Decision Making with Dempster-Shafer Theory of Evidence using Geometric Operators

José M. Merigó, and Montserrat Casanovas

***Abstract*—**We study the problem of decision making with Dempster-Shafer belief structure. We analyze the previous work developed by Yager about using the ordered weighted averaging (OWA) operator in the aggregation of the Dempster-Shafer decision process. We discuss the possibility of aggregating with an ascending order in the OWA operator for the cases where the smallest value is the best result. We suggest the introduction of the ordered weighted geometric (OWG) operator in the Dempster-Shafer framework. In this case, we also discuss the possibility of aggregating with an ascending order and we find that it is completely necessary as the OWG operator cannot aggregate negative numbers. Finally, we give an illustrative example where we can see the different results obtained by using the OWA, the Ascending OWA (AOWA), the OWG and the Ascending OWG (AOWG) operator.

***Keywords*—**Decision making, aggregation operators, Dempster-Shafer theory of evidence, Uncertainty, OWA operator, OWG operator.

I. INTRODUCTION

THE Dempster-Shafer theory of evidence was introduced by Dempster [1-2] and by Shafer [3]. Since then, it has been used in a lot of situations [4-5]. It provides a unifying framework for representing uncertainty as it can include in the same formulation the cases of risk and ignorance.

When using the Dempster-Shafer framework in decision making, we need to aggregate the decision information. One of the most common aggregation methods is the ordered weighted averaging (OWA) operator introduced by Yager [6]. Since its appearance, the OWA operator has been used in a wide range of applications such as [7-24]. It provides a parameterized family of aggregation operators that includes the arithmetic mean, the maximum and the minimum. Recently, Chiclana *et al* [25] have developed a geometric version of the OWA operator, the ordered weighted geometric (OWG) operator. Since its appearance, the OWG operator has been extensively analysed by different authors [26-36]. Basically, it consists in combining in the same aggregation the OWA operator with the geometric mean.

In [37], Yager suggested the use of the OWA aggregation in the Dempster-Shafer belief structure as a more general formulation for decision making in the face of evidential knowledge. This problem has also been studied in [38-39]. In this paper, we suggest the possibility of using the OWG operator in situations of decision making under uncertainty where the Dempster-Shafer belief structure plays a major role. We also propose the use of different types of orderings depending on the specific problem found. Basically, we suggest a descending order for situations where the highest value is the best result and an ascending order for situations where the smallest value is the best result.

In order to do so, the paper is organized as follows. In Section 2, we briefly comment the OWA and the OWG operator. In Section 3, we briefly comment the main concepts of the Dempster-Shafer belief structure. In Section 4, we study the process to follow in decision making with Dempster-Shafer belief structure. We analyze the process using OWA operators in the aggregation as suggested by Yager [37]. The difference with Yager's work is that we distinguish between aggregations that use a Descending OWA (DOWA) operator or an Ascending OWA (AOWA) operator. In Section 5, we propose the use of the OWG operator in the aggregation step of the decision making process with Dempster-Shafer belief structure. In Section 6, we give an illustrative example where we can see the different results obtained by using the OWA and the OWG operators in decision making with Dempster-Shafer belief structure. Finally, we summarize the main conclusions of the paper in Section 7.

II. AGGREGATION OPERATORS

A. Ordered Weighted Averaging Operator

The OWA operator was introduced in [6] and it provides a parameterized family of aggregation operators which have been used in many applications [7-24]. In the following, we provide a definition of the OWA operator as introduced by Yager [6].

Definition 1. An OWA operator of dimension n is a mapping F: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n having the properties:

(1) $w_j \in [0, 1]$
(2) $\sum_{j=1}^{n} w_j = 1$

and such that:

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J.M. Merigó is with the Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034, Barcelona, Spain (corresponding author: +34-93-4021962; fax: +34-93-4024580; e-mail: jmerigo@ub.edu).
M. Casanovas is with the Department of Business Administration, University of Barcelona, Av. Diagonal 690, 08034, Barcelona, Spain (e-mail: mcasanovas@ub.edu).

$$\text{OWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \quad (1)$$

where b_j is the jth largest of the a_i.

From a generalized perspective of the reordering step, we have to distinguish between the DOWA operator and the AOWA operator. We should note that we consider for both cases a situation where the highest value is the best result. The DOWA operator is defined as in definition 1.

Definition 2. An AOWA operator of dimension n is a mapping H: $R^n \rightarrow R$ that has an associated weighting vector W of dimension n having the properties:

(1) $w_j \in [0, 1]$
(2) $\sum_{j=1}^{n} w_j = 1$

and such that:

$$\text{AOWA}(a_1, a_2, \ldots, a_n) = \sum_{j=1}^{n} w_j c_j \quad (2)$$

where c_j is the jth lowest of the a_i. As it can be seen, the elements c_j (j= 1, 2, …, n) are ordered in an increasing way [19]: $c_1 \leq c_2 \leq \ldots \leq c_n$.

The OWA operator is a mean or averaging operator. This is a reflection of the fact that the operator is commutative, monotonic, bounded and idempotent. It can also be demonstrated that the OWA operator has as special cases the maximum, the minimum and the average criteria [6]. Other types of aggregations with the OWA operator can be seen in [10,12-13,16,17-18,20,23].

Another factor to consider, are the two measures introduced by Yager [6] for characterizing the weighting vector and the type of aggregation it performs. The first measure, the attitudinal character, is defined as:

$$\alpha(W) = \sum_{j=1}^{n} \left(\frac{n-j}{n-1} \right) w_j \quad (3)$$

It can be shown that $\alpha \in [0, 1]$. The more of the weight located near the top of W, the closer α to 1 and the more of the weight located toward the bottom of W, the closer α to 0. Note that for the optimistic criteria $\alpha(W) = 1$, for the pessimistic criteria $\alpha(W) = 0$, and for the average criteria $\alpha(W) = 0.5$.

The second measure introduced in [6], is called the entropy of dispersion of W and it is used to provide a measure of the information being used. It is defined as:

$$H(W) = -\sum_{j=1}^{n} w_j \ln(w_j) \quad (4)$$

That is, if $w_j = 1$ for some j, known as step-OWA [16], then $H(W) = 0$, and the least amount of information is used. If $w_j = 1/n$ for all j, then $H(W) = \ln n$, and the amount of information used is maximum.

These two measures can also be studied with the AOWA operator. Then, the attitudinal character $\alpha(W)$ is defined as:

$$\alpha(W) = \sum_{j=1}^{n} \left(\frac{n-j}{n-1} \right) w_j \quad (5)$$

It can be shown that here we also get $\alpha \in [0, 1]$. The more of the weight located near the top of W, the closer α to 0, and the more of the weight located toward the bottom of W, the closer α to 1. Note that for the optimistic criteria $\alpha(W) = 1$, for the pessimistic criteria $\alpha(W) = 0$, and for the average criteria $\alpha(W) = 0.5$. An interesting result found when comparing the descending and the ascending version of the measure is that they are dual [15] between them. That is: $\alpha_{OWA}(W) = 1 - \alpha_{AOWA}(W)$. For the entropy of dispersion, we will get the same index as in [6], so the measure is the same for both the DOWA and the AOWA operator although the reordering step is different.

B. Ordered Weighted Geometric Operator

The OWG operator was introduced by Chiclana *et al* [25] and it provides a family of aggregation operators similar to the OWA operator. It consists in combining the OWA operator with the geometric mean. In the following, we provide a definition of the OWG operator as introduced by Xu [34].

Definition 3. An OWG operator of dimension n is a mapping F: $R^{+n} \rightarrow R^+$ that has an associated weighting vector W of dimension n having the properties:

(1) $w_j \in [0, 1]$
(2) $\sum_{j=1}^{n} w_j = 1$

and such that:

$$\text{OWG}(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} b_j^{w_j} \quad (6)$$

where b_j is the jth largest of the a_i, and R^+ is the set of positive real numbers.

From a generalized perspective of the reordering process in the OWG operator, we have to distinguish between the Descending OWG (DOWG) operator and the Ascending OWG (AOWG) operator [34]. The DOWG operator is defined as in definition 3.

Definition 4. An AOWG operator of dimension n is a mapping H: $R^{+n} \rightarrow R^+$ that has an associated weighting vector $W = (w_1, w_2, \ldots, w_n)^T$, having the properties:

(1) $w_j \in [0,1]$
(2) $\sum_{j=1}^{n} w_j = 1$

and such that:

$$\text{AOWG}(a_1, a_2, \ldots, a_n) = \prod_{j=1}^{n} c_j^{w_j} \quad (7)$$

where c_j is the jth largest of the a_i, and R^+ is the set of positive real numbers. As it can be seen, the elements c_j ($j = 1, 2, \ldots, n$) are ordered in an increasing way: $c_1 \leq c_2 \leq \ldots \leq c_n$.

As it is seen in [25,34], the OWG operator has the following properties:

(1) Commutative: any permutation of the arguments has the same evaluation.
(2) Monotonic: If $a_i \geq d_i \quad \forall_i \Rightarrow$ OWG($a_1, \ldots, a_n$) $\geq$ OWG($d_1, \ldots, d_n$).
(3) Bounded: Min$\{a_i\} \leq$ OWG($a_1, \ldots, a_n$) $\leq$ Max$\{a_i\}$.
(4) Idempotent: OWG($a_1, \ldots, a_n$) $= a$, if $a_i = a$, $\forall i$

By choosing a different manifestation of the weighting vector, we are able to obtain different types of aggregation operators [25]. For example, with the DOWG operator we get:

(1) Optimistic criteria: $w_1 = 1$ and $w_j = 0 \quad \forall_{j \neq 1} \Rightarrow$ OWG($a_1, \ldots, a_n$) = Max$\{a_i\}$
(2) Pessimistic criteria: $w_n = 1$ and $w_j = 0 \quad \forall_{j \neq n} \Rightarrow$ OWG($a_1, \ldots, a_n$) = Min$\{a_i\}$
(3) Geometric mean: $w_j = 1/n \quad \forall_j \Rightarrow$ OWG($a_1, \ldots, a_n$) = $\prod_{i=1}^{n} (a_i)^{1/n}$

Other types of aggregations that could be obtained with the DOWG operator are the olympic geometric average and the geometric median. For the olympic geometric average, that it is based on the olympic average [16,20], we should say the following: if $w_1 = w_n = 0$; and $w_j = 1/(n\text{-}2) \quad \forall_{j \neq 1,n}$; then:

$$\text{OWG}(a_1, \ldots, a_n) = \prod_{j=2}^{n-1} (b_j)^{\frac{1}{n-2}} \tag{8}$$

where b_j is the jth largest of the a_i.

For the geometric median, based on the OWA median [17], we should distinguish between cases where the number of arguments is odd or even. If it is odd, then OWG($a_1, \ldots, a_n$) = $b_{(n+1)/2}$, where $b_{(n+1)/2}$ is the $[(n+1)/2]$th largest of the a_i. If the number of arguments n is even, then we shall call $b_{n/2}$ the lower median and $b_{(n/2)+1}$ the upper median. The lower median represents the $(n/2)$th largest of the a_i and the upper median represents the $[(n/2) + 1]$th largest of the a_i. Then, we can define the median in a different number of ways according to our interests or our attitudinal character. For example: OWG($a_1, \ldots, a_n$) = (lower median + upper median) / 2. Other examples for obtaining the geometric median could consist in select the lower median, the upper median, or the whole range between the lower and the upper median.

Using the same methodology as for the DOWG operator, if we look for different types of AOWG operators [34], we get the following:

(1) Optimistic criteria: $w_n = 1$ and $w_j = 0 \quad \forall_{j \neq n} \Rightarrow$ AOWG($a_1, \ldots, a_n$) = Max$\{a_i\}$
(2) Pessimistic criteria: $w_1 = 1$ and $w_j = 0 \quad \forall_{j \neq 1} \Rightarrow$ AOWG($a_1, \ldots, a_n$) = Min$\{a_i\}$
(3) Geometric mean: $w_j = 1/n \quad \forall_j \Rightarrow$ AOWG($a_1, \ldots, a_n$) = $\prod_{i=1}^{n} (a_i)^{1/n}$

Other types of aggregations that could be obtained with the AOWG operator are the olympic geometric average and the geometric median. For the olympic geometric average we get the same result as with the DOWG operator although the reordering is different. Then, the formulation is as follows: if $w_1 = w_n = 0$; and $w_j = 1/(n\text{-}2) \quad \forall_{j \neq 1,n}$; then:

$$\text{AOWG}(a_1, \ldots, a_n) = \prod_{j=2}^{n-1} (b_j)^{\frac{1}{n-2}} \tag{9}$$

where b_j is the jth smallest of the a_i.

For the geometric median, we also get the same result when the number of arguments is odd although the reordering of the arguments is different. That is: AOWG($a_1, \ldots, a_n$) = $b_{(n+1)/2}$, where $b_{(n+1)/2}$ is the $[(n+1)/2]$th smallest of the a_i. When the number of arguments is even, then, we find some differences. With the AOWG operator we shall call $b_{n/2}$ the lower median and $b_{(n/2)+1}$ the upper median but now the lower median is the $(n/2)$th smallest of the a_i and the upper median is the $[(n/2) + 1]$th smallest of the a_i. With this information, we can obtain the median in a different number of ways. For example: AOWG($a_1, \ldots, a_n$) = (lower median + upper median) / 2. Other examples for obtaining the median are the lower median, the upper median, or the whole range between the lower and the upper median.

III. Dempster-Shafer Belief Structure

The Dempster-Shafer belief structure was introduced by Dempster [1-2] and by Shafer [3]. Since then, a lot of new developments have been developed about it [4-5,37-39]. It provides a unifying framework for representing uncertainty as it can include in the same formulation the cases of risk and ignorance. Obviously, the case of certainty is also included as it can be seen as a particular case of risk or ignorance. For the case of risk, we find a situation of certainty when the probability of some outcome is one. For the case of ignorance, we find a situation of certainty when there is only one element in the set of events. Apart from these traditional cases, the Dempster-Shafer framework allows to represent various other forms of information a decision maker may have about the states of nature.

A Dempster-Shafer belief structure defined on a space X consists of a collection of n nonnull subsets of X, B_j for $j = 1, \ldots, n$, called focal elements and a mapping m, called the basic assignment function, defined as:

$$m: 2^X \rightarrow [0, 1]$$

such that:

(1) $\sum_{j=1}^{n} m(Bj) = 1$.

(2) $m(A) = 0, \ \forall A \neq B_j$.

As we said before, the cases of risk and ignorance are included as special cases of belief structure in the Dempster-Shafer framework. For the case of risk, a belief structure is called Bayesian belief structure [3] if it consists of n focal elements such that $B_j = \{x_j\}$, where each focal element is a singleton. Then, we can see that we are in a situation of decision making under risk environment as $m(B_j) = P_j = \text{Prob}\ \{x_j\}$.

For the case of ignorance, the belief structure consists in only one focal element B, where $m(B)$ essentially is the decision making under ignorance environment as this focal element comprises all the states of nature. Thus, $m(B) = 1$. Other special cases of belief structures such as the consonant belief structure or the simple support function are studied in [3].

The two measures associated with these belief structures are the measures of plausibility and belief [3]. The plausibility measure Pl is defined as Pl: $2^X \rightarrow [0, 1]$ such that:

$$\mathrm{Pl}(A) = \sum_{A \cap B_j \neq \varnothing} m(B_j) \tag{10}$$

The belief measure Bel is also defined as Bel: $2^X \rightarrow [0, 1]$ such that:

$$\mathrm{Bel}(A) = \sum_{B_j \subseteq A} m(B_j) \tag{11}$$

Bel(A) represents the exact support to A and Pl(A) represents the possible support to A. With these two measures we can form the interval of support to A as [Bel(A),Pl(A)]. This interval can be seen as the lower and upper bounds of the probability to which A is supported. From this we see that Pl(A) ≥ Bel(A) for all A. Another interesting aspect about these two measures is that they are connected by Bel(A) = 1 – Pl($\bar{A}$) or by Pl(A) = 1 – Bel($\bar{A}$), where $\bar{A}$ is the complement of A.

IV. Decision Making with Dempster-Shafer Belief Structure

The problem of selecting an appropriate alternative in situations in which our knowledge about the state of nature is in the form of a belief structure, has been studied by different authors such as [37-39]. In [37], Yager proposed a more generalized methodology by using the OWA operator. In the following, we are going to summarize the process as suggested by Yager.

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, ..., A_q\}$ with states of nature $\{S_1, ..., S_n\}$. C_{ij} is the payoff to the decision maker if he selects alternative A_i and the state of nature is S_j. In addition, the knowledge of the state of nature is captured in terms of a belief structure m. The focal elements of m are $B_1, ..., B_r$ and associated with each of these is a weight $m(B_k)$. The objective of the problem is to select the alternative which best satisfies the payoff to the decision maker. In order to do that, we should follow the following steps:

(1) Determine the payoff matrix.
(2) Determine the belief function m about the states of nature and the decision makers degree of optimism α.
(3) Calculate the collection of weights, w, to be used in the OWA aggregation function for each different cardinality of focal elements.
(4) Determine the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{C_{ij} \mid S_j \in B_k\}$.
(5) Calculate the aggregated payoff, V_{ik} = OWA(M_{ik}), using Eq. 1, for all the values of i and k.
(6) For each alternative, calculate the generalized expected value, C_i, where:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k) \tag{12}$$

(7) Select the alternative with the largest C_i as the optimal.

In this process, we could also introduce the AOWA operator instead of the traditional OWA operator. The reason for considering both aggregations is because of the different types of decisions we can find depending on the problem analyzed. Basically, we could distinguish between situations where the highest payoff is the best alternative such as in situations where we analyze benefits, and situations where the highest payoff is the worst alternative such as in situations where we analyze costs. Then, if for a situation of costs we use the OWA operator, our aggregation would not reflect correctly this situation as it would take first the worst result and the weighting vector is prepared for taking the best one. This problem can be demonstrated with an example.

Example 1. Assume we want to aggregate the following arguments (20, 50, 30, 40) and (10, 0, 70, 60), representing the costs of two projects depending on the state of the nature that will happen in the future. Assume the decision maker is very conservative, then, he will have a low degree of optimism α. Assume he uses the following weighting vector: W = (0.1, 0.2, 0.3, 0.4). As we can see, this weighting vector gives more importance to the last weight because this one is supposed to aggregate the worst case. As the decision maker is conservative, he wants to consider a pessimistic scenario in order to select between these two projects. If we look to the projects before making the aggregation, we could see that the first one is more conservative as its possible results are more stable than the second project. Then, we should expect that the aggregation would give us a result that shows that the first project is the best one. But if we use the OWA operator (Eq. 1), that will not be the case.

OWA(20, 50, 30, 40) = 50 × 0.1 + 40 × 0.2 + 30 × 0.3 + 20 × 0.4 = 30

OWA(10, 0, 70, 60) = 70 × 0.1 + 60 × 0.2 + 10 × 0.3 + 0 × 0.4 = 22

As we can see, we should select the second alternative as it gives us a smaller expected cost. But as it was mentioned before, this is wrong because the arguments are more unstable than the first project and the decision maker was supposed to be conservative. The explanation is that the traditional OWA operator takes first the highest value, and in this case, the highest value is the worst possible scenario. With this example, it is very easy to see that these results are wrong. Then, this problem can be extended to more complex situations where it is not so easy to see directly the conflict of using the OWA operator in a situation of costs.

In order to aggregate correctly in a situation of costs, we need to use the AOWA operator because then we consider first the lowest value which is the best result and so on. In this example, we will get (Eq. 2):

AOWA(20, 50, 30, 40) = 20 × 0.1 + 30 × 0.2 + 40 × 0.3 + 50 × 0.4 = 40

AOWA(10, 0, 70, 60) = 0 × 0.1 + 10 × 0.2 + 60 × 0.3 + 70 × 0.4 = 48

As we can see, with the AOWA operator we decide to select the first alternative as it has the lowest expected cost. This is in accordance with our intuition because the decision maker is selecting the project with more stable results as he is conservative.

From this example, we can conclude that we should use the traditional OWA operator or also called DOWA operator, in situations concerning benefits. But in situations where we analyze costs, we should use the AOWA operator. This conclusion could also be extended for other situations different from benefits and costs. In general, we should use the traditional OWA when we want to aggregate arguments where the highest value is the best result. And we should use the AOWA operator when we want to aggregate arguments where the smallest value is the best result. We should note that for the particular case of costs, we could solve the problem by using negative numbers instead of the AOWA operator as it is shown in [38]. But for other cases, this alternative would not be possible such as in the OWG operator [34] or would be inadequate such as in regret methods [21].

For the case of decision making with Dempster-Shafer belief structure, if we use the AOWA operator, we should make the following changes in the decision process:

In *Step 3*, when determining the collection of weights, w, to be used in the AOWA aggregation function for each different cardinality of focal elements, we should consider that now the attitudinal character $\alpha(W)$ is defined by Eq. (5).

In *Step 5*, when determining the aggregated payoff, we should use V_{ik} = AOWA(M_{ik}), using Eq. 2, for all the values of i and k.

In *Step 7*, we should select the alternative with the lowest C_i as the optimal because the best result is the one which predicts the lowest expected values.

V. Using the OWG Operators in Decision Making with Dempster-Shafer Belief Structure

An alternative method when taking decisions with Dempster-Shafer belief structure is possible by using the OWG operator in the aggregation instead of the OWA operator. The motivation for using the OWG operator is because there are some cases where we could prefer to aggregate with a geometric operator instead of the traditional methods used previously. Here, the procedure will be the same as for the case with the OWA operator [37] with the difference that now we will use the OWG operator in the aggregation step. Then, we can summarize the procedure as follows:

Assume we have a decision problem in which we have a collection of alternatives $\{A_1, ..., A_q\}$ with states of nature $\{S_1, ..., S_n\}$. C_{ij} is the payoff to the decision maker if he selects alternative A_i and the state of nature is S_j. In addition, the knowledge of the state of nature is captured in terms of a belief structure m. The focal elements of m are $B_1, ..., B_r$ and associated with each of these is a weight $m(B_k)$. The objective of the problem is to select the alternative which best satisfies the payoff to the decision maker. In order to do that, we should follow the following steps:

(1) Determine the payoff matrix.
(2) Calculate the belief function m about the states of nature.
(3) Determine the collection of weights, w, to be used in the OWG aggregation function for each different cardinality of focal elements.
(4) Determine the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of i and k. Hence $M_{ik} = \{C_{ij} \mid S_j \in B_k\}$.
(5) Calculate the aggregated payoff, V_{ik} = OWG(M_{ik}), using Eq. 3, for all the values of i and k.
(6) For each alternative, calculate the generalized expected value, C_i, with:

$$C_i = \sum_{k=1}^{r} V_{ik} m(B_k) \tag{13}$$

(7) Select the alternative with the largest C_i as the optimal.

As in the case with OWA operators, we could use in the aggregation step the AOWG operators. The reason for using them is also because they are necessary when analysing a situation of costs or a situation where the smallest value is the best result. We should note that in this case, it is completely necessary because the OWG operator cannot aggregate negative numbers [34]. Then, the alternative method that could be used with the OWA operator [38] is not applicable.

Then, if we use the AOWG operators in the decision process, we should make the following changes to the 7 steps mentioned before:

In *Step 5*, when determining the aggregated payoff, we should use V_{ik} = AOWG(M_{ik}), using Eq. 4, for all the values of *i* and *k*.

In *Step 7*, we should select the alternative with the lowest C_i as the optimal because now we assume the best result is the lowest one.

VI. Illustrative Example

In the following, we are going to develop an example in order to understand numerically all the procedures commented previously. We will distinguish four cases: the aggregation with the OWA operator, with the AOWA, with the OWG operator and with the AOWG operator. We will use the same payoff matrix for all the cases. But we have to implicitly assume that when using the OWA and the OWG operator we are considering a decision with benefits and when using the AOWA and the AOWG, a decision with costs.

Step 1: Assume we have the following payoff matrix shown in Table 1.

TABLE I
PAYOFF MATRIX

	S_1	S_2	S_3	S_4	S_5
A_1	20	30	10	40	40
A_2	35	20	10	50	25
A_3	30	40	30	30	10
A_4	20	35	40	25	20

Step 2: Assume the decision maker has a degree of optimism of 0.6 or 60% when using OWA operators and assume the following belief function *m* about the states of nature:

Focal element

$B_1 = \{S_1, S_2, S_5\} = 0.4$

$B_2 = \{S_2, S_3, S_4, S_5\} = 0.3$

$B_3 = \{S_1, S_3\} = 0.3$

Step 3: Assume we have used one of the different methods existing [10,13] for determining the OWA weights or the OWG weights and we have obtained the following weighting vector for different number of arguments.

Weighting vector

$W_2 = (0.6, 0.4)$

$W_3 = (0.4, 0.4, 0.3)$

$W_4 = (0.3, 0.3, 0.3, 0.1)$

Step 4: Determine the payoff collection, M_{ik}, if we select alternative A_i and the focal element B_k occurs, for all the values of *i* and *k*. Then, we calculate the bags M_{ik}.

A_1: $M_{11} = \langle 20, 30, 40 \rangle$; $M_{12} = \langle 30, 10, 40, 40 \rangle$;
$M_{13} = \langle 20, 10 \rangle$.

A_2: $M_{21} = \langle 35, 20, 25 \rangle$; $M_{22} = \langle 20, 10, 50, 25 \rangle$;
$M_{23} = \langle 35, 10 \rangle$.

A_3: $M_{31} = \langle 30, 40, 10 \rangle$; $M_{32} = \langle 40, 30, 30, 10 \rangle$;
$M_{33} = \langle 30, 30 \rangle$.

A_4: $M_{41} = \langle 20, 35, 20 \rangle$; $M_{42} = \langle 35, 40, 25, 20 \rangle$;
$M_{43} = \langle 20, 40 \rangle$.

From the fifth step, we will distinguish 4 cases: the aggregation with the OWA, with the AOWA, with the OWG and with the AOWG operator.

Step 5: Calculate the aggregated payoff, V_{ik}, using Eq. 1 for the OWA operator, using Eq. 2 for the AOWA, using Eq. 6 for the OWG operator and using Eq. 7 for the AOWG operator, for all the values of *i* and *k*. The results are shown in Table 2.

TABLE II
AGGREGATED PAYOFF

	OWA	*AOWA*	*OWG*	*AOWG*
V_{11}	32	28	31.03	27.02
V_{12}	34	28	31.94	24.2
V_{13}	16	14	15.15	13.19
V_{21}	28	25	27.35	24.45
V_{22}	29.5	21.5	26.26	19.03
V_{23}	25	20	21.2	16.5
V_{31}	30	24	27.01	20.47
V_{32}	31	25	29.3	22.2
V_{33}	30	30	30	30
V_{41}	26	23	25.01	22.37
V_{42}	32	28	31.14	27.11
V_{43}	32	28	30.31	26.39

Step 6: For each alternative, calculate the generalized expected value, C_i, using Eq. 12 for the OWA and the AOWA operator and Eq. 13 for the OWG and the AOWG operator. The results obtained for the different operators are shown in Table 3.

TABLE III
GENERALIZED EXPECTED VALUE

	OWA	*AOWA*	*OWG*	*AOWG*
A_1	27.8	23.8	26.54	22.02
A_2	27.55	22.45	25.18	20.43
A_3	30.3	26.1	28.59	23.84
A_4	29.6	26	28.44	24.99

Step 7: For the OWA and the OWG operator, we will select alternative 3 as it gives the highest expected value. For the AOWA and the AOWG operator, we will select alternative 2 because in these cases we assume that the best result is the smallest one.

VII. Conclusions

In this paper, we have suggested the use of the OWG operator in decision making with Dempster-Shafer belief structure. We have distinguished between aggregations with an ascending or a descending order. We have seen that there are some problems to aggregate with descending operators when we are in a situation where the smallest value is the best result. We have demonstrated this situation with an example. Although there are alternative solutions with the OWA operator such as using negative numbers, we have seen that for the OWG operator it is completely necessary to use an ascending order because this operator cannot aggregate negative numbers. We have developed the decision making process distinguishing in the aggregation step between the OWA operator, the AOWA, the OWG and the AOWG operator. Finally, an illustrative example has been given by using the four different cases in the aggregation step.

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José M. Merigó (M'08) was born in Barcelona (Spain) in 1980. He is an assisstant professor in the Department of Business Administration at the University of Barcelona. He holds a master degree in Business Administration from the University of Barcelona and a Bachelor of Science and Social Science in Economics from the Lund University (Sweden). Currently, he is developing his PhD thesis in Business Administration in the Department of Business Administration of the University of Barcelona.

He has written more than 40 papers in journals and conference proceedings including articles in *Fuzzy Economic Review*, *International Journal of Computational Intelligence* and *International Journal of Information Technology*. He is on the editorial board of the *Association for Modelling and Simulation in Enterprises* (AMSE). He has served as a reviewer in different journals such as *IEEE Transactions on Fuzzy Systems* and *European Journal of Operational Research*. His current research interests include Decision Making, Aggregation Operators and Uncertainty.

Montserrat Casanovas was born in Barcelona (Spain) in 1947. She is a full professor in the Department of Business Administration at the University of

Barcelona. She holds a master degree in Business Administration from the University of Barcelona and a PhD degree in Business Administration from the Autonomous University of Barcelona. She is also an Auditor, a Financial Analyst and an European Financial Advisor.

She has written more than 70 papers in journals and conference proceedings including articles in *Fuzzy Economic Review*, *International Journal of Computational Intelligence* and *International Journal of Information Technology*. She is the general secretary of the Catalonian School of Economists. She has also participated in several journals and in the scientific comitee of different conferences. She has served as a reviewer in different journals such as *European Journal of Operational Research*. Her current research interests include Corporate Finance, Financial Markets, Decision Making and Uncertainty.