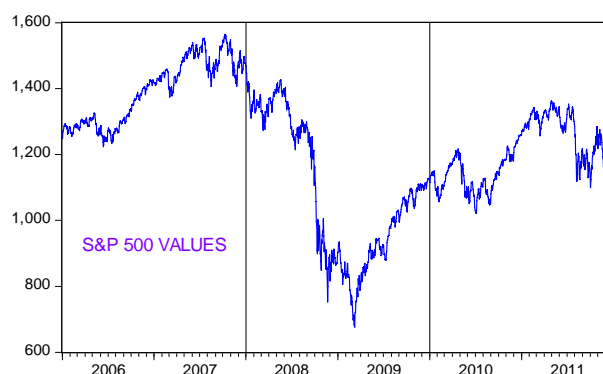


Fig. 1 S&P 500 corresponding values (2006-2011)

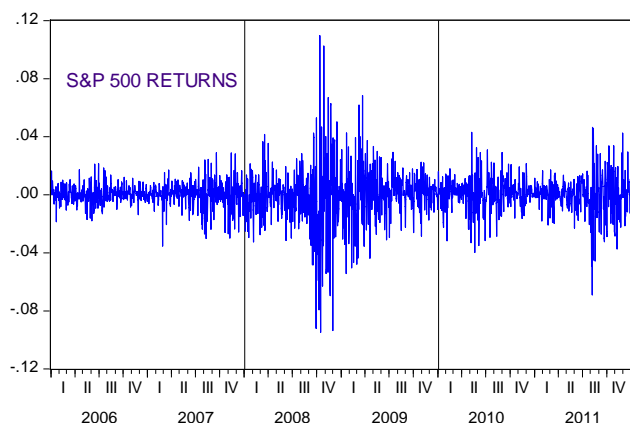


Fig. 2 S&P 500 corresponding returns (2006-2011)

C. Construction of Variables

The data were not subject to cleaning, as index price represents the whole market and it is comprised of the most liquid stocks. Therefore, it should not exhibit any sharp jumps or periods of low trading. The variable realized volatility was constructed in two ways. First, we start from (3) however with standard definition of returns as defined in (1). Moreover, we use its logarithmic form of returns using exactly definition according (3) as described in [7]. For every time point recorded, logarithmic return was squared and then summed over one day. Variables were annualized using the number of observations in 2006.

The descriptive statistics of realized volatility for S&P 500 can be seen in Table I. Standard form of realized volatility as well as its logarithmic form are not normally distributed, with the logarithmic version being closer to it.

TABLE I
 DESCRIPTIVE STATISTICS OF REALIZED VOLATILITY FOR S&P 500

	Pre-crisis period	Crisis period	Post-crisis period
<i>RV</i>			
Mean	2,798	13,819	5,872
Median	2,571	13,276	5,714
St. Dev.	3,541	22,814	5,177
Kurtosis	61,730	297,638	156,519
<i>J-B</i> stat.	11745,862	31764,797	16871,083
Prob.	0,000	0,000	0,000
<i>logRV</i>			
Mean	0,552	1,837	1,284
Median	0,485	1,696	1,076
St. Dev.	0,796	1,582	1,164
Kurtosis	4,873	9,595	6,481
<i>J-B</i> stat.	18,83	95,53	35,38
Prob.	0,000	0,000	0,000

V. MODEL ESTIMATION

Before processing further, we find it important to summarize shortly some results obtained in previous studies. First, in [8] it was estimated the initial HAR-RV model in its standard form with all parameters being significant and the daily coefficient having the highest value and monthly coefficient the lowest. Subsequently, in [10] it was estimated HAR-RV-GARCH model with a GARCH (1, 1) component. The authors found out that all parameters are highly significant with the highest coefficient at weekly component, followed by daily and monthly ones.

Following the recent literature on the realized volatility, we estimated the following two models: HAR-RV as defined by (9) and HAR-RV-GARCH represented by (10) - (12). The HAR-RV-GARCH model was estimated with its GARCH (1, 1) component which seemed to be the most appropriate, as indicated by LM test. Also, the use of GARCH (1, 1) is suggested in [10]. HAR-RV models were estimated by ordinary least squares whereas in case of HAR-RV-GARCH models the maximum likelihood method was used.

As mentioned in the previous section, realized volatility was constructed in standard and logarithmic form as well. All HAR-RV models were estimated on the in-sample data and the results were applied on the in-sample period to compare the forecasting accuracy of the models.

According [10], the HAR-RV models can be classified as models with generated regressor. It usually leads to the covariance matrix of the disturbance term to be non-spherical, with both non-zero off-diagonal and non-constant diagonal elements. These kinds of models generally show heteroscedasticity and serial correlation in disturbance components. To overcome this problem, we use Newey-West correction which should assure that the standard errors, *t*-statistics, and respective *p*-values are correctly estimated.

Selected results of the estimation of the HAR-RV models are reported in Table II. The numbers in parentheses are the standard errors, parameters significant at 5% are marked by symbol *, *SC* means Schwarz information criterion.

TABLE II
 REGRESSION RESULTS FOR S&P 500

	Pre-crisis period		Crisis period		Post-crisis period	
	HAR-RV	HAR-RV GARCH	HAR-RV	HAR-RV GARCH	HAR-RV	HAR-RV GARCH
<i>RV</i>						
β_0	0,109*	0,0781*	1,779*	0,754*	0,484*	0,756*
	(0,022)	(0,013)	(0,085)	(0,113)	(0,092)	(0,083)
β^d	0,427*	0,268*	0,821*	0,678*	0,619*	0,510*
	(0,116)	(0,053)	(0,183)	(0,098)	(0,109)	(0,040)
β^w	0,315*	0,416*	0,202*	0,381*	0,411*	0,494*
	(0,089)	(0,027)	(0,071)	(0,087)	(0,037)	(0,091)
β^m	0,121*	0,209*	0,082	-0,106	0,230	0,174*
	(0,097)	(0,008)	(0,028)	(0,005)	(0,063)	(0,027)

R^2	0,482	0,469	0,279	0,265	0,602	0,519
SC	5,827	5,951	4,419	4,893	5,081	5,448
<i>logRV</i>						
β_0	-0,029*	-0,032*	0,840*	0,791*	0,736*	0,573*
	(0,009)	(0,008)	(0,132)	(0,126)	(0,112)	(0,145)
β^d	0,353*	0,369*	0,592*	0,558*	0,297*	0,304*
	(0,064)	(0,073)	(0,076)	(0,085)	(0,078)	(0,064)
β^w	0,553*	0,561*	0,225*	0,302*	0,228*	0,301*
	(0,104)	(0,116)	(0,098)	(0,086)	(0,073)	(0,043)
β^m	0,197*	0,244*	0,251	0,243*	0,238*	0,205*
	(0,124)	(0,126)	(0,052)	(0,015)	(0,117)	(0,022)
R^2	0,679	0,673	0,334	0,327	0,612	0,602
SC	2,852	3,024	3,667	3,581	6,867	6,771

The standard HAR-RV models suggest a significant impact of last day volatility on today's volatility which can be quantified in a range from 27% to 82%. Significance of monthly component is not confirmed in all the HAR-RV type of models, which on the other hand, regard monthly term as important in crisis period. It seems that information criterions favor HAR-RV-GARCH type of model, even though it has lower R^2 than the values obtained by HAR-RV model.

Results of the standard HAR-RV estimation suggest that the major determinant of volatility one day ahead is the present daily volatility value. Its impact is significantly higher in the global financial crisis period compared to non-crisis periods. Moreover, weekly volatility is important for all types of models even in crisis period and the relevance of volatility computed on monthly basis is rejected in selected regressions in the crisis period only.

Comparing the two versions of realized volatility measurement, the logarithmic form performs very similarly and exhibit a lot better patterns than the standard version. By closer inspection of criterion values, one may favor the logarithmic version more as it has most of the criteria significantly lower compared standard form. It is also important to note that in the standard form model, residuals still include some linear and nonlinear relationships, while in the logarithmic form only non-linear patterns are present.

To sum up, it seems that investigation of in-sample criteria provide quite clear implications. However, despite the pros and cons of both approaches, the final model would have been selected in the out-of-sample performance investigation.

VI. MODEL PERFORMANCE

The estimated models performance was evaluated on the basis of the following criteria: Root Mean Square Error (RMSE), Mean Absolute Error (MAE), Mean Absolute Percentage Error (MAPE), and Theil's Inequality coefficient (U). The criteria for assessing the quality of prediction are defined as follows:

$$RMSE = \sqrt{\frac{1}{T} \sum_{i=1}^T (\hat{\sigma}_i - \sigma_i)^2}, \quad (13)$$

$$MAE = \frac{1}{T} \sum_{i=1}^T |\hat{\sigma}_i - \sigma_i|, \quad (14)$$

$$MAPE = \frac{1}{T} \sum_{i=1}^T \frac{|\hat{\sigma}_i - \sigma_i|}{\sigma_i}, \quad (15)$$

$$Theil's\ U = \frac{\sqrt{\frac{1}{T} \sum_{i=1}^T (\sigma_i - \hat{\sigma}_i)^2}}{\sqrt{\frac{1}{T} \sum_{i=1}^T \hat{\sigma}_i^2} + \sqrt{\frac{1}{T} \sum_{i=1}^T \sigma_i^2}}, \quad (16)$$

where σ_i denotes the actual value of volatility and $\hat{\sigma}_i$ means a value estimated by HAR-RV family models for T predicted periods.

RMSE penalizes larger errors more than MAE; therefore RMSE tends to be higher than MAE. In other words, the greater difference between them means the greater the variance in the individual errors in the sample. MAPE is appropriate for comparison of accuracy among models as it is measure free. Theil's Inequality Coefficient also measures a degree to which one time series differs from another. It is particularly useful for comparison of various models. The value ranges from 0 to 1 and the lowest means the better quality. Values close to 1 indicate that naive forecast is of similar accuracy. According to [11], we focus mostly on RMSE and Theil's Inequality Coefficient.

In the in-sample model comparison, which is reported in Table III, HAR-RV-GARCH is the leading model and is more accurate in the standard version of RV as well as its logarithmic form as it scores best in all key criteria. Results we achieved are in harmony with [10].

TABLE III
 IN SAMPLE PERFORMANCE OF HAR-RV MODELS FOR S&P 500

	Pre-crisis period		Crisis period		Post-crisis period	
	HAR-RV	HAR-RV-GARCH	HAR-RV	HAR-RV-GARCH	HAR-RV	HAR-RV-GARCH
<i>RV</i>						
RMSE	1,523	1,520	2,732	2,492	1,672	1,654
MAE	0,825	0,752	1,209	1,204	0,938	0,936
MAPE	0,427	0,364	0,801	0,799	0,470	0,428
Theil's x100	22,720	22,382	38,662	38,014	26,671	25,692
<i>logRV</i>						
RMSE	0,302	0,301	0,475	0,471	0,338	0,328
MAE	0,188	0,184	0,285	0,284	0,197	0,193
MAPE	0,847	0,819	1,451	1,402	0,977	0,951
Theil's x 100	7,812	7,209	11,830	11,409	8,072	7,619

Moreover, our analysis clearly shows that logarithmic HAR-RV models clearly outperform the standard HAR-RV models. However, despite the more than satisfactory results, we should keep in mind that there still exists a non-linear relationship among residuals that cannot be rejected.

Turning to the comparison of investigated data period, results of our calculations clearly show that the quality of models performance was the highest in the period before the global financial crisis occurred in 2006-2007 years followed by post-crisis period defined over 2010-2011 years. Lowest quality results were achieved during the crisis period in 2008-2009 years as expected before analysis.

It would have been also interesting to compare HAR models used in our paper with their extensions like HAR-RV-J models since the practical properties of high frequency data often indicate jumps in volatility thus breaking the assumption the price process exhibits only continuous sample path.

What is surprising in general, the ability of the HAR-RV model to achieve good results with only a few parameters even in crisis period. However, we are aware that actual forecasting performance of the models estimated means to compare the models on the basis of truly out of sample forecast. That would have been a next step of our future analysis.

VII. CONCLUSION

By projecting a dynamic process on its past values aggregated over different time horizons, the HAR-RV family models are a general and flexible approach to fit the autocorrelation function of any persistent process in a very simple and tractable way. In this paper, we have surveyed the nature, construction, and properties of the HAR class models for realized volatility estimation and prediction.

The HAR-RV type models seem to successfully achieves the purpose of modeling, the long memory behavior of volatility in a very simple and parsimoniously way. In spite of the simplicity of its structure and estimation, the HAR-RV model shows remarkably good in-sample forecasting performance.

The paper deals with the topic of daily realized volatility of S&P 500 stock market index in particular. It first describes the Heterogeneous Market Hypothesis and realized volatility measurement which is a headstone of HAR-RV models and then turns to the application on high frequency data in the period 2006-2011 years. We tried to explain behavior of three groups of market agents and to quantify differences in in-sample forecasting performance of HAR-RV models in different periods. The investigation is conducted on 2008 and 2009 data set, thus covering also the period of the global financial crisis.

The empirical analysis uncovers some features that are common for the U.S. stock market indices. Regressions show that the most appropriate and accurate model for prediction of future realized volatility is HAR-RV-GARCH model estimated in logarithmic form. In some cases, monthly volatility component is not a significant determinant of future

volatility. This fact contrasts with previous literature which finds monthly volatility component significant. We suggest that this result corresponds clearly with the time frame in which it is analyzed, as it covers the period of the global financial crisis that hit the U.S. market significantly.

In this context, the insignificant monthly component probably detects an outflow of trading strategies in one month horizon, or in other words, outflow of market participants trading on a monthly basis in crisis period. This can apparently reflect unwillingness of market participants to trade in longer horizons, probably due to high degree of uncertainty at the markets. Prediction of future daily realized volatility is therefore driven especially by daily and weekly volatility component. However, this fact probably will not be a long term characteristics of the U.S. stock market indices, but it would be necessary to investigate it over a longer period.

To summarize, HAR-RV models identify shorter term variables as driving future daily volatility. We consider the empirical results to have important implications for risk management.

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