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# On the Fp-Normal Subgroups of Finite Groups

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***Abstract*—**Let $G$ be a finite group, and let $F$ be a formation of finite group. We say that a subgroup $H$ of $G$ is $F_p$-normal in G if there exists a normal subgroup T of $G$ such that HT is a permutable Hall subgroup of G and $(H \cap T)H_G / H_G$ is contained in the F-hypercenter $Z_\infty^F(G/H_G)$ of $G/H_G$. In this note, we get some results about the $F_p$-normal subgroups and then use them to study the structure of finite groups.

***Keywords*—**Finite group, $F_p$ -normal subgroup, Sylow subgroup, Maximal subgroup



## I. Introduction

ALL groups considered in this paper are finite, and $G$ denotes a finite group. The notation and terminology are standard, as in [1].Recall that, for a class $F$ of groups, a chief factor $H/K$ of a group $G$ is called $F$ -central (see [2]) if $[G/K](G/C_G(H/K)) \in F$ . The symbol $Z_\infty^F(G)$ denotes the $F$ -hypercenter of a group $G$, that is, the product of all such normal subgroups $H$ of $G$ whose $G$ -chief factors are $F$ - central. A subgroup $H$ of $G$ is said to be $F$ -hypercenter [3] in $G$ if $H \le Z_\infty^F(G)$ A class $F$ of groups is called a formation if it is closed under a homomorphic image and a subdirect product. It is clear that every group $G$ has a smallest normal subgroup (called $F$ -residual of $G$ and denoted by $G^F$ ) with quotient in $F$. A formation $F$ is said to be saturated if it contains every group $G$ with $G/\phi(G) \in F$. We use $U$ and $S$ to denote the formations of all supersoluble groups and soluble groups, respectively. Recall that a subgroup $H$ of $G$ is said to be complemented in $G$ if $G$ has a subgroup $B$ such that $G = AB$ and $A \cap B = 1$ (see [4]). A subgroup $H$ of $G$ is said to be $F_h$ -normal [5] (or $c$ -normal [6], $F_c$ -normal [7] or $F_n$ -supplemented [8] ) in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a normal Hall subgroup of $G$, and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$. By using the above subgroups, people have obtained some interesting results (see [5,7,8,9]}). As a development of this topic, we now introduce the following new concept.

*Definition 1.1* Let $F$ be a class of groups. A subgroup $H$ of $G is said to be $F_p$ -normal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a permutable Hall subgroup of $G$, and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$
Obviously, all normal subgroups, $c$ -normal subgroups, $F_n$ -supplemented subgroups and $F_h$ -normal subgroups are all $F_p$ -normal in $G$ for any nonempty saturated formation $F$. However, the following example shows that the converse is not true. For example, if a subgroup $H$ is $c$ -normal in $G$, then there exists a normal subgroup $K$ such that $G = HK$ and
$(H \cap T)H_G / H_G = 1 \le Z_\infty^F(G/H_G)$

However, the following example shows that the converse is not true.

*Example 1.2*. Let $S_4$ be the symmetric group of degree 4 , $P$ be the Sylow 3-subgroup of $S_4$ and $Z$ be a group of order $p$ with $p \ne 2,3$. Let $G = Z\iota S_4 = [K]S_4$ be a regular wreath product, where $K$ is the base group of the regular wreath product $G$. Then $PK$ is a permutable Hall subgroup and $P \cap K = 1$. Hence $P$ is $F_p$ -normal in $G$ for any nonempty saturated formation $F$. However, it is easy to see that $P$ is not normal, $c$ -normal, $S_h$ -normal and is not $U_c$ -supplemented in $G$ (in fact, for example, $G$ is the only normal subgroup of $G$ such that $PG = G$ and $P_G = 1$. However, since the unique minimal normal subgroup $K$ of $G$ is not cyclic, $P \cap G = P \not\le Z_\infty^S(G)$. Thus, $P$ is not $S_h$ -normal in $G$ ).

In this paper, we study the properties of $F_p$ -normal subgroups and use them to give some new characterizations of some classes of groups. Some previously known results are generalized.

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## II. PRELIMINARIES

A formation $F$ is said to be $S$ -closed (resp. $S_n$ -closed) if it contains all subgroups (resp. all normal subgroups) of all its groups.

We cite the following lemmas which will be useful in the sequel.

*Lemma 2.1* [9, Lemma 2.1] Let $G$ be a group and $A \le G$ . Let $F$ be a non-empty saturated formation and $Z = Z_\infty^F(G)$ . Then

(1) If $A$ is normal in $G$ , then $AZ / A \le Z_\infty^F(G/A)$;

(2) If $F$ is $S$ -closed, then $Z \cap A \le Z_\infty^F(G)$;

(3) If $F$ is $S_n$ -closed and $A$ is normal in $G$ , then $Z \cap A \le Z_\infty^F(G)$;

(4) If $G \in F$ , then $Z = G$ .

*Lemma 2.2* Let $F$ be a saturated formation containing $U$ and $G$ a group with a normal subgroup $E$ such that $G/E \in F$ . If \$E\$ is cyclic, then $G \in F$ .

Proof. We assume that $E$ is a minimal normal subgroup of $G$ such that $E \not\subseteq \phi(G)$. Let $M$ be a maximal subgroup such that $G = [E]M$ and let $C = C_G(E)$ . Then $C \cap M = M_G \lhd G$, and so we have $G/M_G = [EM_G / M_G](M/M_G) \in F$ . Hence $G \cong G/(E \cap M_G) \le G/E \times G/M_G$ is supersoluble. Thus $G \in F$ .
This completes the proof. □

*Lemma 2.3* [14, Lemma 2.6] Let $N$ be a soluble normal subgroup of $G$ . If $N \cap \phi(G) = 1$, then $F(N)$ is the direct product of minimal normal subgroups of $G$ contained in $N$ .

*Definition 2.4* Let $H$ be a subgroup of $G$ . A subgroup $K$ of $G$ is said to be a supersoluble supplement of $H$ in $G$ if $K$ is a supersoluble subgroup of $G$ such that $G = HK$

*Lemma 2.5* [5, Lemma 2.6] Suppose that $H$ has a supersoluble supplement in $G$ .

(1) If $N \lhd G$, then $HN/N$ has a supersoluble supplement in $G/N$ .

(2) If $H \le K \le G$, then $H$ has a supersoluble supplement in $K$ .

*Lemma 2.6* [5, Lemma 2.7] Suppose that $G$ has a unique minimal normal subgroup $N$ and $\phi(G) = 1$. If $N$ is soluble, then there exist a maximal subgroup $M$ of $G$ such that $G = [N]M$ , and $N = O_p(G) = F(G) = C_G(N)$ for some prime $p$ .

*Definition 2.7* Let $F$ be a class of groups. A subgroup $H$ of $G$ is said to be $F_p$ -normal in $G$ if there exists a normal subgroup $T$ of $G$ such that $HT$ is a permutable Hall subgroup of $G$ , and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$ .

*Lemma 2.8* Let $G$ be a group and $H \le K \le G$ . Then

(1) $H$ is $F_p$ -normal in $G$ if and only if $G$ has a normal subgroup $T$ such that $HT$ is a permutable Hall subgroup of $G$ , $H_G \le T$ and $(H/H_G) \cap (T/H_G \le Z_\infty^F(G/H_G)$ .

(2) Suppose that $H$ is normal in $G$ . Then $K/H$ is $F_p$ -normal in $G/H$ if and only if $K$ is $F_p$ -normal in $G$ .

(3) Suppose that $H$ is normal in $G$ . Then for every $F_p$ -normal subgroup $E$ in $G$ satisfying $(|H|,|E|) = 1$, the subgroup $HE/H$ is $F_p$ -normal in $G/H$ .

(4) If $H$ is $F_p$ -normal in $G$ and $F$ is hereditary, then $H$ is $F_p$ -normal in $K$ .

(5) If $K$ is normal in $G$ and $F$ is normally hereditary, then $H$ is $F_p$ -normal in $K$ .

(6) If $G \in F$ , then every subgroup of $G$ is $F_p$ -normal in $G$ .

Proof. (1) Assume that $H$ is $F_p$ -normal in $G$ and let $T$ be a normal subgroup of $G$ such that $HT$ is a permutable Hall subgroup of $G$ and $(H \cap T)H_G / H_G \le Z_\infty^F(G/H_G)$. Let $T_0 = TH_G$. Then $T_0$ is normal in $G$ , $HT_0 = HTH_G = HT$ is a permutable Hall subgroup of $G$ and

$$(T_0/H_G) \cap (H/H_G) = (T_0 \cap H)/H_G = (T \cap H)H_G / H_G \le Z_\infty^F(G/H_G).$$

(2) First assume that $K/H$ is $F_p$ -normal in $G/H$ . Then by (1), $G/H$ has a normal subgroup $T/H$ such that $(K/H)(T/H)$ is a permutable Hall subgroup of $G/H$ , $(K/H)_{G/H} \le T/H$ and

$$(T/H)/(K/H)_{G/H} \cap (K/H)/(K \cap H)_{G/H} \le Z_\infty^F((G/H)/(K/H)_{G/H}).$$

Then $T$ is normal in $G$ and $KT$ is a permutable Hall subgroup of $G$. Besides, since

$(T/H)/(K/H)_{G/H} \cap (K/H)/(K/H)_{G/H}$
$= (T/H)/(K_G/H) \cap (K/H)/(K_G/H)$
$= ((T\cap K)/H)/(K_G/H)$
and
$Z_\infty^F((G/H)/(K/H)_{G/H}) = Z_\infty^F((G/H)/(K_G/H))$,
we have
$(T\cap K)/K_G = (T/K_G)\cap(K/K_G) \le Z_\infty^F(G/K_G)$.
Hence, $K$ is $F_p$-normal in $G$. Analogously, one can show that if $K$ is $F_p$-normal in $G$, then $K/H$ is $F_p$-normal in $G/H$.

(3) Assume that $E$ is $F_p$-normal in $G$. Then by (2), $G$ has a normal subgroup $T$ such that $ET$ is a permutable Hall subgroup of $G$, $E_G \le T$ and $(E/E_G)\cap(T/E_G) \le Z_\infty^F(G/E_G)$. We will prove that $HE/H$ is $F_p$-normal in $G/H$. By (1), we only need to show that $HE$ is $F_p$-normal in $G$. Since $(|H|,|E|)=1$, we have $H \le T$ and so $T\cap HE = H(T\cap E) \le HZ$, where $Z/E_G = Z_\infty^F(G/E_G)$. Hence from the $G$-isomorphism
$HZ/HE_G = HE_GZ/HE_G$
$\cong Z/(Z\cap HE_G) = Z/E_G(Z\cap H),$
we have $HZ/HE_G \le X/HE_G = Z_\infty^F(G/HE_G)$ and so $(HE\cap T)/HE_G \le X/HE_G$. Let $D = HE_G$. By Lemma 2.3,
$(X/HE_G)(D/HE_G)/(D/HE_G) \le Z_\infty^F(G/HE_G)$.
Therefore,
$(TD/D)\cap(HE/D) = D(T\cap HE)/D \le Z_\infty^F(G/D)$
and so $HE$ is $F_p$-normal in $G$.

(4) Let $T$ be a normal subgroup of $G$ such that $HT$ is a permutable Hall subgroup of $G$, $H_G \le T$ and $(H/H_G)\cap(T/H_G) \le Z_\infty^F(G/H_G)$. Let $T_1 = H_G(H\cap K)$. Since $K = K\cap HT = H(K\cap T)$, we have $K = HT_1$. By Lemma 2.2(3), $T_1$ is normal in $K$. Besides,
$(T_1/H_G)\cap(H/H_G)$
$= H_G(H\cap T\cap K)/H_G$
$\le Z/H_G = Z_\infty^F(G/H_G)\cap K/H_G$.
Assume that $F$ is hereditary. Then by Lemma 2.3(2), $Z/H_G \le Z_\infty^F(G/H_G)$. By Lemma 2.3(1),
$(Z/H_G)(HK/H_G)/(HK/H_G)$
$\le Z_\infty^F((K/H_G)/(H_K/H_G))$
and so $(T_1/H_K)\cap(H/H_K) \le Z_\infty^F(K/H_K)$. Hence, $H$ is $F_p$-normal in $K$

(5) See the proof of (4).

(6) Assume that $G\in F$ and let $H$ be an arbitrary subgroup of $G$. By Lemma 2.3(6), $Z = Z_\infty^F(G) = G$, and so by Lemma 2.3(1), $Z_\infty^F(G/H_G) = G/H_G$. Hence, $H/H_G \le Z_\infty^F(G/H_G)$.

This completes the proof. □

*Lemma 2.9* Let $R$ be a soluble minimal normal subgroup of $G$. If there exists a maximal subgroup $R_1$ of $R$ such that $R_1$ is $U_p$-normal in $G$, then $R$ is a group of prime order.

Proof. Assume that $|R|$ is not a prime. Then $R_1 \ne 1$. Since $R$ is a minimal normal subgroup $R_1$ of $R$, $(R_1)_G = 1$. Then, by hypothesis, there exists a normal subgroup $K$ of $G$ such that $R_1K$ is a permutable Hall subgroup of $G$, $R_1\cap K \le Z_\infty^F(G)$ and $R_1K$ is subnormal in $G$. Hence $R \le R_1K$ since the minimality of $R$. Since $R\cap K \lhd G$, $R\cap K = 1$ or $R\cap K = R$. If $R\cap K = 1$, then $R = R\cap R_1K = R_1(R\cap K) = R_1$, a contradiction. If $R\cap K = R$, then $R\le K$. Hence, $R_1 \le K$ and therefore, $1\ne R_1 = R_1\cap K \le Z_\infty^U(G)$. Thus $1\ne R_1 \le Z_\infty^U(G)\cap R \lhd G$. It follows, from the minimality of $R$, that $R \le Z_\infty^U(G)$. Consequently, $|R|$ is a prime.

The final contradiction completes the proof. □

## III. Characterization of Supersoluble Groups

*Theorem 3.1* Let $F$ be a saturated formation containing $U$. Then $G\in F$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H\in F$, and every maximal subgroup of every Sylow subgroup of $F(H)$, which has no supersoluble supplement in $G$, is $U_p$-normal in $G$.

Proof. The necessity part is obvious. We only need to prove the sufficiency part. Suppose that the assertion is false and let $G$ be a contraexample with $|G||H|$ minimal. Let $P$ be an arbitrary Sylow $p$-subgroup of $F(H)$. Since $P$ char $F(H)$ char $H \lhd G$. We proceed the proof via the following steps.

*Step 1.* $\phi(G)=1$.

If $\phi(G)\neq 1$, then $\phi(G)\leq F(H)$. Let $R=\phi(G)$. Clearly, $(G/R)/(H/R)\cong G/H\in F$. By [10, III, Theorem 3.5], we have that $F(H/R)=F(H)/R$. Let $P/R$ be a Sylow $p$-subgroup of $F(H)/R$, and $P_1/R$ be a maximal subgroup of $P/R$. Then $P_1$ is a maximal subgroup of $P$. By hypothesis, $P_1$ either has supersoluble supplement in $G$ or is $U_p$-normal in $G$. It follows from Lemmas 2.5(1) and 2.8(2), $P_1/R$ either has supersoluble supplement in $G/R$ or is $U_p$-normal in $G/R$. Now Let $Q/R$ be a maximal subgroup of some Sylow $q$-subgroup of $F(H)/R$, where $q\neq p$. Then $Q=Q_1R$, where $Q_1$ is a maximal subgroup of the Sylow $q$-subgroup of $F(H)$. By hypothesis, $Q_1$ either has supersoluble supplement in $G$ or is $U_p$-normal in $G$. It follows from Lemmas 2.5(1) and 2.8(3), that $Q_1R/R$ either has supersoluble supplement in $G/R$ or is $U_p$-normal in $G/R$. Hence $(G/R,H/R)$ satisfies the hypothesis. The minimal choice of $(G,H)$ inplies that $G/R\in F$. Since $G/\phi(G)=G/R$ and $F$ is a saturated formation, we have $G\in F$, a contradiction.

*Step 2.* $P=Z_1\times Z_2\times\ldots\times Z_m$, where every $Z_i$ is a normal subgroup of order $p$ of $G$.

Since $P\lhd G$, by Step 1, $P\cap\phi(G)=1$. Hence by Lemma 2.3, $P=Z_1\times Z_2\times\ldots\times Z_m$, where all $Z_i$ are minimal normal subgroups of $G$. Next, we prove that all $Z_i$ are of prime order $p$.

Assume that $|Z_i|>p$ for some $i$. Without loss of generality, let $|Z_1|>p$. Let $Z_1^*$ be a maximal subgroup of $Z_1$. Then $Z_1^*\times Z_2\times\ldots Z_m=P_1$ is a maximal subgroup of a Sylow $p$-subgroup $P$ of $F(H)$. Set $T=Z_2\times\ldots\times Z_m$, then $(P_1)_G=T$. If $P_1$ has a supersoluble supplement $K$ in $G$, then $G=P_1K=Z_1^*Z_2\ldots Z_mK=Z_1^*TK$. Since $T\lhd G$, $TK$ is a subgroup of $G$ and so $|G:TK|\leq|Z_1^*|<|Z_1|$. Since

$$(Z_1\cap TK)^G=(Z_1\cap TK)^{Z_1TK}$$
$$=(Z_1\cap TK)^{TK}=Z_1\cap TK,$$

we have $Z_1\cap TK\lhd G$. Hence, we have either $Z_1\cap TK=1$ or $Z_1\cap TK=Z_1$. If $Z_1\cap TK=1$, then we have $|G:TK|=|Z_1|>|Z_1^*|$, a contradiction.
If $Z_1\cap TK=Z_1$, then $Z_1\leq TK$ and so $G=TK$. Since $T\lhd G$, then $K\cong G/T=TK/T=K/T\cap K$ is supersoluble as $K$ is a supersoluble supplement and $Z_1\cong Z_1T/T$ is chief factor of $G/T$. It follows that $|Z_1|=|Z_1T/T|=p$, which is a contradiction. Thus $P_1$ is $U_p$-normal in $G$, and so by Lemma 2.8(1), there exists a normal subgroup $N$ of $G$ such that $(P_1)_G\leq N$, $P_1N$ is a permutable Hall subgroup of $G$, and $(P_1\cap N)/(P_1)_G\leq Z_\infty^U(G/(P_1)_G)$. Hence $P_1N=Z_1^*Z_2\ldots Z_mN=Z_1^*(P_1)_GN=Z_1^*N$. We only think the following two cases.

*Case 1.* $Z_1^*\cap N=1$.

In this case, $(Z_1^*)_G\leq Z_1^*\cap N$ and so $(Z_1)_G=1\leq N\lhd Z_\infty^F(G/(Z_1^*)_G)$, $Z_1^N=P_1N$ is a permutable subgroup of $G$, which implies that $Z_1^*$ is $U_p$-normal in $G$. Hence By Lemma 2.9, $Z_1$ is a cyclic group of order $p$, a contradiction.

*Case 2.* $Z_1^*\cap N\neq 1$.

In this case, $1<Z_1\cap N\lhd G$ since $Z_1$ and $N$ are both normal in $G$. By the minimality of $Z_1$, we have either $Z_1\cap N=1$ or $Z_1\cap N=Z_1$. We assume that $Z_1\cap N=1$. Hence $Z_1^*\cap N=1$, a contradiction. Hence $Z_1\cap N=Z_1$ and so $Z_1\leq N$. Thus $P_1N=Z_1^*N=N$. Consequently, $P_1\leq N$. It follows that

$$P_1/(P_1)_G=(P_1\cap N)/(P_1)_G\leq Z_\infty^F(G/(P_1)_G)\cap P/(P_1)_G$$

. If $P_1=(P_1)_G$, then $Z_1^*=1$, which contradicts $Z_1^*\cap N\neq 1$. Hence $(P_1)_G<P_1$ and so

$$1\neq P_1/(P_1)_G=(P_1\cap N)/(P_1)_G$$
$$\leq Z_\infty^F(G/(P_1)_G)\cap P_1/(P_1)_G.$$

Since $P=Z_1\times\ldots Z_m=Z_1T=Z_1(P_1)_G$, then we have that $P/(P_1)_G=P/P_1\cong Z_1$ and $P/(P_1)_G$ is a chief factor of

$G$ because of the minimality of $Z_1$. This implies that $Z_\infty^F(G/(P_1)_G) \cap P/(P_1)_G$, and thereby, $P/(P_1)_G \le Z_\infty^F(G/(P_1)_G)$. It means that $|P/(P_1)_G| = p$. Therefore $P_1/(P_1)_G = 1$, which contradicts $P_1/(P_1)_G \ne 1$.

*Step 3.* $P = F(H)$.

Obviously, $P \le F(H)$. Assume that $P < F(H)$, then since $F(H)$ is nilpotent, there exists a Sylow $q$-subgroup $Q$ of $F(H)$ and $Q \lhd G$. From Step 2, $P = Z_1 \times \ldots \times Z_m$ and $P = R_1 \times \ldots \times R_s$, where $Z_i, R_j$ are a normal subgroup of $G$ of order $p, q$ respectively. Obviously, $F(H)/Z_i \le F(H/Z_i)$ and $F(H)/R_j \le F(H/R_j)$. From Step 1, we have $G/Z_i \in F$ and $G/R_j \in F$. It follows that $G/1 \cong G/Z_i \times G/R_j$, a contradiction.

*Step 4.* $G/F(H) \in F$.

From Steps 2 and 3, $P = F(H) = Z_1 \times Z_2 \times \ldots \times Z_m$, where every $Z_i$ is a normal subgroup of $G$ of order $p$. Since $G/C_G(Z_i)$ is isomorphic to a subgroup of $Aut(Z_i)$, $G/C_G(Z_i)$ is cyclic and so it lies in $U$ for each $i$. It follows that $G/\cap_{i=1}^m C_G(Z_i) \in U$ Obviously, $C_G(F(H)) = \cap_{i=1}^m C_G(Z_i)$. Hence, $G/C_G(F(H)) \in U \subseteq F$. Consequently, $G/(H \cap C_G(F(H))) = G/C_H(F(H)) \in F$. Since $F(H)$ is abelian and $H$ is soluble, we have that $F(H) \le C_G(F(H)) \le F(H)$. Thus, $F(H) = C_G(F(H)) = P$ and so $G/F(H) \in F$.

*Step 5*. If $K$ is a miniml normal subgroup of $G$ contained in $H$, then $G/K \in F$.

Since $H$ is soluble, then $K \le F(H)$. By Lemmas 2.5(1) and 2.8(2), we have that every maximal subgroup of $F(H)/K$ either has a supersoluble supplement in $G/K$ or is $U_p$-normal in $G/K$ and $(G/K)/(H/K) \in F$. It follow that $(G/K, F(H)/K)$ satisfies the hypothesis. The minimal choice of $(G,H)$ implies that $G/K \in F$.

*Step 6*. Final contradiction.

Since $F$ is a saturated formation, then by Steps 2 and 4, we have that $F(H)$ is the uniue minimal ormal subgroup of $G$ contained in $H$, and $F(H) = R_1$ is a cyclic group of order $p$. Hence by Step 4 and Lemma 2.2, $G \in F$, a contradiction. The final contradiction completes the proof. □

*Corollary 3.2* [5] Let $F$ be a saturated formation containing $U$. Then $G \in F$ if and only if there exists a soluble normal subgroup $H$ of $G$ such that $G/H \in F$, and every maximal subgroup of every Sylow subgroup of $F(H)$, which has no supersoluble supplement in $G$, is $U_p$-normal in $G$.

*Corollary 3.3* [12] Let $G$ be a soluble group having a normal subgroup $H$ such that $G/H$ is supersoluble. If every maximal subgroup of any Sylow subgroup of $F(H)$ $ either is normal in $G$ or has a supersoluble supplement in $G$, then $G$ is supersoluble.

*Corollary 3.4* [14] Let $G$ be a group. If $H$ is a soluble normal subgroup of $G$ with supersoluble quotient $G/H$ and all maximal subgroups of all Sylow subgroups of $F(H)$ are $c$-normal in $G$, then $G$ is supersoluble.

*Corollary 3.5* [15] If $G$ is a soluble group and all maximal subgroups of Sylow subgroups of $F(G)$ are normal in $G$, then $G$ is supersoluble
.

*Corollary 3.6* [16] Let $F$ be a saturated formation containing $U$. Suppose that $G$ is a group with a soluble normal subgroup $H$ such that $G/H \in F$. If all maximal subgroups of any Sylow subgroup of $F(H)$ are $c$-normal in $G$, then $G \in F$.

## IV. Characterization of Soluble Groups

*Theorem 4.1* Let $p$ be the smallest prime dividing |G|, and let P be some Sylow p-subgroup P. Then group $G$ is soluble if and only if every maximal subgroup of P is $U_p$-normal in $G$.

Proof. Let P be an arbitrary Sylow subgroup of $G$. If $G$ is soluble, then $G/P_G$ is also soluble and so $Z_\infty^S(G/P_G) = G/P_G$. Hence, the necessity part obviously holds. We now prove the sufficiency part. Suppose that the assertion is false, and let $G$ be a counterexample of minimal order. Then by the well-known Feit-Thompson's theorem, we have p=2. Now, we proceed the proof by the following steps.

*Step 1.* $O_2(G) = 1$.

Let $N = O_2(G) \neq 1$. Obviously, P/N is a Sylow 2-subgroup of G/N. Let M/N be a maximal subgroup of P/N. Then M is a maximal subgroup of P and so M is is Up-normal in G by hypothesis. By Lemma 2.8(2), M/N is Up-normal in G/N. The minimality of G implies that G/N is soluble. It follow that $G$ is soluble, a contradiction.

*Step 2.* $O_{2'}(G) = 1$.

Let $D = O_{2'}(G) \neq 1$. Then PD/D is a Sylow 2-subgroup of G/D. Let M/D be a maximal subgroup of PD/D. Then there exists a maximal subgroup $P_1$ of P such that $M = P_1 D$ and so $P_1$ is Up-normal in G by hypothesis. By Lemma 2.8(3), M/D is Up-normal in G/D. Hence G/D is soluble by the minimal choice of $G$, and so $G$ is soluble, a contradiction.

*Step 3.* Final contradiction.

Let $P_1$ be a maximal subgroup of P. By the hypothesis, there exists a normal subgroup K of $G$ such that $P_1K$ is a permutable Hall subgroup of $G$ and $(P_1 \cap K)(P_1)_G/(P_1)_G \leq Z_\infty^S(G/(P_1)_G)$. By Steps 1 and 2, we have $(P_1)_G = 1$ and $Z_\infty^S(G) = 1$. This induces that $P_1 \cap K = 1$. Since $P_1K$ is a permutable Hall subgroup of $G$, $|K_2| = 2$, where $K_2$ is some Sylow 2-subgroup of K. By \cite[Theorem 10.1.9]{robin}, we see that K is 2-nilpotent, and so K has a normal 2-complement $K_{2'}$. Since $K_{2'}$ char $K \lhd G$, $K_{2'} \lhd G$. Hence, by Step 2, $K_{2'} = 1$, and so $|K| = 2$, which contradicts Step 1.

This completes the proof. □

*Corollary 4.2* Let M be a maximal subgroup of $G$ with |G : M| = r, where r is a prime. Let p be the smallest prime dividing |M|. If there exists a Sylow p-subgroup P of M such that every maximal subgroup of P is Sp-normal in $G$, then $G$ is soluble.

Proof. By Feit-Thompson's theorem, we may assume that 2||G|. If r =2, then M is normal in $G$. By Lemma 2.8(4), every maximal subgroup of P is Sp-normal in M. Hence, by Theorem 4.1, M is soluble. It follows that $G$ is soluble. If r ≠ 2, then p =2$, and so P is a Sylow 2-subgroup of $G$. By Theorem 4.1, we have that $G$ is soluble.

This completes the proof. □

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