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Some exact solutions of the (2+1)-dimensional breaking soliton equation using the three-wave method

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Abstract—This paper considers the (2+1)-dimensional breaking soliton equation in its bilinear form. Some exact solutions to this equation are explicitly derived by the idea of three-wave solution method with the assistance of Maple. We can see that the new idea is very simple and straightforward.

Keywords—Soliton solution, Computerized symbolic computation, Painlevé analysis, (2+1)-dimensional breaking soliton equation, Hirota's bilinear form.

I. Introduction

We can find many important phenomena and dynamic processes in physics, mechanics, chemistry and biology which can be represented by nonlinear partial differential equations. The study of exact solutions of nonlinear evolution equations plays an important role in soliton theory and explicit formulas of nonlinear partial differential equations play an essential role in the nonlinear science. Also, the explicit formulas may provide physical information and help us to understand the mechanism of related physical models.

In recent years, many kinds of powerful methods have been proposed to find solutions of nonlinear partial differential equations, numerically and/or analytically, e.g., the variational iteration method [1], [2], [3], the Adomian decomposition method [4], [5], the homotopy perturbation method [6], [7], [8], [9], [10], parameter expansion method [11], [12], [13], spectral collocation method [14], [15], [16], [17], [18], homotopy analysis method [19], [20], [21], [22], [23], [24], [25], and the Exp-function method [26], [27], [28], [29], [30], [31].

In this paper, by means of the idea of the three-wave method, we will obtain some exact solutions for the (2+1)-dimensional breaking soliton equation in its bilinear form. The paper is organized as follows: in the following section we have a brief review on the three-wave method. In Section III we obtain some exact solutions for the (2+1)-dimensional breaking soliton equation. In Section IV we obtain some soliton solutions for the (2+1)-dimensional Bogoyavlenskii's breaking soliton equation. Finally the paper is concluded in Section V.

II. Methodology

Dai et al. [32], suggested the three-wave method for nonlinear evolution equations. The basic idea of this method applies the Painlevè analysis to make a transformation as

$$u = T(f) \tag{1}$$

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for some new and unknown function f. Then we use this transformation in a high dimensional nonlinear equation of the general form

$$F(u, u_t, u_x, u_y, u_z, u_{xx}, u_{yy}, u_{zz}, \cdots) = 0, \tag{2}$$

where $u = u(x, y, z, t)$ and F is a polynomial of u and its derivatives. By substituting (1) in (2), the first one converts into the Hirota's bilinear form, which it will solve by taking a special form for f and assuming that the obtained Hirota's bilinear form has three-wave solutions, then we can specify the unknown function f, (for more details see [32], [33]).

III. (2+1)-dimensional breaking soliton equation

In this section, we investigate explicit soliton solutions of the following (2+1)-dimensional breaking soliton equation given in [34]

$$u_{xxxy} - 2\,u_y\,u_{xx} - 4\,u_x\,u_{xy} + u_{xt} = 0. \tag{3}$$

Equation (3) is used to describe the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with a long wave along the x-axis, which was first described by Calogero and Degasperis in 1977. To solve eq. (3) authors in [34] used of N-soliton solution. In this paper, we use the idea of three-wave method [32], [33], to solve equation (3). By this idea we obtain some analytic solutions for the problem. The process of the method is very easy and more simple than the method of Zheng et al. [34]. To solve eq. (3), we introduce a new dependent variable w by

$$w = -2(\ln f)_x \tag{4}$$

where $f(x, y, t)$ is an unknown real function which will be determined. Substituting eq. (4) into eq. (3), we have

$$\begin{aligned} 2(\ln f)_{xxt} + 2(\ln f)_{xxxxy} + 16(\ln f)_{xx}\,(\ln f)_{xxy} + \\ 8(\ln f)_{xxx}\,(\ln f)_{xy} = 0, \end{aligned} \tag{5}$$

which can be integrated once with respect to x to give

$$\begin{aligned} &2(\ln f)_{xt} + 2(\ln f)_{xxxy} + 12(\ln f)_{xx}\,(\ln f)_{xy} \\ &+4\partial_x^{-1}((\ln f)_{xx}\,(\ln f)_{xxy} - (\ln f)_{xxx}\,(\ln f)_{xy}) = 0. \end{aligned} \tag{6}$$

Thus, eq. (6) can be written as

$$(D_x D_t + D_y D_x^3) f \cdot f + 4\,f^2\,\partial_x^{-1}(D_x(\ln f)_{xx} \cdot (\ln f)_{xy}) = 0, \tag{7}$$

where the D-operator, e.g. for two-variable functions is defined by

$$D_x^m D_t^n f(x,t)\cdot g(x,t)=$$

$$(\tfrac{\partial}{\partial x_1}-\tfrac{\partial}{\partial x_2})^m(\tfrac{\partial}{\partial t_1}-\tfrac{\partial}{\partial t_2})^n[f(x_1,t_1)g(x_2,t_2)]\Big|_{\substack{x_1=x_2=x\\ t_1=t_2=t}}$$

to reduce eq. (7) to

$$(D_xD_t+D_yD_x^3)f\cdot f=0, \tag{8}$$

we follow the assumption of [34] by assuming

$$\partial_x^{-1}(D_x(\ln f)_{xx}\cdot(\ln f)_{xy})=0,$$

where $\partial_x^{-1}\partial_x=\partial_x\partial_x^{-1}=1$. Now we suppose the solution of eq. (8) as

$$f(x,y,t)=\mathrm{e}^{-\xi_1}+\delta_1\,\cos(\xi_2)+\delta_2\,\cosh(\xi_3)+\delta_3\,\mathrm{e}^{\xi_1} \tag{9}$$

where

$$\xi_i=a_ix+b_iy+c_it,\quad i=1,2,3 \tag{10}$$

and a_i, c_i, δ_i are some constants to be determined later. Substituting eq. (9) into eq. (8), and equating all coefficients of $\sin(a_2\,x+b_2\,y+c_2\,t)$, $\cos(a_2\,x+b_2\,y+c_2\,t)$, $\sinh(a_3\,x+b_3\,y+c_3\,t)$ and $\cosh(a_3\,x+b_3\,y+c_3\,t)$ to zero, we get the set of algebraic equation for a_i,b_i,c_i, δ_i, $(i=1,2,3)$

$$\begin{gathered}
-3\,a_1{}^2\,b_1\,a_3-a_1{}^3\,b_3-3\,b_3a_3{}^2\,a_1-\\
a_3{}^3\,b_1-c_3\,a_1-c_1\,a_3=0,\\
3\,a_1\,b_1\,a_3{}^2+c_1\,a_1+a_3\,c_3+a_1{}^3b_1+\\
b_3\,a_3{}^3+3\,b_3\,a_3\,a_1{}^2=0,\\
-a_2\,c_2+b_2\,a_2{}^3+a_1{}^3\,b_1+c_1\,a_1-\\
3\,b_2\,a_2\,a_1{}^2-3\,a_1\,b_1a_2{}^2=0,\\
a_1{}^3\,b_2+c_2\,a_1+3\,a_1{}^2\,b_1\,a_2+\\
c_1\,a_2-a_2{}^3\,b_1-3\,b_2\,a_2{}^2\,a_1=0,\\
-a_2{}^3b_3+c_2\,a_3+c_3\,a_2+a_3{}^3b_2-\\
3\,b_2a_2{}^2a_3+3\,b_3a_3{}^2a_2=0,\\
a_3\,c_3+b_3a_3{}^3-a_2\,c_2-3\,b_3\,a_3a_2{}^2+\\
b_2\,a_2{}^3-3\,b_2\,a_2a_3{}^2=0,\\
16\,a_1{}^3b_1\delta_3+4\,c_1\,a_1\delta_3\,-\delta_1{}^2c_2\,a_2+\\
\delta_2{}^2c_3\,a_3+4\,\delta_1{}^2a_2{}^3b_2+4\,\delta_2{}^2a_3{}^3b_3=0.
\end{gathered} \tag{11}$$

Solving the system of equations (11) with the aid of Maple, we obtain the following cases:

A. Case 1:

$$\begin{gathered}
a_1=a_3,a_2=0,b_1=-b_3,b_3=-\tfrac{c_3}{a_3{}^2},\\
c_1=-c_3,c_2=-a_3{}^2\,b_2,\delta_1=0,\delta_3=\tfrac{\delta_2{}^2}{4},
\end{gathered} \tag{12}$$

for some arbitrary real constants a_3 ,c_3 ,b_2 and δ_2. Substitute eq. (12) into eq. (4) with eq. (9), we obtain the solution as

$$f(x,y,t)=\mathrm{e}^{-\xi_1}+\delta_2\cosh(\xi_2)+\delta_3\mathrm{e}^{\xi_1}$$

and

$$u(x,y,t)=\frac{-2(-a_3\mathrm{e}^{-\xi_1}+\delta_2\sinh(\xi_2)\,a_3+\delta_3a_3\mathrm{e}^{\xi_1})}{\mathrm{e}^{-\xi_1}+\delta_2\cosh(\xi_2)+\delta_3\mathrm{e}^{\xi_1}} \tag{13}$$

for

$$\xi_1=a_3x-b_3y-c_3t,\quad \xi_2=a_3x+b_3y+c_3t$$

$$b_3=-\tfrac{c_3}{a_3{}^2},\quad \delta_3=\tfrac14\,\delta_2{}^2.$$

If $\delta_3>0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t)=\frac{-2\,a_3\left(2\,\sqrt{\delta_3}\sinh(\xi_1-\theta)+\delta_2\sinh(\xi_2)\right)}{2\,\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_2\cosh(\xi_2)}$$

for

$$\theta=\frac12\,\ln(\delta_3)\quad,\quad \delta_3=\frac14\,\delta_2{}^2.$$

If $\delta_3<0$, then we obtain the exact breather cross-kink solution

$$u(x,y,t)=\frac{-2a_3\left(2\,\sqrt{-\delta_3}\cosh(\xi_1-\theta)+\delta_2\sinh(\xi_2)\right)}{2\,\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_2\cosh(\xi_2)}$$

for

$$\theta=\frac12\,\ln(-\delta_3)\quad,\quad \delta_3=\frac14\,\delta_2{}^2.$$

B. Case 2:

$$a_1=a_3,b_1=b_3,c_1=c_3=-4\,b_3a_3{}^2,\delta_1=0$$

$$c_2=-\tfrac12\,\tfrac{b_3\left(a_2{}^4+6\,a_3{}^2a_2{}^2-3\,a_3{}^4\right)}{a_2a_3},b_2=-\tfrac12\,\tfrac{b_3\left(a_2{}^2+3\,a_3{}^2\right)}{a_2a_3} \tag{14}$$

for some arbitrary real constants $a_3,a_2,b_3,\delta_i,\ i=1,2.$ Substitute eq. (14) into eq. (4) with eq. (9), we obtain the solution as follows

$$f(x,y,t)=\mathrm{e}^{-\xi_1}+\delta_2\cosh(\xi_1)+\delta_3\mathrm{e}^{\xi_1}$$

and

$$u(x,y,t)=\frac{-2(-a_3\mathrm{e}^{-\xi_1}+\delta_2\sinh(\xi_1)\,a_3+\delta_3a_3\mathrm{e}^{\xi_1})}{\mathrm{e}^{-\xi_1}+\delta_2\cosh(\xi_1)+\delta_3\mathrm{e}^{\xi_1}} \tag{15}$$

for

$$\xi_1=a_3x+b_3y-4\,b_3a_3{}^2t.$$

If $\delta_3>0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t)=\frac{-2\,a_3\left(2\,\sqrt{\delta_3}\sinh(\xi_1-\theta)+\delta_2\sinh(\xi_1)\right)}{2\,\sqrt{\delta_3}\cosh(\xi_1-\theta)+\delta_2\cosh(\xi_1)}$$

for

$$\theta=\frac12\,\ln(\delta_3).$$

If $\delta_3<0$ then we obtain the exact breather cross-kink solution

$$u(x,y,t)=\frac{-2\,a_3\left(2\,\sqrt{-\delta_3}\cosh(\xi_1-\theta)+\delta_2\sinh(\xi_1)\right)}{-2\,\sqrt{-\delta_3}\sinh(\xi_1-\theta)+\delta_2\cosh(\xi_1)}.$$

for

$$\theta=\frac12\,\ln(-\delta_3).$$

IV. (2+1)-DIMENSIONAL BOGOYAVLENSKII'S BREAKING SOLITON EQUATION

In this part, we obtain some explicit formula of solutions of the following (2+1)-dimensional Bogoyavlenskii's breaking soliton equation given in [35]

$$u_{xxxy} + 4\,u_y\,u_{xx} + 4\,u_x\,u_{xy} + u_{xt} = 0. \tag{16}$$

To solve eq. (16) author in [35] used the Bilinear Bäcklund transformation and explicit solution. In this paper, we use the idea of three-wave method [32], [33], to solve equation (16). By this idea we obtain some analytic solutions for the problem. To solve eq. (16), we introduce a new dependent variable w by

$$u = \partial_x\, w. \tag{17}$$

After this, the breaking soliton equation can be written as

$$w_{xxxxy} + 4\,w_{xy}\,w_{xxx} + 4\,w_{xx}\,w_{xxy} + w_{txx} = 0, \tag{18}$$

which can be integrated once with respect to x to give the potential form of breaking soliton equation

$$w_{xxxy} + 4\,w_{xx}\,w_{xy} + w_{tx} = 0, \tag{19}$$

by using the Hirota's bilinear method [36] and D-operator, we set

$$u = \frac{3}{2}(\ln f)_x \tag{20}$$

where $f(x,y,t)$ is an unknown real function which will be determined. Substituting eq. (20) into eq. (16), we obtain the following Hirota's bilinear form

$$(D_xD_t + D_yD_x^3)f\cdot f = 0. \tag{21}$$

Now we suppose the solution of eq. (21) as

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_1\,\cos\left(\xi_2\right) + \delta_2\,\cosh\left(\xi_3\right) + \delta_3\,\mathrm{e}^{\xi_1} \tag{22}$$

where

$$\xi_i = a_ix + b_iy + c_it, \quad i = 1,2,3 \tag{23}$$

and a_i, c_i, δ_i are some constants to be determined later. Substituting eq. (22) into eq. (21), and equating all coefficients of $\sin\left(a_2\,x + b_2\,y + c_2\,t\right)$, $\cos\left(a_2\,x + b_2\,y + c_2\,t\right)$, $\sinh\left(a_3\,x + b_3\,y + c_3\,t\right)$ and $\cosh\left(a_3\,x + b_3\,y + c_3\,t\right)$ to zero, we get the set of algebraic equation for a_i,b_i,c_i, δ_i, $(i = 1,2,3)$

$$\begin{aligned}
&-3\,{a_1}^2\,b_1\,a_3 - {a_1}^3\,b_3 - 3\,b_3{a_3}^2\,a_1 - \\
&\qquad {a_3}^3\,b_1 - c_3\,a_1 - c_1\,a_3 = 0,\\
&3\,a_1\,b_1\,{a_3}^2 + c_1\,a_1 + a_3\,c_3 + {a_1}^3b_1 + \\
&\qquad b_3\,{a_3}^3 + 3\,b_3\,a_3\,{a_1}^2 = 0,\\
&-a_2\,c_2 + b_2\,{a_2}^3 + {a_1}^3\,b_1 + c_1\,a_1 - \\
&\qquad 3\,b_2\,a_2\,{a_1}^2 - 3\,a_1\,b_1{a_2}^2 = 0,\\
&{a_1}^3\,b_2 + c_2\,a_1 + 3\,{a_1}^2\,b_1\,a_2 + c_1\,a_2 - \\
&\qquad {a_2}^3\,b_1 - 3\,b_2\,{a_2}^2\,a_1 = 0,\\
&-{a_2}^3b_3 + c_2\,a_3 + c_3\,a_2 + {a_3}^3b_2 - \\
&\qquad 3\,b_2{a_2}^2a_3 + 3\,b_3{a_3}^2a_2 = 0,\\
&a_3\,c_3 + b_3{a_3}^3 - a_2\,c_2 - 3\,b_3\,a_3{a_2}^2 + \\
&\qquad b_2\,{a_2}^3 - 3\,b_2\,a_2{a_3}^2 = 0,\\
&16\,{a_1}^3b_1\delta_3 + 4\,c_1\,a_1\delta_3\, - {\delta_1}^2c_2\,a_2 + {\delta_2}^2c_3\,a_3 + \\
&\qquad 4\,{\delta_1}^2{a_2}^3b_2 + 4\,{\delta_2}^2{a_3}^3b_3 = 0.
\end{aligned} \tag{24}$$

Solving the system of equations (24) with the aid of Maple, we obtain the following cases:

A. Case 1:

$$\begin{aligned}
&a_1 = a_3, a_2 = 0, b_1 = -b_3, b_3 = -\tfrac{c_3}{{a_3}^2},\\
&c_1 = -c_3, c_2 = -{a_3}^2\,b_2, \delta_1 = 0, \delta_3 = \tfrac{{\delta_2}^2}{4},
\end{aligned} \tag{25}$$

for some arbitrary real constants a_3 ,c_3 ,b_2 and δ_2. Substitute eq. (25) into eq. (20) with eq. (22), we obtain the solution as

$$f\,(x,y,t) = \mathrm{e}^{-\xi_1} + \delta_2\cosh\left(\xi_2\right) + \delta_3\mathrm{e}^{\xi_1}$$

and

$$u\,(x,y,t) = \frac{3}{2}\,\frac{-a_3\mathrm{e}^{-\xi_1} + \delta_2\sinh\left(\xi_2\right)a_3 + \delta_3a_3\mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_2\cosh\left(\xi_2\right) + \delta_3\mathrm{e}^{\xi_1}} \tag{26}$$

for

$$\xi_1 = a_3x - b_3y - c_3t\,, \quad \xi_2 = a_3x + b_3y + c_3t$$

$$b_3 = -\tfrac{c_3}{{a_3}^2}, \quad \delta_3 = \tfrac{1}{4}\,{\delta_2}^2.$$

If $\delta_3 > 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{3}{2}\,\frac{a_3\left(2\sqrt{\delta_3}\sinh\left(\xi_1-\theta\right) + \delta_2\sinh\left(\xi_2\right)\right)}{2\sqrt{\delta_3}\cosh\left(\xi_1-\theta\right) + \delta_2\cosh\left(\xi_2\right)}.$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_3) \quad , \quad \delta_3 = \frac{1}{4}\,{\delta_2}^2.$$

If $\delta_3 < 0$, then we obtain the exact breather cross-kink solution

$$u\,(x,y,t) = \frac{3}{2}\,\frac{a_3\left(2\sqrt{-\delta_3}\cosh\left(\xi_1-\theta\right) + \delta_2\sinh\left(\xi_2\right)\right)}{2\sqrt{-\delta_3}\sinh\left(\xi_1-\theta\right) + \delta_2\cosh\left(\xi_2\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_3) \quad , \quad \delta_3 = \frac{1}{4}\,{\delta_2}^2.$$

B. Case 2:

$$\begin{array}{c} a_1 = a_3, b_1 = b_3, c_1 = c_3 = -4\, b_3 {a_3}^2, \delta_1 = 0 \\ c_2 = -\frac{1}{2}\,\frac{b_3\left({a_2}^4 + 6\, {a_3}^2 {a_2}^2 - 3\, {a_3}^4\right)}{a_2 a_3}, b_2 = -\frac{1}{2}\,\frac{b_3\left({a_2}^2 + 3\, {a_3}^2\right)}{a_2 a_3} \end{array} \quad (27)$$

for some arbitrary real constants $a_3, a_2, b_3, \delta_i,\ \ i = 1, 2$. Substitute eq. (27) into eq. (20) with eq. (22), we obtain the solution as follows

$$f\left(x,y,t\right) = \mathrm{e}^{-\xi_1} + \delta_2 \cosh\left(\xi_1\right) + \delta_3 \mathrm{e}^{\xi_1}$$

and

$$u\left(x,y,t\right) = \frac{3}{2}\,\frac{-a_3 \mathrm{e}^{-\xi_1} + \delta_2 \sinh\left(\xi_1\right) a_3 + \delta_3 a_3 \mathrm{e}^{\xi_1}}{\mathrm{e}^{-\xi_1} + \delta_2 \cosh\left(\xi_1\right) + \delta_3 \mathrm{e}^{\xi_1}} \quad (28)$$

for

$$\xi_1 = a_3 x + b_3 y - 4\, b_3 {a_3}^2 t.$$

If $\delta_3 > 0$ then we obtain the exact breather cross-kink solution

$$u\left(x,y,t\right) = \frac{3}{2}\,\frac{a_3\left(2\sqrt{\delta_3}\sinh\left(\xi_1 - \theta\right) + \delta_2 \sinh\left(\xi_1\right)\right)}{2\sqrt{\delta_3}\cosh\left(\xi_1 - \theta\right) + \delta_2 \cosh\left(\xi_1\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(\delta_3).$$

If $\delta_3 < 0$ then we obtain the exact breather cross-kink solution

$$u\left(x,y,t\right) = \frac{3}{2}\,\frac{a_3\left(2\sqrt{-\delta_3}\cosh\left(\xi_1 - \theta\right) + \delta_2 \sinh\left(\xi_1\right)\right)}{-2\sqrt{-\delta_3}\sinh\left(\xi_1 - \theta\right) + \delta_2 \cosh\left(\xi_1\right)}$$

for

$$\theta = \frac{1}{2}\,\ln(-\delta_3).$$

V. Conclusions

In this paper, using the idea of three-wave method we obtained some explicit solutions for the (2+1)-dimensional breaking soliton and the (2+1)-dimensional Bogoyavlenskii's breaking soliton equations. By comparison of three-wave method and another analytic methods, like HAM, HTA and EHTA methods, we can see that the new idea is very easy and straightforward which can be applied on another nonlinear partial differential equations.

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