Extended Cubic B-spline Interpolation Method 
Applied to Linear Two-Point Boundary Value Problems

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Abstract—Linear two-point boundary value problem of order two is solved using extended cubic B-spline interpolation method. There is one free parameter, λ, that control the tension of the solution curve. For some λ, this method produced better results than cubic B-spline interpolation method.

Keywords—two-point boundary value problem, B-spline, extended cubic B-spline.

I. INTRODUCTION

Consider the general form of linear two-point boundary value problem

\[ u''(x) + p(x)u'(x) + q(x)u(x) = r(x), \]
\[ x \in [a,b], \quad u(a) = \alpha, \quad u(b) = \beta. \] (1)

This problem has a unique solution, \( u(x) \), if \( p, q, r \in C^1 \) and \( q(x) < 0 \) [1]. Generally, this problem is difficult to solve analytically. Some of the most frequently used numerical methods are shooting, finite difference, finite element and finite volume methods [1], [2]. These methods, although requiring little computational time, evaluate the approximated solutions only at the collocation points, \( u(x_i) \) for \( i = 0, 1, ..., n \).

A different approach of solving linear two-point boundary value problem has first been suggested by Bickley in 1968 [3]. He used cubic spline interpolation to model the solution curve and applied the differential equation as well as the boundary conditions to solve for the unknown constants. As a result, a set of equations could be produced approximating the analytical solution. Further work on this approach can be found in [4], [5]. Thirty years later, Caglar et al. proposed the use of cubic B-spline interpolation to solve this problem. The basis function of B-spline is constructed using piecewise polynomial function that satisfies \( C^2 \) continuity. The definition and properties of the function as well as their approach can be found in [6] and the references therein. Continuing with this work, we applied the same procedure using extended cubic B-spline interpolation to solve the problem.

Extended B-spline is a generalization of B-spline. One free parameter, λ, is introduced within the basis function that can be used to change the shape of the produced curve. The value of λ is varied systematically and the results were analyzed. The value of λ producing the least error is identified. One example is provided at the end.

II. EXTENDED CUBIC B-SPLINE BASIS FUNCTION

For a finite interval \([a,b]\), let \( \{x_i\}_{i=0}^n \) be a partition of the interval with uniform step size, \( h \). We can extend the partition using

\[ h = \frac{b-a}{n}, \quad x_0 = a, \quad x_i = x_0 + ih, \quad i = \pm 1, \pm 2, \pm 3, ... \]

Extended cubic B-spline basis function is constructed by linear combination of the cubic B-spline basis function [7]. Here, blending function of degree 4, \( EB_{3,4}(x) \), is considered and the resulting function is shown in (2).

\[
\begin{align*}
\frac{1}{24h^4} \left[ b_1(x), \quad x \in [x_i, x_{i+1}] , \\
b_{i+1}(x), \quad x \in [x_{i+1}, x_{i+2}] , \\
b_{i+2}(x), \quad x \in [x_{i+2}, x_{i+3}] , \\
b_{i+3}(x), \quad x \in [x_{i+3}, x_{i+4}] \right]
\end{align*}
\] (2)

Extended cubic B-spline basis will degenerate into cubic B-spline basis when \( \lambda = 0 \). For \( \lambda \in [-8, 1] \), B-spline and extended B-spline share the same properties: local support, non-negativity, partition of unity and \( C^2 \) continuity.

III. EXTENDED CUBIC B-SPLINE INTERPOLATION

Given \( \{x_i\} \), the extended cubic B-spline function, \( S(x) \) is a linear combination of the extended cubic B-spline basis function,

\[
S(x) = \sum_{i=-3}^{n-1} C_i EB_{3,4}(x), \quad x \in [x_0, x_n],
\] (3)

where \( C_i \) are unknown real coefficients. Since \( EB_{3,4}(x_i) \) has a support on \([x_i, x_{i+4}]\), there are three nonzero basis

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function evaluated at each \( x_i : EB_{3, i-3}(x_i), EB_{3, i-2}(x_i) \) and \( EB_{3, i-1}(x_i) \). Thus, from (3), for \( i = 0, 1, ..., n \),

\[
S(x_i) = C_{i-3}E_{4, i-3}(x_i) + C_{i-2}E_{4, i-2}(x_i) + C_{i-1}E_{4, i-1}(x_i),
\]

\[
S'(x_i) = C_{i-3}E_{4, i-3}'(x_i) + C_{i-2}E_{4, i-2}'(x_i) + C_{i-1}E_{4, i-1}'(x_i),
\]

\[
S''(x_i) = C_{i-3}E_{4, i-3}''(x_i) + C_{i-2}E_{4, i-2}''(x_i) + C_{i-1}E_{4, i-1}''(x_i),
\]

Returning to the two-point boundary value problem stated in (1), \( S(x) \) is presumed to be the approximation of its solution, \( u(x) \). Substituting \( S(x) \) into (1), the equation becomes

\[
\begin{align*}
    u''(x) + p(x)u'(x) + q(x)u(x) &= r(x), \\
    x &\in [a, b], \\
    u(a) &= \alpha, \\
    u(b) &= \beta.
\end{align*}
\]

Substituting (4), (5) and (6) into (7) would result in a system of linear equations of order \((n + 3) \times (n + 3)\). The \( C_i \)'s are solved from the system and are substituted in (3). The resulting equation becomes the approximated analytical solution for (1).

### IV. Varying \( \lambda \)

The value of \( \lambda \) is varied systematically in the neighborhood of zero using brute force with suitable step size. At each trial, Max-norm and \( L^2 \)-norm for the solution are calculated. The values of \( \lambda \) with the lowest norms are identified. Suppose that the true and approximated solution of (1) are \( u(x) \) and \( S(x) \), respectively. The norms are calculated using the following equations:

\[
\begin{align*}
    \text{Max-norm} &= \max_{i=0}^{n} |S(x_i) - u(x_i)|, \\
    L^2\text{-norm} &= \sum_{i=0}^{n} |S(x_i) - u(x_i)|^2.
\end{align*}
\]

### V. NUMERICAL EXAMPLE AND CONCLUSION

**Problem 5.1** [6]

\[
u''(x) - u'(x) = -e^{x^{-1}} - 1, \quad x \in [0, 1], \quad u(0) = u(1) = 0.
\]

Exact solution: \( u(x) = x(1 - e^{x^{-1}}) \).

Problem 5.1 was solved using extended cubic B-spline interpolation method. The numerical results are shown in Table I. The first row is the norms when \( \lambda = 0 \), that is, for cubic B-spline interpolation method. Using \( \lambda = 2.9762 \times 10^{-3} \), the approximated analytical solution is given in (8). The plots of \( S(x) \) and \( u(x) \) along with the error are presented in Figure 1.

<table>
<thead>
<tr>
<th>( \lambda )</th>
<th>Max-Norm</th>
<th>( L^2 )-Norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>2.8996 \times 10^{-4}</td>
<td>6.6069 \times 10^{-4}</td>
</tr>
<tr>
<td>2.9762 \times 10^{-3}</td>
<td>3.1415 \times 10^{-6}</td>
<td>7.2625 \times 10^{-6}</td>
</tr>
<tr>
<td>2.9776 \times 10^{-3}</td>
<td>3.2452 \times 10^{-6}</td>
<td>7.2555 \times 10^{-6}</td>
</tr>
</tbody>
</table>

These results show that extended cubic B-spline has potential to approximate the solution of two-point boundary value problems better than B-spline. Here, we used the exact solution of the problem as a reference to find good values of \( \lambda \). Therefore, future work will focus on finding the values of \( \lambda \) that produce better approximation from the differential equation in (1) itself without using the exact solution. This study confirmed that for some problems, these values do exist.

### REFERENCES


