Numerical Method based on Initial Value-Finite Differences for Free Vibration of Stepped Thickness Plates

Ahmed M. Farag, Wael F. Mohamed, Atef A. Ata, and Burhamy M. Burhamy

***Abstract*—**The main objective of the present paper is to derive an easy numerical technique for the analysis of the free vibration through the stepped regions of plates. Based on the utilities of the step by step integration initial values IV and Finite differences FD methods, the present improved Initial Value Finite Differences (IVFD) technique is achieved. The first initial conditions are formulated in convenient forms for the step by step integrations while the upper and lower edge conditions are expressed in finite difference modes. Also compatibility conditions are created due to the sudden variation of plate thickness. The present method (IVFD) is applied to solve the fourth order partial differential equation of motion for stepped plate across two different panels under the sudden step compatibility in addition to different types of end conditions. The obtained results are examined and the validity of the present method is proved showing excellent efficiency and rapid convergence.

***Keywords*—**Vibrations, Step by Step Integration, Stepped plate, Boundary.

I. Introduction

PLATES with varying thickness are extensively used in modern structures due to their unique functions. For example, stepped plates possess a number of attractive features, such as material saving, weight reduction, stiffness enhancing, designated strengthening and fundamental vibration frequency increasing. With the availability of inexpensive and high performance computers, theoretical analysis is employed to optimize stepped plates in practical engineering designs [1]-[2]-[3]. In particular, buckling and free vibration analysis of stepped plates has attracted much attention in the past few decades [2]. A few numbers of theoretical approaches have been formulated for this class of problems. Although, Chopra [4] has attempted an exact solution for a simply supported stepped thickness plate with two panels, Warburton [4] pointed out that the continuity conditions used by Chopra were incorrect. He offered a modified analytical technique for two paneled stepped plate with different properties of orthotropy. However, dealing with the proposed plates by analytical method (Levy-type) was successful [2], but it was restricted to limited edges boundary conditions. On the other hand there was[1] a flexibility of applying variety of boundary condition with power matrix exponential method. Other developed numerical methods like Finite Strip Method (FSM) [5], or a developed one like finite strip transition matrix method (FSTM) [4] are used to get dynamical solutions. Since analytical solutions of the partial differential equation of motion for some plate problems are so complicated, numerical and approximate methods can be used [6]-[7]-[8]. In the engineering applications of the theory of plates, Finite Differences (FD) method is a classical field [9]-[10]-[11] while among the numerical techniques, finite differences method is one of the most general. Recently, despite the existence of a broad variety of numerical method for the structural analysis, finite differences method is still regarded as a numerical method for straightforward analysis of specialized problems [12]. On the other hand, the initial value method is one of the fast approximate methods, which proves a good convergence with the exact solution [13]-[14].In the present paper, an improved technique (IVFD) combining the initial value method (IV) and finite differences method (FD) is achieved to analyze the flexural vibration of stepped plate. A selected mesh grid is applied for the partial differential equation of plate motion where the initial value method is utilized in one direction. In the other direction, the partial derivatives are replaced with quantities of the finite differences method. The developed IVFD method is successfully applied for free vibration analysis of stepped thickness plates with different combinations of boundary conditions. The natural frequency parameters are calculated and the efficiency of the method is validated by comparing the deduced results with those published in the literature.

II. Initial Value Finite Differences Mathematical Formulation

The governing partial differential equation of small transverse deflection of a thin isotropic elastic plate, of thickness h, for free vibration [15] has the form:

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} + \frac{\rho h}{D} w_{,tt} = 0 \tag{1}$$

where $D = \dfrac{Eh^3}{12(1-\upsilon^2)}$ is the cylindrical stiffness of the plate,

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w is the plate deflection measured along the z axis, E is the modulus of elasticity and υ is Poisson's ratio of the plate material.Also, $w_{,xxxx}$ and $w_{,yyyy}$ are the fourth-order partial derivatives of w with respect to x and y respectively. By analogy, the other partial derivatives of w are written according to the independent variables.To investigate the vibration modes, the displacement is defined as time harmonic function as:

$$w(x, y, t) = \hat{w}(x, y)e^{i\omega t} \tag{2}$$

where ω is the circular natural frequency. Substitution of (2) in (1) yields the equation governing the dynamic stationary free vibration behavior of Kirchhoff plates [16]-[17]-[18]:

$$\hat{w}_{,xxxx} + 2\hat{w}_{,xxyy} + \hat{w}_{,yyyy} - \lambda^2 \hat{w} = 0 \tag{3}$$

where $\lambda^2 = \frac{\rho h \omega^2}{D}$ is the plate natural frequency parameter.

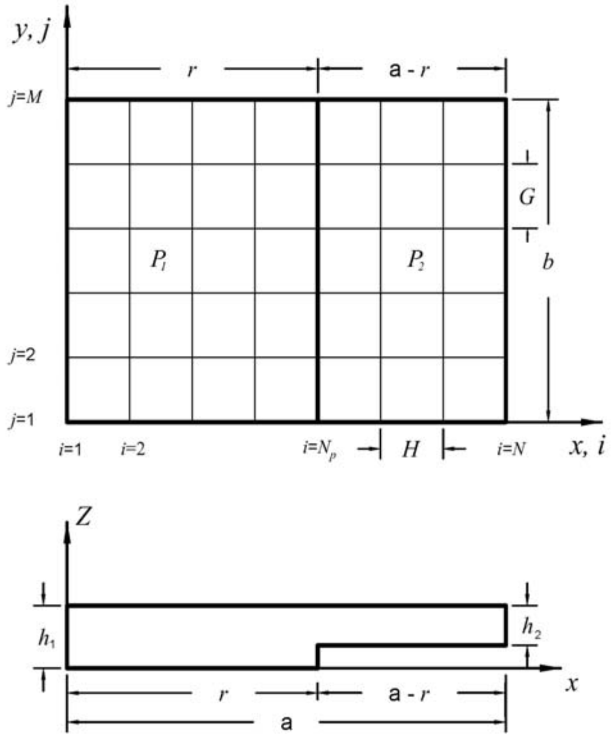

Fig. 1 Model of stepped thickness plate

The investigated model is thin elastic isotropic plate of two panels p_1 and p_2 with two different thicknesses h_1 and h_2 respectively as shown in Fig. 1. The dimensions of the plate are a and b in x and y directions respectively, and the plate aspect ratio is $\beta = \frac{a}{b}$. Also, $\gamma = \frac{r}{(a - r)}$ is called the panel width ratio. A mesh of $M \times N$ numbers of divisions is constructed with equal divisions in each direction, $H = a/N$ in x-direction and $G = b/M$ in y-direction, as shown in Fig. 1.

In the present paper a step by step integration technique is used by initial value method in the x-direction. The partial derivatives $\hat{w}_{,xxyy}$, $\hat{w}_{,xxxx}$ at any point (i, j), in (3), are represented in y-direction by the lower displacement derivatives $\hat{w}$, $\hat{w}_{,xx}$ of four neighborhood points [19]-[20], while the derivatives $\frac{\partial^n \hat{w}}{\partial x^n}$; $n = 1,2,3,4$ in x-direction are retained.Using the finite differences rules[21], one can obtain:

$$\left(\hat{w}_{,xxyy}\right)_{(i,j)} = \frac{1}{G^2}\left[\left(\hat{w}_{,xx}\right)_{(i,j+1)} - 2\left(\hat{w}_{,xx}\right)_{(i,j)} + \left(\hat{w}_{,xx}\right)_{(i,j-1)}\right] \tag{4}$$

$$\left(\hat{w}_{,yyyy}\right)_{(i,j)} = \frac{1}{G^4}\left[\hat{w}_{(i,j-2)} - 4\hat{w}_{(i,j-1)} + 6\hat{w}_{(i,j)} - 4\hat{w}_{(i,j+1)} + \hat{w}_{(i,j+2)}\right] \tag{5}$$

Then by substitution from (4) and (5) into (3), the nodal equation of motion of isotropic plates is transformed to:

$$\left(\hat{w}_{,xxxx}\right)_{(i,j)} = \lambda^2\hat{w} - \frac{2}{G^2}\left[\left(\hat{w}_{,xx}\right)_{(i,j+1)} - 2\left(\hat{w}_{,xx}\right)_{(i,j)} + \left(\hat{w}_{,xx}\right)_{(i,j-1)}\right] - \frac{1}{G^4}\left[\hat{w}_{(i,j-2)} - 4\hat{w}_{(i,j-1)} + 6\hat{w}_{(i,j)} - 4\hat{w}_{(i,j+1)} + \hat{w}_{(i,j+2)}\right] \tag{6}$$

Trapezoidal rule is used as an integration method to solve the differential equation in x-direction, by which the derivatives applied at a point (i, j) are represented as:

$$\hat{w}^{(n)}_{(i+1,j)} = \hat{w}^{(n)}_{(i,j)} + \frac{H}{2}\left[\hat{w}^{(n+1)}_{(i,j)} + \hat{w}^{(n+1)}_{(i+1,j)}\right] \quad ; n = 0,1,2,3 \tag{7}$$

where, $\hat{w}^{(n)}_{(i,j)}$ is the n-order derivative of the displacement $\hat{w}$ at a point (i, j) with respect to x.

The procedure of the step by step integration is illustrated by a flow chart diagram, see Fig. 2. This procedure can be explained in the following steps:

Step1*: Initial Values at Starting Line* $i = 1$

According to the proposed initial conditions at the edge $i = 1$, two quantities of $\left(\frac{\partial^n \hat{w}}{\partial x^n}\right)_{(i,j)}$; $n = 0,1,2,3$, $j = 1,2,3,..., M$ are known. The other two values have to be assumed. Here are $2M$-initial values corresponding to $2M$-homogenous solutions. Elementary, for one homogenous solution, only one non-trivial value of $2M$-initial values is assumed while the other values are considered zeros. Applying (6) at any point $(1, j)$; $j = 1,2,3,..., M$, the magnitude of the fourth derivative $\left(\hat{w}_{,xxxx}\right)_{(1,j)}$ are determined at $i = 1$.

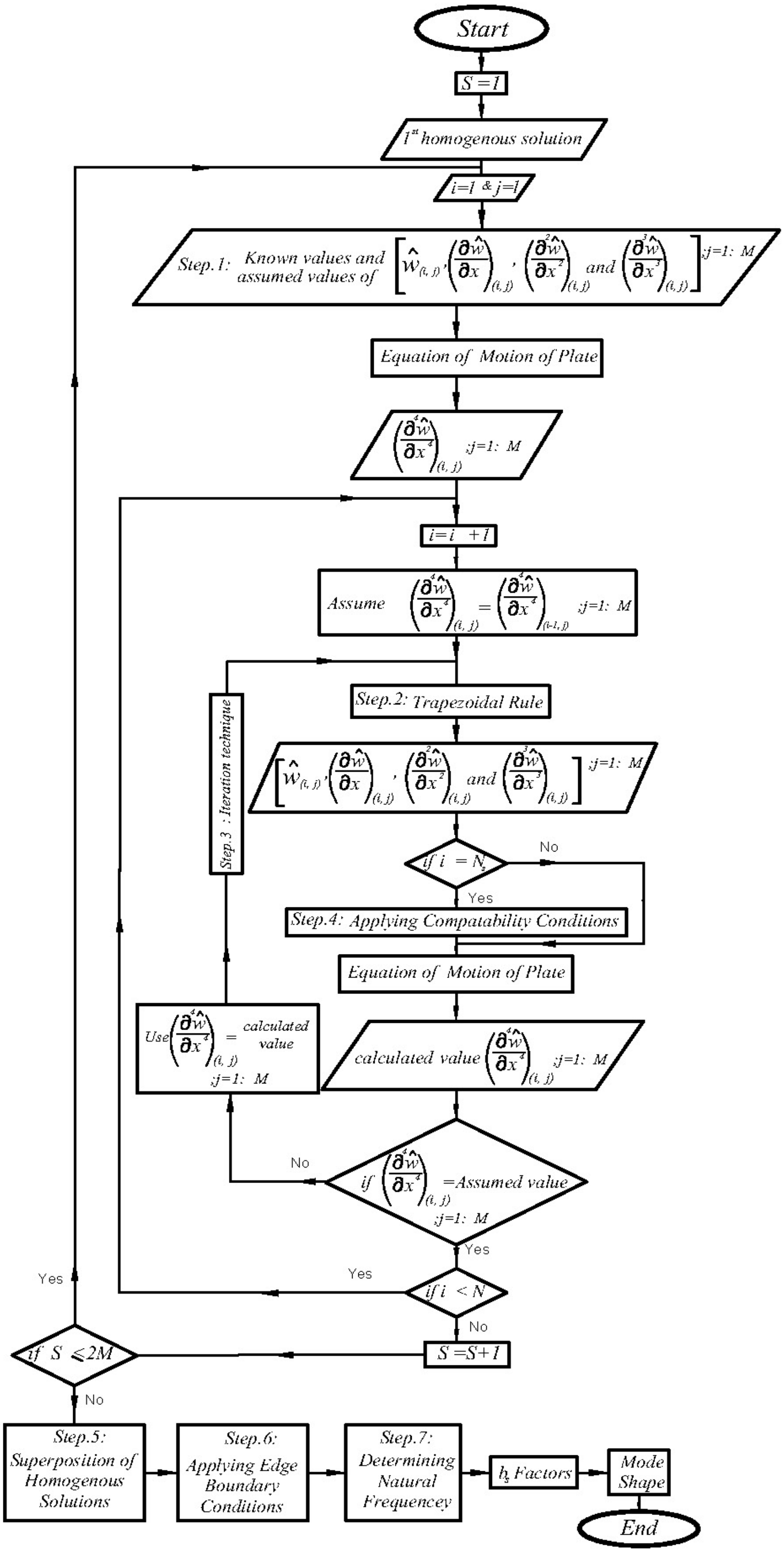

Fig. 2 Flow chart diagram of IVFD technique

***Step2**: Trapezoidal Rules for Step by Step Integration Technique*

Beginning with the assumption $\left(\hat{w}_{,xxxx}\right)_{(i+1,j)} = \left(\hat{w}_{,xxxx}\right)_{(i,j)}$ = $j=1,2,3,...,M$ and using the trapezoidal rules of (7), one can calculate the first trail values of $\left(\frac{\partial^n \hat{w}}{\partial x^n}\right)_{(i+1,j)}$; $n=0,1,2,3$, $j=1,2,3,...,M$ from the corresponding obtained values $\left(\frac{\partial^n \hat{w}}{\partial x^n}\right)_{(i,j)}$; $n=0,1,2,3$, $j=1,2,3,...,M$. Consequently by substituting the obtained quantities from trapezoidal rule into (6), the values of the fourth derivative $\left(\hat{w}_{,xxxx}\right)_{(i,j)}$ for all js can be determined at the current line.

***Step3**: Iteration Technique*

For the determined value of the fourth derivative $\left(\hat{w}_{,xxxx}\right)_{(i,j)}$, from (6), which is not coincident with the assumed one, the new determined value is taken as the assumed value. The procedure of step 2 has to be repeated for all $j=1,2,3,...,M$ until the assumed value agrees as closely as with the deduced one for all js .

***Step4**: Compatibility Conditions*

Steps 2 and 3 are applied to the next lines until $i = N_p$ is reached, where N_p is the line number at the sudden of plate.

Consequently, compatibility conditions [1]-[2]-[3] have to be applied at line $i = N_s$. These conditions can be represented in the form:

$$\begin{Bmatrix} \hat{w}_{(i,j)} \\ \left(\hat{w}_{,x}\right)_{(i,j)} \\ M_{x(i,j)} \\ V_{x(i,j)} \end{Bmatrix}_{p_1} = \begin{Bmatrix} \hat{w}_{(i,j)} \\ \left(\hat{w}_{,x}\right)_{(i,j)} \\ M_{x(i,j)} \\ V_{x(i,j)} \end{Bmatrix}_{p_2} \tag{8}$$

The suffixes p_1 and p_2 refer to the values before and values after the sudden step respectively. Expressing the moment and shear in finite differences forms, one gets [15]:

$$\begin{Bmatrix} \hat{w}_{(i,j)} \\ \left(\hat{w}_{,x}\right)_{(i,j)} \\ \left(\hat{w}_{,xx}\right)_{(i,j)} \\ \left(\hat{w}_{,xxx}\right)_{(i,j)} \end{Bmatrix}_{p_2} = \begin{Bmatrix} \hat{w}_{(i,j)} \\ \left(\hat{w}_{,x}\right)_{(i,j)} \\ k\left(\hat{w}_{,xx}\right)_{(i,j)} + c_1 D_1 \\ k\left(\hat{w}_{,xxx}\right)_{(i,j)} + c_2 D_2 \end{Bmatrix}_{p_1} \tag{9}$$

where $c_1 = \dfrac{(\alpha^3-1)\upsilon}{G^2}$, $c_2 = \dfrac{(\alpha^3-1)(2-\upsilon)}{G^2}$, $k = \alpha^3 == \left(\dfrac{h_1}{h_2}\right)^3$

The magnitude $\alpha = \dfrac{h_1}{h_2}$ is the stepped plate thickness ratio and

$D_1 = [\hat{w}_{(i,j-1)} + 2\hat{w}_{(i,j)} - \hat{w}_{(i,j+1)}]$,

$D_2 = [\left(\hat{w}_{,x}\right)_{(i,j+1)} - 2\left(\hat{w}_{,x}\right)_{(i,j)} + \left(\hat{w}_{,x}\right)_{(i,j-1)}]$

The previous steps 2 and 3 are applied to the next lines i; $i = N_p : N$ until the deflection $\hat{w}$ and their partial derivatives $\dfrac{\partial^n \hat{w}}{\partial x^n}$; $n=1,2,3$ are calculated at the points of the terminal edge $i = N$.

***Step5:**Superposition of Homogenous Solutions*

Because of existing $2M$ -assumed initial values at the beginning edge $i=1$, the terminal boundary conditions are not satisfied for each homogenous solution. At the terminal edge $i = N$, there are $2M$ -homogeneous solutions (each homogeneous solution corresponds each assumed initial value at point $(1, j)$.

There are $2M$ -sets of displacement quantities at the points (i, j) of the mesh (each corresponds to one of the $2M$ -assumed initial values). The superposition method is used to derive each individual solution. The true superimposed solution $\hat{w}^{(n)}$ of plate is the sum of the $2M$ -homogeneous solutions such as:

$$\hat{w}^{(n)} = \sum_{s=1}^{2M} b_s \hat{w}_s^{(n)} \; ; s=1,2,3,....,2M \; , n=0,1,2,3,4 \tag{10}$$

where:

$\hat{w}_s^{(n)}$ is the n -derivative of the displacement of the homogenous solutions and b_s ; $s=1,2,3,....,2M$ are unknowns coefficients to be determined from the boundary conditions.

***Step6:** Boundary Condition at the end Edge* $i = N$

The individual solutions of the end edge $i = N$ will not satisfy boundary condition, so it is necessary to force the superimposed solutions to coincides with the boundary conditions. For example, the boundary conditions for simply supported plate are:

$$\hat{w}_{(N,j)}^{(0)} = \hat{w}_{(N,j)}^{(2)} = 0 \; ; j=1,2,3,...,M \tag{11}$$

The unknown b_s; $s=1,2,3,....,2M$ factors can be determined by satisfying the boundary conditions of the terminal edge $i = N$ .Boundary conditions will be expressed in a matrix form such as:

$$[\Re_s][b_s] = [0] ; \; s=1,2,3,....,2M \tag{12}$$

where $\Re_s$ is $2M \times 2M$ known matrix and b_s is $2M$ -

unknown vector.

Step7:*The Circular Natural Frequency*

For a non-trivial solution of (12), the determinate of $[\Re_s]$ must be zero[22]-[23]-[24].All values that satisfy zero determinate of $[\Re_s]$ are the natural frequencies ω of plate. The corresponding displacement of obtained natural frequency is the mode shape [25]-[26]-[27]such as:

$$(\hat{w})_{(i,j)} = \sum_{s=1}^{2M} b_s (\hat{w}_s)_{(i,j)} \quad ; \; s = 1,2,3,....,2M \qquad (13)$$

III. Formulation of the Boundary Conditions

Boundary conditions, at the edges of the rectangular plate, are formulated in convenient forms to deal with the IVFD method. The assumed initial values at the initial edges are depending on the boundary conditions along this edge.

The assigned initial values at the first edge for all js for different boundary conditions [25] are:

i. *Simply supported plate edge*, S.

a- At ($i = 1, N$):

$$\hat{w}_{(i,j)} = (\hat{w}_{,xx})_{(i,j)} = 0$$

The other two magnitudes $\hat{w}_{(i,j)}$, $(\hat{w}_{,xxx})_{(i,j)}$ have to be assumed non-trivial values.

b- At ($j = 1, M$):

$$\hat{w}_{(i,j-1)} = -\hat{w}_{(i,j+1)} \text{ and } \hat{w}_{(i,j)} = 0$$

ii. *Clamped supported plate edge*, C.

a- At ($i = 1, N$):

$$\hat{w}_{(i,j)} = (\hat{w}_{,x})_{(i,j)} = 0$$

The other two magnitudes $(\hat{w}_{,xx})_{(i,j)}$, $(\hat{w}_{,xxx})_{(i,j)}$ have to be assumed non-trivial values.

b- At ($j = 1, M$):

$$\hat{w}_{(i,j-1)} = \hat{w}_{(i,j+1)} \text{ and } \hat{w}_{(i,j)} = 0$$

iii. *Free plate edge*, F.

For F-edges at($i = 1, N$),the values of $V_{x(i,j)}$ and $M_{x(i,j)}$ must bevanish.The magnitudes $(\hat{w})_{(1,j)}$ and $(\hat{w}_{,x})_{(1,j)}$ at the initial edges, have to be assumed.

IV. Numerical Verification and Discussion

Several natural frequency parameters of different modes are calculated for different cases of plate composed of two panels with unequal thickness and panel widths. Different aspect ratio β, panel width ratio γ and various magnitude of thickness ratio α are investigated. The natural frequency parameter is pointed out for every case where Poisson's ratio is taken as $\upsilon = 0.3$. Tables I and II show the values of normalized dimensionless natural frequency $\Gamma_{mn} = \left(\frac{\lambda_{mn}^2}{\pi^2}\right)$ parameter for two cases of boundary conditions of square plates SSSS and CSCS where S, C refers to simply supported and clamped edges respectively. The plate panel width ratio is taken as $\gamma = 0.5$ with different values of thickness ratio α.The obtained results are compared with the exact values showing good agreements.

TABLE I
Normalized Dimensionless Natural Frequency Parameter Γ_{mn}

Boundary conditions	α	Γ_{mn}	Method		
			IVFD	Farag [1]	Xiang[2]
SSSS	0.5	Γ_{11}	2.8168	2.9015	2.9015
		Γ_{12}	7.0926	7.1156	7.1156
		Γ_{13}	13.5798	13.7848	13.7850
CSCS	0.5	Γ_{11}	4.0352	4.1711	4.1711
		Γ_{12}	9.9327	9.9047	9.9047
		Γ_{13}	17.8633	18.0453	18.0450

TABLE II
Normalized Dimensionless Natural Frequency Parameter Γ_{mn}

Boundary conditions	α	Γ_{mn}	Method		
			IVFD	Farag [1]	Xiang[2]
SSSS	2/3	Γ_{11}	2.4124	2.4470	2.4471
		Γ_{12}	6.2622	6.2229	6.2229
		Γ_{13}	11.8552	11.9124	11.9480
CSCS	2/3	Γ_{11}	3.5094	3.5609	3.5610
		Γ_{12}	8.7183	8.7199	8.7199
		Γ_{13}	15.5305	15.6215	15.6210

The first fundamental natural frequency parameter for the case of full clamped plate is calculated when the thickness ratios α vary from0.1to 1.0, and the panel width ratio is taken as $\gamma = 0.75$, as shown in Table III. The results show that the natural frequency parameters increase by decreasing the thickness ratio α.

TABLEIII
Dimensionless Natural Frequency Parameter λ_{mn}^2

IVFD	α	0.1	0.2	0.3	0.4	0.5
	λ_{11}^2	50.54	49.78	48.76	46.91	44.726
	α	0.6	0.7	0.8	0.9	1.0
	λ_{11}^2	41.8	40.2	38.15	36.59	35.536

Another case is studied for CCCC square plate of Panel width ratio $\gamma = 0.5$ and thickness ratio $\alpha = 2.0$ for different aspect ratios β as shown in Table IV. As seen from the results, the natural frequency parameter decreases whenever the aspect ratio increases.

TABLE IV
DIMENSIONLESS NATURAL FREQUENCY PARAMETER λ^2_{mn}

	IVFD				
	$\alpha = 2.0$				
β	1.2	1.4	1.6	1.8	2.0
λ^2_{11}	22.5549	20.1156	18.649	17.6919	17.0598
λ^2_{12}	49.8966	48.0585	47.4841	46.5267	46.0672
λ^2_{21}	41.9315	34.1196	29.1031	25.7716	23.4357
λ^2_{22}	66.1331	59.0487	54.7599	51.122	51.1603
λ^2_{31}	70.8815	55.1428	44.4972	38.1021	33.3154
λ^2_{32}	86.5435	77.1999	68.1626	61.842	56.6745

The plate mode shapes of a full clamped CCCC plate with thickness ratios $\alpha = 2.0$ and $\alpha = 1.0$, panel width ratio $\gamma = 0.5$ and aspect ratio $\beta = 1.6$, are plotted as shown in Figs. 3-a,b and 4-a,b in the appendix.

V. CONCLUSION

The developed method is successfully applied to free vibration analysis of stepped thickness plates with different combinations of boundary conditions. It has been shown that the natural frequencies calculated using the IVFD method agrees closely with the results in the published literature. It has also been found that the solutions converge rapidly with the increase in the number of grid lines in the direction of the step-by-step integration technique. However, it is noticed that the number of the grid lines in the direction of the initial value method have to be chosen for the other direction of finite difference method. This is due to the efficiency of the two different techniques used in the two directions. The method can be developed to extend in the future work to deal with the dynamic problems of different shapes of plates as well as stepped plates with hollow and circular stepped plate.

APPENDIX

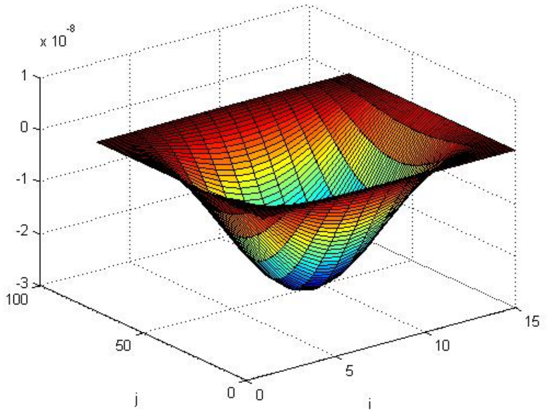

Fig. 3(a) 1st Mode shape of $\lambda^2 = 26.1928$ and $\alpha = 1.0$

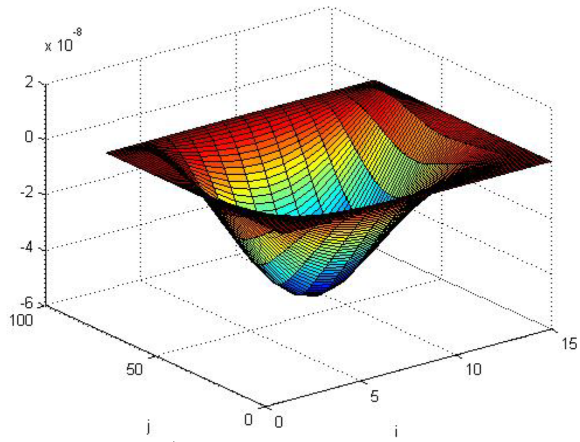

Fig. 3(b) 1st Mode shape of $\lambda^2 = 18.649$ and $\alpha = 2.0$

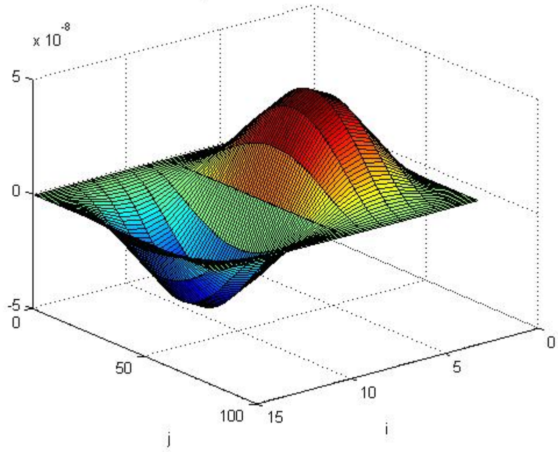

Fig. 4(a) 2nd Mode shape of $\lambda^2 = 37.9107$ and $\alpha = 1.0$

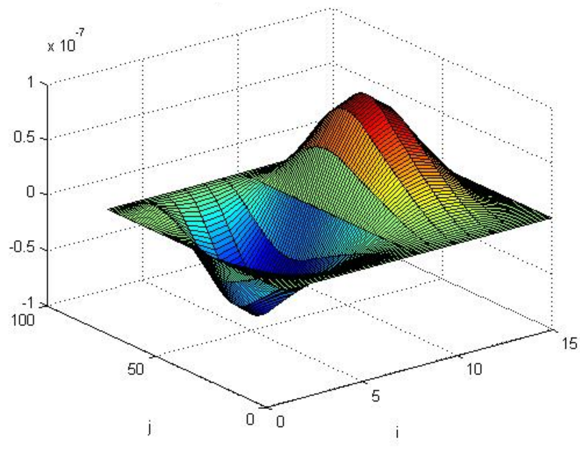

Fig. 4(b) 2nd Mode shape of $\lambda^2 = 29.1031$ and $\alpha = 2.0$

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