World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:5, No:12, 2011

An adaptive least-squares mixed finite element method for pseudo-parabolic integro-differential equations

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Abstract—In this article, an adaptive least-squares mixed finite element method is studied for pseudo-parabolic integro-differential equations. The solutions of least-squares mixed weak formulation and mixed finite element are proved. A posteriori error estimator is constructed based on the least-squares functional and the posteriori errors are obtained.

Keywords—Pseudo-parabolic integro-differential equation; least-squares mixed finite element method; Adaptive method; A posteriori error estimates.

I. Introduction

In this article, we consider the following pseudo-parabolic integro-differential equation

$$\begin{cases} u_t - \nabla\cdot(a(x)\nabla u_t + b(x)\nabla u \\ \quad + d(x)\displaystyle\int_0^t \nabla u(x,\tau)d\tau) + qu = f(x,t), & x\in\Omega, \quad t\in J, \\ u(x,t) = 0, & x\in\partial\Omega, \ t\in J, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{1}$$

where Ω is a bounded convex polygonal domain in $R^d(d = 1,2,3)$ with $Lipschitz$ continuous boundary $\partial\Omega$, $J = (0,T]$ is the time interval with $0 < T < \infty$. $q = q(x) \geq 0$ and $f = f(x,t)$ in (1) are given functions. We shall make the following assumptions on the coefficients c, a, b and d: there exist some positive constants c_*, c^*,a_*, a^*,b_*, b^* and d_*, d^* such that $0 < c_* \leq c(x) \leq c^*, 0 < a_* \leq a(x) \leq a^*, 0 < b_* \leq b(x) \leq b^*, 0 < d_* \leq d(x) \leq d^*$.

Evolution integro-differential equations are a class of important evolution partial differential equations, and have a lot of applications in many physical problems, such as the transport of reactive and passive contaminates in aquifers. Ewing et al. [1] studied finite volume element approximations for two-dimensional parabolic integro-differential equations. Lin et al. [2] proposed the Ritz-Volterra projection onto finite element spaces for integro-differential and related equations. H^1-Galerkin mixed element methods were studied for parabolic, pseudo-parabolic, and pseudo-hyperbolic integro-differential equations in [3], [4], [5], [6]. Refs.[7], [8] analysed the space-time finite element methods for second-order and fourth-order parabolic partial integro-differential equations. Refs.[9], [10] studied the least-squares mixed finite element method for parabolic partial integro-differential equations. Cui [11] studied pseudo-parabolic, and pseudo-hyperbolic integro-differential equations, proposed a new Sobolev-Volterra projection. Zhang [12] investigated the superconvergence properties of finite element approximations to parabolic and hyperbolic integro-differential equations.

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Manuscript received DEC 14, 2011.

In recent years, a lot of researchers have studied the least-squares mixed finite element methods for partial differential equations. Luo et al. [13] proposed a nonlinear Galerkin/Petrov-least squares mixed element method for the stationary Navier-Stokes equations. Yang [14] studied the least-square mixed finite element methods for nonlinear parabolic problems. Chen et al. [15] studied least-squares mixed finite element method for degenerate elliptic problems. Duan and Lin [17] studied least-squares mixed finite elements for elasticity. Cai et al. [16] proposed and analysed an adaptive least-squares mixed finite element method for the stress-displacement formulation of linear elasticity Gu and Li [18] studied an adaptive least-squares mixed finite element method for nonlinear parabolic problems. In this article, we study an adaptive least-squares mixed finite element method for the pseudo-parabolic integro-differential equations, and obtain a posteriori error estimates.

II. A Least-Squares Formulation

Introduce the auxiliary variable $\sigma = -(a(x)\nabla u_t + b(x)\nabla u + d(x)\int_0^t \nabla u(x,\tau)d\tau)$ to have

$$\begin{cases} u_t + div\sigma + qu = f, & x\in\Omega, \quad t\in J, \\ \sigma + a(x)\nabla u_t + b(x)\nabla u \\ \quad + d(x)\displaystyle\int_0^t \nabla u(x,\tau)d\tau = 0, & x\in\Omega, \quad t\in J, \\ u = 0, & x\in\partial\Omega, \ t\in J, \\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{2}$$

We differentiate the second equation of system (2) with respect to time t to obtain

$$\sigma_t = -a(x)\nabla u_{tt} - b(x)\nabla u_t - d(x)\nabla u. \tag{3}$$

Taking $\omega = u_t$, we can derive the following equivalent system of (2)

$$\begin{cases} \omega + div\sigma + qu = f, & x\in\Omega,\quad t\in J,\\ \sigma_t + a(x)\nabla\omega_t + b(x)\nabla\omega + d(x)\nabla u = 0, & x\in\Omega,\quad t\in J,\\ \omega - u_t = 0, & x\in\Omega,\quad t\in J,\\ u = 0, & x\in\partial\Omega,\ t\in J,\\ u(x,0) = u_0(x), & x\in\Omega, \end{cases} \tag{4}$$

Using the finite difference to discretize (4) in t. Let $\tau > 0, \tau^n = n\tau, n = 0, 1\cdots, N = T/\tau$. Let $\omega^n = (u^n - u^{n-1})/\tau, n \geq 1$. We can rewrite the system (4) :

$$\begin{cases} \omega^n + div\sigma^n + qu^n = f^n, & x\in\Omega,\quad t\in J,\\ \dfrac{\sigma^n-\sigma^{n-1}}{\tau} + a(x)\dfrac{\nabla\omega^n - \nabla\omega^{n-1}}{\tau} & \\ + b(x)\nabla\omega^n + d(x)\nabla u^n = 0, & x\in\Omega,\quad t\in J,\\ \omega^n = \dfrac{u^n - u^{n-1}}{\tau}, & x\in\Omega,\ t\in J,\\ u^n = 0, & x\in\partial\Omega,\ t\in J,\\ u^n(x,0) = u_0^n(x), & x\in\Omega, \end{cases} \tag{5}$$

Introduce Sobolev spaces:

$$L \doteq H^1(\Omega) = \{p\in L^2(\Omega) : \nabla p\in L^2(\Omega)^2\},$$
$$H \doteq H(div,\Omega) = \{\mathbf{w}\in L^2(\Omega)^2 : div\mathbf{w}\in L^2(\Omega)\}.$$

We can use the least-squares mixed finite element method for (5). The least-squares minimization problem is: find $(u^n, \sigma^n, \omega^n) \in X := L\times H\times L$ such that

$$I(u^n,\sigma^n,\omega^n) = \inf_{(\nu,\xi,\eta)\in X} I(\nu,\xi,\eta), \tag{6}$$

where I is the quadratic functional defined by

$$\begin{aligned} I(\nu^n,\xi^n,\eta^n) =& \|\eta^n + div\xi^n + q\nu^n - f^n\|^2_{0,\Omega}\\ &+ \|\frac{\xi^n-\xi^{n-1}}{\tau} + a(x)\frac{\nabla\eta^n - \nabla\eta^{n-1}}{\tau}\\ &+ b(x)\nabla\eta^n + d(x)\nabla\nu^n\|^2_{0,\Omega}. \end{aligned}$$

Set $J^n = [t^{n-1}, t^n]$. The corresponding variational statement is: find $(u^n,\sigma^n,\omega^n) : J^n \to X$

$$B(u^n,\sigma^n,\omega^n;\nu,\xi,\eta) = 0,\ \forall(\nu,\xi,\eta)\in X, \tag{7}$$

where

$$\begin{aligned} &B(u^n,\sigma^n,\omega^n;\nu,\xi,\eta)\\ &= \tau(\omega^n + div\sigma^n + qu^n, \eta + div\xi + q\nu)_{0,\Omega}\\ &+ (\sigma^n - \sigma^{n-1} + a(x)(\nabla\omega^n - \nabla\omega^{n-1}) + \tau b(x)\nabla\omega^n\\ &+ \tau d(x)\nabla u^n, \xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}. \end{aligned} \tag{8}$$

Theorem 2.1: The bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ is continuous and coercive. That is that there exist positive constants α and β such that

$$\begin{aligned} &B(u^n,\sigma^n,\omega^n;\nu,\xi,\eta)\\ \leq&\alpha(\|\omega^n\|^2_{0,\Omega} + \|div\sigma^n\|^2_{0,\Omega} + \|u^n\|^2_{0,\Omega} + \|\sigma^n\|^2_{0,\Omega}\\ &+ \|\nabla u^n\|^2_{0,\Omega} + \|\nabla\omega^n\|^2_{0,\Omega})^{\frac{1}{2}}(\|\eta\|^2_{0,\Omega} + \|div\xi\|^2_{0,\Omega}\\ &+ \|v\|^2_{0,\Omega} + \|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega})^{\frac{1}{2}}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} B(\nu,\xi,\eta;\nu,\xi,\eta) \geq& \beta(\|\eta\|^2_{0,\Omega} + \|div\xi\|^2_{0,\Omega} + \|v\|^2_{0,\Omega}\\ &+ \|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega}). \end{aligned} \tag{10}$$

Proof: **(I). For the upper bound**

Noting that the functions $a(x), b(x), d(x)$ and $q(x)$ are bounded, we have

$$\begin{aligned} &B(\nu,\xi,\eta;\nu,\xi,\eta)\\ =&\tau(\eta + div\xi + q\nu, \eta + div\xi + q\nu)_{0,\Omega}\\ &+ (\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu,\\ &\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}\\ =&\tau\|\eta + div\xi + q\nu\|^2_{0,\Omega} + \|\xi + a(x)\nabla\eta\\ &+ \tau b(x)\nabla\eta + \tau d(x)\nabla\nu\|^2_{0,\Omega}\\ \leq&c(\|\eta\|^2_{0,\Omega} + \|div\xi\|^2_{0,\Omega} + \|v\|^2_{0,\Omega} + \|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega}\\ &+ \|\tau b(x)\nabla\eta\|^2_{0,\Omega} + \|\tau d(x)\nabla\nu\|^2_{0,\Omega})\\ \leq&c(\|\eta\|^2_{0,\Omega} + \|div\xi\|^2_{0,\Omega} + \|v\|^2_{0,\Omega}\\ &+ \|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega}). \end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 2.1.

(II).For the lower bound

Note that the bounded functions $a(x), b(x), d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$\begin{aligned} &B(\nu,\xi,\eta;\nu,\xi,\eta)\\ =&\tau(\eta + div\xi + q\nu, \eta + div\xi + q\nu)_{0,\Omega}\\ &+ (\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu,\\ &\xi + a(x)\nabla\eta + \tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}\\ =&\tau(\eta,\eta)_{0,\Omega} + \tau(div\xi, div\xi)_{0,\Omega} + \tau(q\nu, q\nu)_{0,\Omega}\\ &+ 2\tau(\eta, div\xi)_{0,\Omega} + 2\tau(\eta, q\nu)_{0,\Omega} + 2\tau(div\xi, q\nu)_{0,\Omega}\\ &+ (\xi,\xi)_{0,\Omega} + a^2(x)(\nabla\eta,\nabla\eta)_{0,\Omega} + \tau^2 b^2(x)(\nabla\eta,\nabla\eta)_{0,\Omega}\\ &+ \tau^2 d^2(x)(\nabla\nu,\nabla\nu)_{0,\Omega} + 2(\xi, a(x)\nabla\eta)_{0,\Omega} + 2\tau(\xi, b(x)\nabla\eta)_{0,\Omega}\\ &+ 2\tau(\xi, d(x)\nabla\nu)_{0,\Omega} + 2\tau(a(x)\nabla\eta, b(x)\nabla\eta)_{0,\Omega}\\ &+ 2\tau(a(x)\nabla\eta, d(x)\nabla\nu)_{0,\Omega} + 2\tau^2(b(x)\nabla\eta, d(x)\nabla\nu)_{0,\Omega}. \end{aligned}$$

$$\begin{aligned} \geq&\tau\|\eta\|^2_{0,\Omega} + \tau\|div\xi\|^2_{0,\Omega} + c_1\|v\|^2_{0,\Omega} + \|\xi\|^2_{0,\Omega} + c_2\|\nabla\eta\|^2_{0,\Omega}\\ &+ c_3\|\nabla\nu\|^2_{0,\Omega} - \epsilon_1(\|\eta\|^2_{0,\Omega} + \|div\xi\|^2_{0,\Omega}) - \epsilon_2(\|\eta\|^2_{0,\Omega}\\ &+ \|\nu\|^2_{0,\Omega}) - \epsilon_3(\|div\xi\|^2_{0,\Omega} + \|\nu\|^2_{0,\Omega}) - \epsilon_4(\|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega})\\ &- \epsilon_5(\|\xi\|^2_{0,\Omega} + \|\nabla\eta\|^2_{0,\Omega}) - \epsilon_6(\|\xi\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega})\\ &- \epsilon_7(\|\nabla\eta\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega}) - \epsilon_8(\|\nabla\eta\|^2_{0,\Omega} + \|\nabla\nu\|^2_{0,\Omega})\\ =&(\tau - \epsilon_1 - \epsilon_2)\|\eta\|^2_{0,\Omega} + (\tau - \epsilon_1 - \epsilon_3)\|div\xi\|^2_{0,\Omega}\\ &+ (c_1 - \epsilon_2 - \epsilon_3)\|\nu\|^2_{0,\Omega} + (1 - \epsilon_4 - \epsilon_5 - \epsilon_6)\|\xi\|^2_{0,\Omega}\\ &+ (c_2 - \epsilon_5 - \epsilon_7 - \epsilon_8)\|\nabla\eta\|^2_{0,\Omega} + (c_3 - \epsilon_6 - \epsilon_7 - \epsilon_8)\|\nabla\nu\|^2_{0,\Omega}. \end{aligned}$$

So we can select constants $\epsilon_1, \epsilon_2, \epsilon_3, \epsilon_4, \epsilon_5, \epsilon_6, \epsilon_7, \epsilon_8, c_1, c_2, c_3$ such that

$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|\eta\|^2_{0,\Omega},$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|div\xi\|^2_{0,\Omega},$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta) \geq c\|\nu\|^2_{0,\Omega},$$

$$B(\nu,\xi,\eta;\nu,\xi,\eta)\geq c\|\xi\|_{0,\Omega}^2,$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta)\geq c\|\nabla\eta\|_{0,\Omega}^2,$$
$$B(\nu,\xi,\eta;\nu,\xi,\eta)\geq c\|\nabla\nu\|_{0,\Omega}^2,$$

So we obtain

$$B(\nu,\xi,\eta;\nu,\xi,\eta)\geq\beta(\|\eta\|_{0,\Omega}^2+\|div\xi\|_{0,\Omega}^2+\|\nu\|_{0,\Omega}^2+\|\xi\|_{0,\Omega}^2+\|\nabla\eta\|_{0,\Omega}^2+\|\nabla\nu\|_{0,\Omega}^2).$$

From the above inequality, we can get the proof of Theorem 2.1. ■

Theorem 2.2: We obtain that Eq.(7) have a unique solution $(u^n,\sigma^n,\omega^n)\in X:=L\times H\times L$.

Proof: Use the similar proof to Ref [18] to obtain the conclusion of Theorem 2.2.

III. Finite Element Approximation

Assume the spaces $L_h\subset L, H_h\subset H$. The completely discrete least-squares mixed finite method approximation is: find $(u_h^n,\sigma_h^n,\omega_h^n)\in X_h:=L_h\times H_h\times L_h$ such that

$$I(u_h^n,\sigma_h^n,\omega_h^n)=\inf_{(\nu_h,\xi_h,\eta_h)\in X} I(\nu_h,\xi_h,\eta_h). \tag{11}$$

where

$$\begin{aligned}&I(\nu_h^n,\xi_h^n,\eta_h^n)\\ &=\sum_{T\in T_h}(\|\eta_h^n+div\xi_h^n+q\nu_h^n-f^n\|_{0,T}^2\\ &+\|\frac{\xi_h^n-\xi_h^{n-1}}{\tau}+a(x)\frac{\nabla\eta_h^n-\nabla\eta_h^{n-1}}{\tau}\\ &+b(x)\nabla\eta_h^n+d(x)\nabla\nu_h^n\|_{0,T}^2).\end{aligned}$$

We defined the bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ as follows

$$\begin{aligned}&B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h)\\ &=\sum_{T\in T_h}(\tau(\omega_h^n+div\sigma_h^n+qu_h^n,\eta_h+div\xi_h+q\nu_h)_{0,T}\\ &+(\sigma_h^n-\sigma_h^{n-1}+a(x)(\nabla\omega_h^n-\nabla\omega_h^{n-1})\\ &+\tau b(x)\nabla\omega_h^n+\tau d(x)\nabla u_h^n,\\ &\xi_h+a(x)\nabla\eta_h+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h)_{0,T}).\end{aligned} \tag{12}$$

■

Theorem 3.1: The bilinear $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ is continuous and coercive, that is that there exist positive constants α_h and β_h such that

$$\begin{aligned}&B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h)\\ &\leq\alpha_h(\sum_{T\in T_h}\|\omega_h^n\|_{0,T}^2+\|div\sigma_h^n\|_{0,T}^2+\|u_h^n\|_{0,T}^2+\|\sigma_h^n\|_{0,T}^2\\ &+\|\nabla u_h^n\|_{0,T}^2+\|\nabla\omega_h^n\|_{0,T}^2)^{\frac12}(\sum_{T\in T_h}\|\eta_h\|_{0,T}^2+\|div\xi_h\|_{0,T}^2\\ &+\|v_h\|_{0,T}^2+\|\xi_h\|_{0,T}^2+\|\nabla\eta_h\|_{0,T}^2+\|\nabla\nu_h\|_{0,T}^2)^{\frac12},\end{aligned} \tag{13}$$

and

$$\begin{aligned}&B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h)\\ &\geq\beta_h\sum_{T\in T_h}(\|\eta_h\|_{0,\Omega}^2+\|div\xi_h\|_{0,\Omega}^2+\|v_h\|_{0,\Omega}^2\\ &+\|\xi_h\|_{0,\Omega}^2+\|\nabla\eta_h\|_{0,\Omega}^2+\|\nabla\nu_h\|_{0,\Omega}^2).\end{aligned} \tag{14}$$

which holds for all $(u_h^n,\sigma_h^n,\omega_h^n)\in X_h:=L_h\times H_h\times L_h,(\nu_h,\xi_h,\eta_h)\in X_h:=L_h\times H_h\times L_h$.

Proof: **(I). For the upper bound**

Since the functions $a(x),b(x),d(x)$ and $q(x)$ are bounded, we have

$$\begin{aligned}&B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h)\\ &=\sum_{T\in T_h}(\tau(\eta_h+div\xi_h+q\nu_h,\eta_h+div\xi_h+q\nu_h)_{0,T}\\ &+(\xi_h+a(x)\nabla\eta_h+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h,\\ &\xi_h+a(x)\nabla\eta_h+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h)_{0,T})\\ &=\sum_{T\in T_h}(\tau\|\eta_h+div\xi_h+q\nu_h\|_{0,T}^2+\|\xi_h+a(x)\nabla\eta_h\\ &+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h\|_{0,T}^2)\\ &\leq c\sum_{T\in T_h}(\|\eta_h\|_{0,T}^2+\|div\xi_h\|_{0,T}^2+\|v_h\|_{0,T}^2+\|\xi_h\|_{0,T}^2\\ &+\|\nabla\eta_h\|_{0,T}^2+\|\tau b(x)\nabla\eta_h\|_{0,T}^2+\|\tau d(x)\nabla\nu_h\|_{0,T}^2)\\ &\leq c\sum_{T\in T_h}(\|\eta_h\|_{0,T}^2+\|div\xi_h\|_{0,T}^2+\|v_h\|_{0,T}^2\\ &+\|\xi_h\|_{0,T}^2+\|\nabla\eta_h\|_{0,T}^2+\|\nabla\nu_h\|_{0,T}^2).\end{aligned}$$

Since the bilinear form is symmetric, this is sufficient for the upper bound in Theorem 3.1.

(II). For the lower bound

Note that the bounded functions $a(x),b(x),d(x)$ and $q(x)$ and use the Holder inequality, the Cauchy inequality to get

$$\begin{aligned}&B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h)\\ &=\sum_{T\in T_h}(\tau(\eta_h+div\xi_h+q\nu_h,\eta_h+div\xi_h+q\nu_h)_{0,T}\\ &+(\xi_h+a(x)\nabla\eta_h+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h,\\ &\xi_h+a(x)\nabla\eta_h+\tau b(x)\nabla\eta_h+\tau d(x)\nabla\nu_h)_{0,T})\\ &=\sum_{T\in T_h}(\tau(\eta_h,\eta_h)_{0,T}+\tau(div\xi_h,div\xi_h)_{0,T}+\tau(q\nu_h,q\nu_h)_{0,T}\\ &+2\tau(\eta_h,div\xi_h)_{0,T}+2\tau(\eta_h,q\nu_h)_{0,T}+2\tau(div\xi_h,q\nu_h)_{0,T}\\ &+(\xi_h,\xi_h)_{0,T}+a^2(x)(\nabla\eta_h,\nabla\eta_h)_{0,T}\\ &+\tau^2b^2(x)(\nabla\eta_h,\nabla\eta_h)_{0,T}+\tau^2d^2(x)(\nabla\nu_h,\nabla\nu_h)_{0,T}\\ &+2(\xi_h,a(x)\nabla\eta_h)_{0,T}+2\tau(\xi_h,b(x)\nabla\eta_h)_{0,T}\\ &+2\tau(\xi_h,d(x)\nabla\nu_h)_{0,T}+2\tau(a(x)\nabla\eta_h,b(x)\nabla\eta_h)_{0,T}\\ &+2\tau(a(x)\nabla\eta_h,d(x)\nabla\nu_h)_{0,T}+2\tau^2(b(x)\nabla\eta_h,d(x)\nabla\nu_h)_{0,T})\\ &\geq\sum_{T\in T_h}(\tau\|\eta_h\|_{0,T}^2+\tau\|div\xi_h\|_{0,T}^2+\tilde{c_1}\|v_h\|_{0,T}^2+\|\xi_h\|_{0,T}^2\\ &+\tilde{c_2}\|\nabla\eta_h\|_{0,T}^2+\tilde{c_3}\|\nabla\nu_h\|_{0,T}^2-\tilde{\epsilon_1}(\|\eta_h\|_{0,T}^2+\|div\xi_h\|_{0,T}^2)\\ &-\tilde{\epsilon_2}(\|\eta_h\|_{0,T}^2+\|\nu_h\|_{0,T}^2)-\tilde{\epsilon_3}(\|div\xi_h\|_{0,T}^2+\|\nu_h\|_{0,T}^2)\end{aligned}$$

$$
\begin{aligned}
&- \tilde{\epsilon_4}(\|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2) - \tilde{\epsilon_5}(\|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2) \\
&- \tilde{\epsilon_6}(\|\xi_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2) - \tilde{\epsilon_7}(\|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h)\|_{0,T}^2 \\
&- \tilde{\epsilon_8}(\|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2)) \\
=& \sum_{T\in T_h} ((\tau - \tilde{\epsilon_1} - \tilde{\epsilon_2})\|\eta_h\|_{0,T}^2 + (\tau - \tilde{\epsilon_1} - \tilde{\epsilon_3})\|div\xi_h\|_{0,T}^2 \\
&+ (\tilde{c_1} - \tilde{\epsilon_2} - \tilde{\epsilon_3})\|\nu_h\|_{0,T}^2 + (1 - \tilde{\epsilon_4} - \tilde{\epsilon_5} - \tilde{\epsilon_6})\|\xi_h\|_{0,T}^2 \\
&+ (\tilde{c_2} - \tilde{\epsilon_5} - \tilde{\epsilon_7} - \tilde{\epsilon_8})\|\nabla\eta_h\|_{0,T}^2 \\
&+ (\tilde{c_3} - \tilde{\epsilon_6} - \tilde{\epsilon_7} - \tilde{\epsilon_8})\|\nabla\nu_h\|_{0,T}^2).
\end{aligned}
$$

So we can select constants $\tilde{\epsilon_1}, \tilde{\epsilon_2}, \tilde{\epsilon_3}, \tilde{\epsilon_4}, \tilde{\epsilon_5}, \tilde{\epsilon_6}, \tilde{\epsilon_7}, \tilde{\epsilon_8}, \tilde{c_1}, \tilde{c_2}, \tilde{c_3}$ such that

$$
\begin{aligned}
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|\eta_h\|_{0,T}^2, \\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|div\xi_h\|_{0,T}^2, \\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|\nu_h\|_{0,T}^2, \\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|\xi_h\|_{0,T}^2, \\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|\nabla\eta_h\|_{0,T}^2, \\
B(\nu,\xi,\eta;\nu,\xi,\eta) &\geq c\sum_{T\in T_h}\|\nabla\nu_h\|_{0,T}^2,
\end{aligned}
$$

So we can obtain

$$
\begin{aligned}
&B(\nu_h,\xi_h,\eta_h;\nu_h,\xi_h,\eta_h) \\
\geq&\beta_h\sum_{T\in T_h}(\|\eta_h\|_{0,T}^2 + \|div\xi_h\|_{0,T}^2 + \|\nu_h\|_{0,T}^2 \\
&+ \|\xi_h\|_{0,T}^2 + \|\nabla\eta_h\|_{0,T}^2 + \|\nabla\nu_h\|_{0,T}^2).
\end{aligned}
$$

We can obtain the proof for Theorem 3.1. ■

Theorem 3.2: It is easy to show that Eq.(12) have a unique solution $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$.

IV. The Least-Squares Functional of an A Posteriori Error Estimator

In this section, we use the similar method to Ref [18]. From the least-squares functional

$$
\begin{aligned}
I(u_h^n,\sigma_h^n,\omega_h^n) = \sum_{T\in T_h}(&\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 \\
&+ \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} \\
&+ b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2).
\end{aligned}
$$

So we obtain

$$
\begin{aligned}
&B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n) \\
=&\sum_{T\in T_h}(\|\omega_h^n - \omega^n + div(\sigma_h^n - \sigma^n) + q(u_h^n - u^n)\|_{0,T}^2 \\
&+ \|\frac{\sigma_h^n - \sigma^n - (\sigma_h^{n-1} - \sigma^{n-1})}{\tau} \\
&+ a(x)\frac{\nabla(\omega_h^n - \omega^n) - \nabla(\omega_h^{n-1} - \omega^{n-1})}{\tau} \\
&+ b(x)\nabla(\omega_h^n - \omega^n) + d(x)\nabla(u_h^n - u^n)\|_{0,T}^2).
\end{aligned}
$$

For obtaining the definition of the a posteriori estimator, we can see that the least-squares functional is the sum of its element-wise contributions

$$
\begin{aligned}
I(u_h^n,\sigma_h^n,\omega_h^n) = \sum_{T\in T_h}(&\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 \\
&+ \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} + a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} \\
&+ b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2) =: \sum_{T\in T_h}\theta_T^2.
\end{aligned}
$$

So we can obtain the following theorem.

Theorem 4.1: The element-wise computation of the least-squares functional constitutes an a posteriori error estimator. That is, for

$$
\begin{aligned}
&\|\omega_h^n + div\sigma_h^n + qu_h^n - f^n\|_{0,T}^2 + \|\frac{\sigma_h^n - \sigma_h^{n-1}}{\tau} \\
&+ a(x)\frac{\nabla\omega_h^n - \nabla\omega_h^{n-1}}{\tau} + b(x)\nabla\omega_h^n + d(x)\nabla u_h^n\|_{0,T}^2 = \theta_T^2.
\end{aligned}
$$

there exist positive constants α_T and β_T such that

$$
\begin{aligned}
\sum_{T\in T_h}\theta_T^2 \geq&\beta_T\sum_{T\in T_h}(\|\omega_h^n - \omega^n\|_{0,T}^2 + \|div(\sigma_h^n - \sigma^n)\|_{0,T}^2 \\
&+ \|u_h^n - u^n\|_{0,T}^2 + \|\sigma_h^n - \sigma^n\|_{0,T}^2 \\
&+ \|\nabla(\omega_h^n - \omega^n)\|_{0,T}^2 + \|\nabla(u_h^n - u^n)\|_{0,T}^2),
\end{aligned}
$$

$$
\begin{aligned}
\sum_{T\in T_h}\theta_T^2 \leq&\alpha_T\sum_{T\in T_h}(\|\omega_h^n - \omega^n\|_{0,T}^2 + \|div(\sigma_h^n - \sigma^n)\|_{0,T}^2 \\
&+ \|u_h^n - u^n\|_{0,T}^2 + \|\sigma_h^n - \sigma^n\|_{0,T}^2 \\
&+ \|\nabla(\omega_h^n - \omega^n)\|_{0,T}^2 + \|\nabla(u_h^n - u^n)\|_{0,T}^2).
\end{aligned}
$$

where α_T and β_T are the same constants as in Theorem 3.1.

V. Error Estimates

In this section, we assume that the spaces L_h, H_h can be the spaces of Raviart-Thomas for $r \geq 0$. The completely discrete least-squares mixed finite element approximation is: find $(u_h^n, \sigma_h^n, \omega_h^n) \in X_h := L_h \times H_h \times L_h$ such that

$$
\begin{aligned}
&B(u_h^n,\sigma_h^n,\omega_h^n;\nu_h,\xi_h,\eta_h) \\
=&\tau(\omega_h^n + div\sigma_h^n + qu_h^n, \eta_h + div\xi_h + q\nu_h)_{0,\Omega} \\
&+ (\sigma_h^n - \sigma_h^{n-1} + a(x)(\nabla\omega_h^n - \nabla\omega_h^{n-1}) + \tau b(x)\nabla\omega_h^n \\
&+ \tau d(x)\nabla u_h^n, \xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega}.
\end{aligned}
$$

We shall split the error as a sum of two terms,

$$u - u_h = (u - \tilde{u}) + (\tilde{u} - u_h).$$

where $\tilde{u}$ is an elliptic projection in L_h of the exact solution u. The projection $\tilde{u} = \tilde{u}(t)$ is defined by

$$(d(x)\nabla(u - \tilde{u}), \nabla\chi) = 0, \forall\chi \in L_h. \qquad (15)$$

It is known that the projection $(\tilde{u}, \tilde{\sigma}, \tilde{\omega}) \in X_h := L_h \times H_h \times L_h$ exits and has the following properties [16], [18], for $1 \leq r \leq k+1$

$$\|u - \tilde{u}\| \leq ch^r\|u\|_r,$$

$$\|(u-\tilde{u})_t\| \le ch^r(\|u\|_r + \|u_t\|_r),$$

$$\|\sigma-\tilde{\sigma}\| \le ch^r\|\sigma\|_r,$$

$$\|div(\sigma-\tilde{\sigma})\| \le ch^r\|\sigma\|_{r+1},$$

$$\|(\sigma-\tilde{\sigma})_t\| \le ch^r(\|\sigma\|_r + \|\sigma_t\|_r),$$

From the bilinear form $B(\cdot,\cdot,\cdot;\cdot,\cdot,\cdot)$ in section 2, we have

$$\begin{aligned} &B(u^n,\sigma^n,\omega^n;\nu,\xi,\eta)\\ =&\tau(\omega^n + div\sigma^n + qu^n, \eta + div\xi + q\nu)_{0,\Omega}\\ &+(\sigma^n - \sigma^{n-1} + a(x)(\nabla\omega^n - \nabla\omega^{n-1})\\ &+\tau b(x)\nabla\omega^n + \tau d(x)\nabla u^n, \xi + a(x)\nabla\eta\\ &+\tau b(x)\nabla\eta + \tau d(x)\nabla\nu)_{0,\Omega}. \end{aligned}$$

Thus,

$$\begin{aligned} &B(u_h^n - u^n, \sigma_h^n - \sigma^n, \omega_h^n - \omega^n; \nu_h, \xi_h, \eta_h)\\ =&\tau(\omega^n - \omega_h^n + div(\sigma^n - \sigma_h^n) + q(u^n - u_h^n),\\ &\eta_h + div\xi_h + q\nu_h)_{0,\Omega} + (\sigma^n - \sigma_h^n - (\sigma^{n-1} - \sigma_h^{n-1})\\ &+a(x)(\nabla(\omega^n - \omega_h^n) - \nabla(\omega^{n-1} - \omega_h^{n-1}))\\ &+\tau b(x)\nabla(\omega^n - \omega_h^n) + \tau d(x)\nabla(u^n - u_h^n),\\ &\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega} = 0. \end{aligned}$$

Let $\theta = \sigma - \tilde{\sigma}, \pi = \sigma_h - \tilde{\sigma}, \rho = u - \tilde{u}, \varepsilon = u_h - \tilde{u}, \rho_t = \omega - \tilde{\omega}, \varepsilon_t = \omega_h - \tilde{\omega}$. We shall use the following notation for the difference operator:

$$\bar{\partial}_t f^n = \frac{f^n - f^{n-1}}{\triangle t},$$

We then have the following estimate:

$$\|\bar{\partial}_t\rho^n\|^2 \le \frac{1}{\triangle t}\int_{t_{n-1}}^{t_n}\|\rho_t\|^2 dt,$$

Lemma 5.1: With $\tilde{u}$ difined in (15), we have

$$\|\nabla\tilde{u}\|_{L\infty} \le c, \forall t \in J$$

$$\|\nabla\tilde{u}_{tt}\| \le c, \forall t \in J$$

Please see [16], [18].

From the above equation $B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h)$ and the boundedness of $a(x), b(x), d(x), q(x)$, we can get

$$\begin{aligned} &B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \nu_h, \xi_h, \eta_h)\\ =&\tau(\rho_t^n - \varepsilon_t^n + div(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n),\\ &\eta_h + div\xi_h + q\nu_h)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1})\\ &+a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1}))\\ &+\tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n),\\ &\xi_h + a(x)\nabla\eta_h + \tau b(x)\nabla\eta_h + \tau d(x)\nabla\nu_h)_{0,\Omega}\\ =&0. \end{aligned}$$

Take $\xi_h = \pi^n, \nu_h = \varepsilon^n, \eta_h = \varepsilon_t^n$ to get

$$\begin{aligned} &B(u^n - u_h^n, \sigma^n - \sigma_h^n, \omega^n - \omega_h^n; \varepsilon^n, \pi^n, \varepsilon_t^n)\\ =&\tau(\rho_t^n - \varepsilon_t^n + div(\theta^n - \pi^n) + q(\rho^n - \varepsilon^n),\\ &\varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega} + (\theta^n - \pi^n - (\theta^{n-1} - \pi^{n-1})\\ &+a(x)(\nabla(\rho_t^n - \varepsilon_t^n) - \nabla(\rho_t^{n-1} - \varepsilon_t^{n-1}))\\ &+\tau b(x)\nabla(\rho_t^n - \varepsilon_t^n) + \tau d(x)\nabla(\rho^n - \varepsilon^n),\\ &\pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}\\ =&0. \end{aligned}$$

The left side of the above equation is

$$\begin{aligned} I_3 =&\tau(\rho_t^n + div\theta^n + q\rho^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}\\ &+(\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1})\\ &+\tau b(x)\nabla\rho_t^n + \tau d(x)\nabla\rho^n, \pi^n + a(x)\nabla\varepsilon_t^n\\ &+\tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}. \end{aligned} \tag{16}$$

The right side of the above equation is

$$\begin{aligned} &I_1 + I_2\\ &= \tau(\varepsilon_t^n + div\pi^n + q\varepsilon^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}\\ &+(\pi^n - \pi^{n-1} + a(x)\nabla(\varepsilon_t^n - \varepsilon_t^{n-1}) + \tau b(x)\nabla\varepsilon_t^n\\ &+\tau d(x)\nabla\varepsilon^n, \pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}. \end{aligned} \tag{17}$$

From the boundedness of $a(x), b(x), d(x)$ and $q(x)$, we obtain

$$\begin{aligned} I_2 =&c(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \nabla\varepsilon_t^n\\ &+\tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} - c(\pi^{n-1} + \nabla\varepsilon_t^{n-1}, \pi^n\\ &+\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ \ge&\frac{c}{2}[(\pi^n + \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n\\ &+\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &-(\pi^{n-1}, \pi^{n-1})_{0,\Omega} - (\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega}]\\ =&\frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega}\\ &-(\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n\\ &+\tau\nabla\varepsilon^n)_{0,\Omega} + 2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &+(\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &+(\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}]. \end{aligned}$$

Use $I_1 \ge \frac{c}{2}I_1, (c \le 2)$ to get

$$\begin{aligned} &I_1 + I_2\\ \ge&\frac{c}{2}[(\pi^n, \pi^n)_{0,\Omega} - (\pi^{n-1}, \pi^{n-1})_{0,\Omega} + (\nabla\varepsilon_t^n, \nabla\varepsilon_t^n)_{0,\Omega}\\ &-(\nabla\varepsilon_t^{n-1}, \nabla\varepsilon_t^{n-1})_{0,\Omega} + 2(\pi^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &+2(\nabla\varepsilon_t^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega} + (\nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n\\ &+\tau\nabla\varepsilon^n, \nabla\varepsilon_t^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &+(\pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n, \pi^n + \tau\nabla\varepsilon_t^n + \tau\nabla\varepsilon^n)_{0,\Omega}\\ &+\tau(\varepsilon_t^n + div\pi^n + q\varepsilon^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega}]\\ =&I_4. \end{aligned}$$

Summing from $n=0$ to N, we obtain

$$\sum_{n=0}^{N} I_4 = c(\|\pi^N\|^2_{0,\Omega} - \|\pi^0\|^2_{0,\Omega}) + c(\|\nabla\varepsilon_t^N\|^2_{0,\Omega} - \|\nabla\varepsilon_t^0\|^2_{0,\Omega}) + c\sum_{n=0}^{N}(\|\pi^n\|^2_{0,\Omega} + \|\nabla\varepsilon_t^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon_t^n\|^2_{0,\Omega}) \le \sum_{n=0}^{N} I_3$$

Let $\|\pi^{N^*}\|^2_{0,\Omega}$ be a maximum of the $\|\pi^n\|^2_{0,\Omega}$ and $\|\nabla\varepsilon_t^{N^*}\|^2_{0,\Omega}$ be a maximum of the $\|\nabla\varepsilon_t^n\|^2_{0,\Omega}$, we obtain

$$\sum_{n=0}^{N} I_4 = c(\|\pi^N\|^2_{0,\Omega} + \|\nabla\varepsilon_t^N\|^2_{0,\Omega}) + c\sum_{n=0}^{N}(\|\pi^n\|^2_{0,\Omega} + \|\nabla\varepsilon_t^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon_t^n\|^2_{0,\Omega}). \tag{18}$$

$$I_3 = \tau(\rho_t^n + div\theta^n + q\rho^n, \varepsilon_t^n + div\pi^n + q\varepsilon^n)_{0,\Omega} + (\theta^n - \theta^{n-1} + a(x)\nabla(\rho_t^n - \rho_t^{n-1}) + \tau b(x)\nabla\rho_t^n + \tau d(x)\nabla\rho^n, \pi^n + a(x)\nabla\varepsilon_t^n + \tau b(x)\nabla\varepsilon_t^n + \tau d(x)\nabla\varepsilon^n)_{0,\Omega}$$

$$\begin{aligned}
=&\tau(\rho_t^n,\varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, div\pi^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} \\
&+ \tau(div\theta^n,\varepsilon_t^n)_{0,\Omega} + \tau(div\theta^n, div\pi^n)_{0,\Omega} \\
&+ \tau(div\theta^n, q\varepsilon^n)_{0,\Omega} + \tau(q\rho^n,\varepsilon_t^n)_{0,\Omega} \\
&+ \tau(q\rho^n, div\pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\
&+ (\theta^n - \theta^{n-1},\pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\theta^n - \theta^{n-1},\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1},\tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\pi^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n,\pi^n)_{0,\Omega} + (\tau b(x)\nabla\rho_t^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n,\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n,\tau d(x)\nabla\varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n,\pi^n)_{0,\Omega} \\
&+ (\tau d(x)\nabla\rho^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n,\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\tau d(x)\nabla\rho^n,\tau d(x)\nabla\varepsilon^n)_{0,\Omega}.
\end{aligned}$$

From (15) and applying Green's formula, we know that

$$(\nabla\rho_t^n,\pi^n)_{0,\Omega} = -(\rho_t^n, div\pi^n)_{0,\Omega},$$
$$(\tau d(x)\nabla\rho^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} = 0,$$

$$(\tau d(x)\nabla\rho^n,\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} = 0,$$
$$(\tau d(x)\nabla\rho^n,\tau d(x)\nabla\varepsilon^n)_{0,\Omega} = 0.$$

So we obtain

$$\begin{aligned}
I_3 =& \tau(\rho_t^n,\varepsilon_t^n)_{0,\Omega} + \tau(\rho_t^n, q\varepsilon^n)_{0,\Omega} + \tau(div\theta^n,\varepsilon_t^n)_{0,\Omega} \\
&+ \tau(div\theta^n, div\pi^n)_{0,\Omega} + \tau(div\theta^n, q\varepsilon^n)_{0,\Omega} \\
&+ \tau(q\rho^n,\varepsilon_t^n)_{0,\Omega} + \tau(q\rho^n, div\pi^n)_{0,\Omega} + \tau(q\rho^n, q\varepsilon^n)_{0,\Omega} \\
&+ (\theta^n - \theta^{n-1},\pi^n)_{0,\Omega} + (\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\theta^n - \theta^{n-1},\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} + (\theta^n - \theta^{n-1},\tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\pi^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (a(x)\nabla(\rho_t^n - \rho_t^{n-1}),\tau d(x)\nabla\varepsilon^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n,\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega} \\
&+ (\tau b(x)\nabla\rho_t^n,\tau d(x)\nabla\varepsilon^n)_{0,\Omega} + (\tau d(x)\nabla\rho^n,\pi^n)_{0,\Omega}.
\end{aligned}$$

We obtain

$$\sum_{n=0}^{N} I_4 = c(\|\pi^N\|^2_{0,\Omega} + \|\nabla\varepsilon_t^N\|^2_{0,\Omega}) + c\sum_{n=0}^{N}(\|\pi^n\|^2_{0,\Omega} + \|\nabla\varepsilon_t^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon^n\|^2_{0,\Omega} + \tau^2\|\nabla\varepsilon_t^n\|^2_{0,\Omega}) \le \sum_{i=1}^{20} T_i.$$

where

$$T_1 = \sum_{n=0}^{N}\tau(\rho_t^n,\varepsilon_t^n)_{0,\Omega}, T_2 = \sum_{n=0}^{N}\tau(\rho_t^n, q\varepsilon^n)_{0,\Omega},$$

$$T_3 = \sum_{n=0}^{N}\tau(div\theta^n,\varepsilon_t^n)_{0,\Omega}, T_4 = \sum_{n=0}^{N}\tau(div\theta^n, div\pi^n)_{0,\Omega},$$

$$T_5 = \sum_{n=0}^{N}\tau(div\theta^n, q\varepsilon^n)_{0,\Omega}, T_6 = \sum_{n=0}^{N}\tau(q\rho^n,\varepsilon_t^n)_{0,\Omega},$$

$$T_7 = \sum_{n=0}^{N}\tau(q\rho^n, div\pi^n)_{0,\Omega}, T_8 = \sum_{n=0}^{N}\tau(q\rho^n, q\varepsilon^n)_{0,\Omega},$$

$$T_9 = \sum_{n=0}^{N}(\theta^n - \theta^{n-1},\pi^n)_{0,\Omega},$$

$$T_{10} = \sum_{n=0}^{N}(\theta^n - \theta^{n-1}, a(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{11} = \sum_{n=0}^{N}(\theta^n - \theta^{n-1},\tau b(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{12} = \sum_{n=0}^{N}(\theta^n - \theta^{n-1},\tau d(x)\nabla\varepsilon^n)_{0,\Omega},$$

$$T_{13} = \sum_{n=0}^{N}(a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \pi^n)_{0,\Omega},$$

$$T_{14} = \sum_{n=0}^{N}(a(x)\nabla(\rho_t^n - \rho_t^{n-1}), a(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{15} = \sum_{n=0}^{N}(a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{16} = \sum_{n=0}^{N}(a(x)\nabla(\rho_t^n - \rho_t^{n-1}), \tau d(x)\nabla\varepsilon^n)_{0,\Omega},$$

$$T_{17} = \sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n, a(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{18} = \sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n, \tau b(x)\nabla\varepsilon_t^n)_{0,\Omega},$$

$$T_{19} = \sum_{n=0}^{N}(\tau b(x)\nabla\rho_t^n, \tau d(x)\nabla\varepsilon^n)_{0,\Omega},$$

$$T_{20} = \sum_{n=0}^{N}(\tau d(x)\nabla\rho^n, \pi^n)_{0,\Omega},$$

From the properties cited in [18] and applying Green's formula, we obtain estimates of T_i:

$$|T_1| \le \sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r}, |T_2| \le \sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_3| \le \sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r}, |T_4| \le \sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_5| \le \sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r}, |T_6| \le \sum_{n=0}^{N}\tau\|\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_7| \le \sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2 + ch^{2r}, |T_8| \le \sum_{n=0}^{N}\tau\|\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_9| \le \sum_{n=0}^{N}\|\pi^n\|_{0,\Omega}^2 + ch^{2r}, |T_{10}| \le \sum_{n=0}^{N}\|\nabla\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{11}| \le \sum_{n=0}^{N}\tau\|\nabla\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{12}| \le \sum_{n=0}^{N}\tau\|\nabla\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{13}| \le \sum_{n=0}^{N}\|div\pi^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{14}| \le \sum_{n=0}^{N}\|\triangle\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{15}| \le \sum_{n=0}^{N}\tau\|\triangle\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{16}| \le \sum_{n=0}^{N}\tau\|\triangle\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{17}| \le \sum_{n=0}^{N}\tau\|\triangle\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{18}| \le \sum_{n=0}^{N}\tau^2\|\triangle\varepsilon_t^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{19}| \le \sum_{n=0}^{N}\tau^2\|\triangle\varepsilon^n\|_{0,\Omega}^2 + ch^{2r},$$

$$|T_{20}| \le \sum_{n=0}^{N}\tau\|div\pi^n\|_{0,\Omega}^2 + ch^{2r},$$

So we have

$$\sum_{n=0}^{N} I_3 \le c(\tau^2 + h^{2r}).$$

Note that

$$\begin{aligned}\|u^N\|_{0,\Omega}^2 =& \sum_{n=1}^{N}\frac{\|u^n\|_{0,\Omega}^2 - \|u^{n-1}\|_{0,\Omega}^2}{\tau}\tau \\ \le& c\sum_{n=1}^{N}(\tau\|u^n\|_{0,\Omega}^2 + \|\omega^n\|_{0,\Omega}^2),\end{aligned}$$

where $\omega^n = \frac{u^n - u^{n-1}}{\tau}$.

So we can obtain

$$\begin{aligned}&\max_{0\le n\le t/\tau}(\|\sigma^n - \sigma_h^n\|_{0,\Omega}^2 + \|u^n - u_h^n\|_{0,\Omega}^2 + \|\omega^n - \omega_h^n\|_{0,\Omega}^2) \\ \le& c(\tau^2 + h^{2r})\end{aligned}$$

Thus, we obtain the following theorem

Theorem 5.2: For $\tau > 0$, we have

$$\begin{aligned}&\max_{0\le n\le t/\tau}(\|\sigma^n - \sigma_h^n\|_{0,\Omega}^2 + \|u^n - u_h^n\|_{0,\Omega}^2 + \|\omega^n - \omega_h^n\|_{0,\Omega}^2) \\ &= O(\tau^2 + h^{2r}).\end{aligned} \tag{19}$$

ACKNOWLEDGMENT

This work is supported by National Natural Science Fund (No. 11061021), the Scientific Research Projection of Higher Schools of Inner Mongolia (No. NJ10006) and YSF of Inner Mongolia University (No. ND0702)

REFERENCES

[1] R Ewing, R Lazarov, Y P Lin. Finite volume element approximations of nonlocal reactive flows in porous media, Numer. Methods Partial Differential Equations, 2000, 16: 285-311.

[2] Y Lin, V Thomee, L Wahlbin. Ritz-Volterra projection onto finite element spaces and application to integro-differential and related equations, SIAM J. Numer. Anal., 1995, 28: 1047-1070.

[3] A K Pani, G Fairweather. H^1-Galerkin mixed finite element methods for parabolic partial integro-differential equations, IMA Journal of Numerical Analysis, 2002, 22: 231-252.

[4] Y Liu, H Li S He. Error estimates of H^1-Galerkin mixed finite element methods for pseudo-hyperbolic partial integro-differential equation, Numerical Mathematics: A Journal of Chinese Universities, 2010, 32(1): 1-20

[5] D Y Shi, H H Wang. An H^1-Galerkin nonconforming mixed finite element method for integro-differential equation of parabolic type, Journal of Mathematical Research and Exposition, 2009, 29(5): 871-881.

[6] Z J Zhou, H Z Chen. H^1-Galerkin mixed finite element method for pseudo-parabolic integro-differential equation, Journal of Shandong Normal University(Natural Science), 2005, 20(2): 3-7.

[7] H Li, Y Liu. Mixed discontinuous space-time finite element method for the fourth-order parabolic integro-differential equations, Mathematica Numerica Sinica, 2007, 29(4): 413-420.

[8] H Li, H Q Wang. The space-time discontinuous finite element method for a semi-linear parabolic integro-differential equation, Mathematica Numerica Sinica, 2006, 28(3): 293-308.

[9] H Guo, H X Rui. Least-squares Galerkin procedures for parabolic integro-differential equations, Appl. Math. Comput., 2004, 150: 749-762.

[10] H Guo, H X Rui. Crank-Nicolson least-squares Galerkin procedures for parabolic integro-differential equations, Appl. Math. Comput., 2006, 180: 622-634.

[11] X Cui. Sobolev-Volterra projection and numerical analysis of finite element methods for integro-differential equations, Acta Mathematicae Applicatae Sinica, 2001, 24(3): 441-454

[12] T Zhang, C J Li. Superconvergence of finite element approximations to parabolic and hyperbolic integro-differential equations, Northeastern Math., 2001, (3): 279-288.

[13] Z D Luo, J Zhu, H J Wang. A nonlinear Galerkin/petrov-least squares mixed element method for the stationary Navier-Stokes equations, Applied Mathematics and Mechanics, 2002, 23(7): 697-706.

[14] D P Yang. Least-square mixed finite element methods for nonlinear parabolic problems, J. Comput. Math., 2002, 20(2): 153-164.

[15] Y P Chen, Y Q Huang, Z H Shen. Least-squares mixed finite element method for degenerate elliptic problems, Numer. Math. Sinica, 2001, 23: 87-94.

[16] Z Q Cai, K Johannes, S Gerhard. An adaptive least-squares mixed finite element method for the stress-displacement formulation of linear elasticity, Numer. Methods Partial Differential Equations, 2005, 21: 132-148.

[17] H Y Duan, Q Lin . Mixed finite elements of least-squares type for elasticity, Computer Methods in Applied Mechanics and Engineering, 2005, 194(9-11), 1093-1112.

[18] H M Gu, H W Li. An adaptive least-squares mixed finite element method for nonlinear parabolic problems, Computational Mathematics and Modeling, 2009, 20(2): 192-206. .