# Delay-range-dependent exponential synchronization of Lur'e systems with Markovian switching

Xia Zhou, Shouming Zhong

*Abstract*—The problem of delay-range-dependent exponential synchronization is investigated for Lur'e master-slave systems with delay feedback control and Markovian switching. Using Lyapunov-Krasovskii functional and nonsingular M-matrix method, novel delay-range-dependent exponential synchronization in mean square criterions are established. The systems discussed in this paper is advanced system, and takes all the features of interval systems, Itô equations, Markovian switching, time-varying delay, as well as the environmental noise, into account. Finally, an example is given to show the validity of the main result.

*Keywords*—Synchronization, Delay-range-dependent, Markov chain, Generalized Itô's formula, Brownian motion, M-matrix.

## I. Introduction

CHAOS is very interesting nonlinear phenomenon and has been intensively studied in the last three decades. It is found to be useful or has great potential in many disciplines. Since Pecora and Carroll [1] addressed the synchronization problem of chaotic systems using a drive-response conception, the subject of chaotic synchronization has received considerable attentions [2-11]. Synchronization has been widely explored in a variety of fields, such as physical, chemical and ecological systems, human heartbeat regulation, secure communications, and so on.

Recently, the effect of delay on synchronization between two chaotic systems has been reported in many literatures due to the unavoidable signal propagation delay. In [12], Yalcin ME, Suykens JAK, and Vandewalle studied the master-slave synchronization of Lur'e systems with time-delay of the form

$$\mathfrak{M}: \begin{cases} \dot{x}(t) = Ax(t) + Bf(Cx(t)) \\ p(t) = Hx(t) \end{cases} \tag{1}$$

$$\mathfrak{S}: \begin{cases} \dot{y}(t) = Ay(t) + Bf(Cy(t)) + u(t) \\ q(t) = Hy(t) \end{cases} \tag{2}$$

$$\mathfrak{G}: u(t) = K(x(t) - y(t)) + M(p(t - \tau_1) - q(t - \tau_1)). \tag{3}$$

with master system $\mathfrak{M}$, slave system $\mathfrak{S}$ and controller $\mathfrak{G}$, where the time delay $\tau_1 > 0$ is constant, state vectors $x, y \in R^n$, outputs of subsystems $p, q \in R^l$, $H, A, B, C$ are real matrices, $f(.)$ is a sector condition. They are derived some delay-independent and delay-dependent synchronization criteria. In [13], Jinde Cao , H.X.Li b, Daniel W.C. Ho, studied the systems (1)–(3), employed model transformation, which leads to some conservative synchronization criteria for inducing additional terms. In [14], Ji Xiang, Yanjun Li, WeiWei used Integral inequality approach studied the same

X. Zhou and S. Zhong are with School of Mathematical sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 611731, China. E-mail: (zhouxia44185@.sohu.com).

systems and again improved synchronization condition. In [15], Tao Li, Jianjiang Yu, and Zhao Wang, they considered the time-varying delay which often arises and may vary in a range, they studied the system of the form

$$\mathfrak{M}: \begin{cases} \dot{x}(t) = Ax(t) + Bf(Cx(t)) \\ p(t) = Hx(t) \end{cases} \tag{4}$$

$$\mathfrak{S}: \begin{cases} \dot{y}(t) = Ay(t) + Bf(Cy(t)) + u(t) \\ q(t) = Hy(t) \end{cases} \tag{5}$$

$$\mathfrak{G}: u(t) = M(p(t - d(t)) - q(t - d(t))). \tag{6}$$

where the time-delay $h_1 \leq d(t) \leq h_2$ and $\dot{d}(t) < \mu$. And derived the delay-range-dependent asymptotical synchronization criteria.

The hybrid systems combine a part of the state that takes values continuously and another part of the state that takes discrete values, an important class of hybrid systems is the semi-linear stochastic differential equation with Markovian switching of the form

$$\mathfrak{M}: \begin{cases} \dot{x}(t) = A(r(t))x(t) + B(r(t))f(Cx(t)) \\ p(t) = Hx(t) \end{cases} \tag{7}$$

$$\mathfrak{S}: \begin{cases} \dot{y}(t) = A(r(t))y(t) + B(r(t))f(Cy(t)) + u(t) \\ q(t) = Hy(t) \end{cases} \tag{8}$$

$$\mathfrak{G}: u(t) = M(p(t - d(t)) - q(t - d(t))). \tag{9}$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2, ..., N\}$. Continuous-time Marlov chains are used to model the abrupt changes in system structure and parameters. If we also take the environmental noise into account, the systems (7)–(9)becomes

$$\mathfrak{M}: \begin{cases} dx(t) = [A(r(t))x(t) + B(r(t))f(Cx(t))]dt \\ \quad\quad + D(r(t))x(t)dw(t) \\ p(t) = Hx(t) \end{cases} \tag{10}$$

$$\mathfrak{S}: \begin{cases} dy(t) = [A(r(t))y(t) + B(r(t))f(Cy(t)) \\ \quad\quad + u(t)]dt + D(r(t))y(t)dw(t) \\ q(t) = Hy(t) \end{cases} \tag{11}$$

$$\mathfrak{G}: u(t) = M(p(t - \delta(t)) - q(t - \delta(t))). \tag{12}$$

In this paper, we will discuss (10)–(12), which is advanced system, and takes all the features of interval systems, Itô equations, Markovian switching, time-varying delay, as well as the environmental noise, into account. Then we will give the delay-range-dependent exponential synchronization criteria.

The rest of this paper is organized as follows. In section 2, we introduce the basic notation, lemma's and some definitions. In section 3, give our main results and corollary's.

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:3, 2010

In section 4, an example is given to show the effectiveness and less conservatism of the proposed criterion.

## II. NOTATION AND PRELIMINARIES

Throughout this article, unless otherwise specified, we use the following notations. Let $|.|$ be the Euclidean norm in $R^n$. If $A$ is a vector or matrix, its transpose is denoted by $A^T$. If $A$ is a matrix, its trace norm is denoted by $|A| = \sqrt{trace(A^T A)}$ while its operator norm is denoted by $\|A\| = \sup\{|Ax| : |x| = 1\}$. If $A$ is a symmetric matrix, denote by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$ its largest and smallest eigenvalue, respectively.

Let $R_+ = [0, \infty)$ and $\tau > 0$. Let $C([-\tau, 0]; R^n)$ denote the family of continuous functions $\varphi$ from $[-\tau, 0]$ to $R^n$ with the norm $\|\varphi\| = \sup_{-\tau \le \theta < 0}|\varphi(\theta)|$. Let $\delta(t) : R_+ \to [0, \tau]$ be a continuous function which will stand for the time delay of the systems discussed in this paper. As a standing hypothesis, we shall always assume that $\delta(t)$ is differentiable and its derivative is bounded by a constant less than one, namely $\dot{\delta}(t) \le \delta_0 < 1, \forall t \ge 0$. Let $\{\Omega, \mathcal{F}, \mathbb{P}\}$ be a complete probability space equipped with some filtration $\{\mathcal{F}_t\}_{t \ge 0}$ satisfying the usual conditions, i.e., the filtration is right continuous and $\mathcal{F}_0$-contains all $\mathbb{P}$-null sets. $C_{\mathfrak{F}_0}^b([-\tau, 0]; R^n)$ : the family of all bounded, $C([-\tau, 0]; R^n)$-valued, $\mathfrak{F}_0$measurable random variables. Let $\mathbb{W}(t)$ be a standard n-dimensional Brownian motion defined on the probability space. Let $r(t), t \ge 0$ be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, ..., N\}$ with generator $\Gamma = (r_{ij})_{N \times N}$ given by

$$P\{r(t+\Delta) = j|r(t) = i\} = \begin{cases} r_{ij}\Delta + o(\Delta), & \text{if } i \ne j \\ 1 + r_{ii}\Delta + o(\Delta), & \text{if } i = j \end{cases}$$

where $\Delta > 0$. Here $r_{ij} \ge 0$ is the transition rate from $i$ to $j$ if $i \ne j$, while $r_{ii} = -\sum_{j \ne i} r_{ij}$. We assume that the Markov chain $r(t)$ is independent of the Brownian motion $w(t)$. It is well known that almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $R_+$. In other words, there is a sequence of stopping times $0 = \tau_0 < \tau_1 < ... < \tau_k \to \infty$ almost surely such that $r(t) = \sum_{k=0}^{\infty} r(\tau_k)I_{[\tau_k, \tau_{k+1})}$, where $I_A$ denotes the indicator function of set $A$.

If $A$ and $B$ are symmetric matrix, by $A > B$ and $A \ge B$ we means that $A - B$ is positive and nonnegative definite, respectively. If $A_1$ is a vector or matrix, by $A_1 \gg 0$ we mean all elements of $A_1$ are positive. If $A_1$ and $A_2$ are vectors or matrices with same dimensions, we write $A_1 \gg A_2$ if and only if $A_1 - A_2 \gg 0$. Moreover, we also adopt here the traditional notation by letting $Z^{N \times N} = \{A = [a_{ij}]_{N \times N} : a_{ij} \le 0, i \ne j\}$. Now defining the synchronization error as $e(t) = y(t) - x(t)$, (10)–(12) has the error-dynamics system of the form

$$\begin{cases} de(t) = [A(r(t))e(t) + B(r(t))\eta(Ce, x) \\ \quad + MHe(t - \delta(t))]dt \\ \quad + D(r(t))e(t)dw(t), \forall t \ge 0 \\ e(t) = \xi(t), t \in [-\tau, 0] \end{cases} \quad (13)$$

where $\xi(t) = \psi(t) - \phi(t), t \in [-\tau, 0]$, $\eta(Ce, x) := f(Ce + Cx) - f(Cx)$ and assume $\eta(Ce, x) \le KCe(t)$. State vectors $x, y \in R^n$, outputs of subsystems $p, q \in R^l$, and matrices $H \in R^{l \times n}, A_r \in R^{n \times n}, B_r \in R^{n \times n_h}, C \in R^{n_h \times n}, D_r \in R^{n \times n}, K \in R^{n \times n_h}$, $r(t)$ is a Markov chain taking values in $S = \{1, 2, ..., N\}$.

The purpose of this paper is to find the condition made the erro-dynamics systems (13) exponential stable in mean square, which means that the system described by (10)–(12) exponential synchronization in mean square.

We still denote by $x(t, \xi)$, it is known that $\{x(t, \xi), r(t)\}_{t \ge 0}$ is a $C([-\tau, 0]; R^n) \times S$-valued Markov process. Its infinitesimal operator $\mathfrak{L}$, acting on functional $V:C([-\tau, 0]; R^n) \times S \times R_+ \to R$, is defined by

$$\mathfrak{L}V(x(t), i, t) = \lim_{\Delta \to 0^+} \frac{1}{\Delta}[E(V(x(t + \Delta), r(t + \Delta), \\ t + \Delta)|x(t), r(t) = i) - V(x(t), i, t)]. \quad (14)$$

*Definition 2.1. The master-slave system (10)–(12) can be exponentially synchronized in mean square, if the trivial solution of equation (13) is exponentially stable in mean square, i.e.*

$$\lim_{t \to \infty} \sup \frac{1}{t} \ln(E(|e(t; \xi)|^2) < 0 \quad (15)$$

*for any initial data $\xi \in C_{\mathfrak{F}_0}^b([-\tau, 0]; R^n)$*

*Definition 2.2. A square matrix $A = [a_{ij}]_{N \times N}$ is called a nonsingular $M$-matrix if $A$ can be expressed in the form $A = sI - B$ with $s > \rho(B)$ while all the elements of $B$ are nonnegative, where $I$ is the identity matrix and $\rho(B)$ the spectral radius of $B$. It is easy to see that a nonsingular $M$-matrix $A$ has nonpositive off-diagonal and positive-diagonal entries, that is $a_{ii} > 0$, while $a_{ij} \le 0, i \ne j$.*

*lemma 2.1. (See Ref.[16]) If $A \in Z^{N \times N}$, then the following statements are equivalent.*
*1) $A$ is a nonsingular $M$-matrix.*
*2) $A$ is semipositive; that is, there exists $x \gg 0$ in $R^N$ such that $Ax \gg 0$.*
*3) $A^{-1}$ exists and its elements are all nonnegative.*
*4) All the leading principal minors of $A$ are positive; that is*

$$\begin{vmatrix} a_{11} & \cdots & a_{1k} \\ \vdots & & \vdots \\ a_{k1} & \cdots & a_{kk} \end{vmatrix} > 0, \text{ for every } k = 1, 2, ..., N$$

*lemma 2.2. Let $x \in R^n, y \in R^n$, matrix $B$ has the suitable dimensional, constant $\varepsilon > 0$ and $Q > 0$. Then, we have*

$$2x^T QBy \le \varepsilon x^T Qx + \varepsilon^{-1} y^T B^T QBy$$

## III. MAIN RESULTS

*Theorem 3.1. Assume that there are two constants $\lambda_1$ and $\lambda_2$ such that*

$$\lambda_1 > \tau\lambda_2 \quad (16)$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:3, 2010

Assume also that there are symmetric matrices $Q_i > 0, E_i \geq 0$ and constants $\varepsilon_i > 0 (1 \leq i \leq N)$ such that

$$\lambda_{\max}(Q_i A_i + A_i^T Q_i + 2Q_i B_i KC + D_i^T Q_i D_i$$
$$+ \sum_{j=1}^{N} r_{ij} Q_j + E_i + \varepsilon_i Q_i) \leq -\lambda_1 \quad (17)$$

$$\lambda_{\max}(\sum_{j=1}^{N} r_{ij} E_j) \leq \lambda_2 \quad (18)$$

$$E_i \geq \frac{1}{1-\delta_0} \varepsilon_i^{-1} H^T M^T Q_i MH \quad (19)$$

for all $i \in S$. Then, for any initial data $\xi \in C_{\mathfrak{F}_0}^b([-\tau, 0]; R^n)$, the solution of (13) has the property that

$$\lim_{t \to \infty} \sup \frac{1}{t} \ln(E(|e(t;\xi)|^2)) < -\lambda < 0 \quad (20)$$

In other words, (13) is exponentially stable in mean square and Lyapunov exponent is not greater than $-\lambda$. Moreover, the positive number $\lambda$ is the unique root to

$$\alpha\lambda + (\lambda_2 + \alpha_1\lambda)\tau e^{\lambda\tau} = \lambda_1 \quad (21)$$

where $\alpha = \max_{1 \leq i \leq N} \lambda_{\max}(Q_i)$,
$\alpha_1 = \max_{1 \leq i \leq N} \lambda_{\max}(E_i)$
*Proof.* Let us first show that (21) does has a unique root $\lambda > 0$. Choose $i$ for $\lambda_{\min}E_i$ to be the smallest of $\lambda_{\min}E_j(1 \leq i \leq N)$, i.e., $\lambda_{\min}E_i = \min_{1 \leq i \leq N} \lambda_{\min}E_j$. Let $v \neq 0$ be the corresponding eigenvector of $E_i$, i.e., $E_i v = \lambda_{\min}(E_i)v$. Then $v^T E_i v = \lambda_{\min}(E_i)|v|^2$. Moreover

$$v^T(\sum_{j=1}^{N} r_{ij} E_j)v = \sum_{j \neq i}^{N} r_{ij} v^T E_j v + r_{ii} v^T E_i v$$
$$\geq \sum_{j \neq i}^{N} r_{ij} \lambda_{\min}(E_j)|v|^2 + r_{ii} \lambda_{\min}(E_i)|v|^2$$
$$\geq \lambda_{\min}(E_i)|v|^2 \sum_{j=1}^{N} r_{ij}$$
$$= 0.$$

Thus

$$\lambda_{\max}(\sum_{j=1}^{N} r_{ij} E_j)|v|^2 \geq v^T(\sum_{j=1}^{N} r_{ij} E_j)v \geq 0$$

Since $|v| > 0$, we have

$$\lambda_{\max}(\sum_{j=1}^{N} r_{ij} E_j) \geq 0$$

so we obtain $\lambda_2 \geq 0$, with (16), we see that $\lambda_1 > 0$, so (21) does has a unique root $\lambda > 0$. Next, we show that the solution of (13) has the property of (20). Fix any initial data $\xi$ and write $e(t;\xi) = e(t)$. Let us define the Lyapunov functional $V : C([-\tau, 0]; R^n) \times S \times R_+ \to R$ by

$$V(e(t), i, t) = e(t)^T Q_i e(t) + \int_{t-\delta(t)}^{t} e(s)^T E_i e(s) ds$$

while by (14)

$$\mathfrak{L}V(e(t), i, t) = 2e(t)^T Q_i[A(r(t))e(t) + B(r(t))\eta(Ce, x)$$
$$+ MHe(t-\delta(t))] + \sum_{j=1}^{N} r_{ij} e(t)^T Q_j e(t)$$
$$+ (D(r(t))e(t))^T Q_i D(r(t))e(t)$$
$$- (1-\dot{\delta}(t))e(t-\delta(t))^T E_i e(t-\delta(t))$$
$$+ \sum_{j=1}^{N} r_{ij} \int_{t-\delta(t)}^{t} e(\theta)^T E_j e(\theta)d\theta$$
$$+ e(t)^T E_i e(t)$$

Using the assumptions,(17), (18), (19), and lemma 2.2, we compute

$$\mathfrak{L}V(e(t), i, t) \leq e(t)^T(Q_i A_i + A_i^T Q_i)e(t) + e(t)^T E_i e(t)$$
$$+ 2e(t)^T Q_i B_i KCe(t) + e(t)^T(D_i^T Q_i D_i)e(t)$$
$$+ 2e(t)^T Q_i MHe(t-\delta(t))$$
$$+ e(t)^T(\sum_{j=1}^{N} r_{ij} Q_j)e(t)$$
$$- (1-\delta_0)e(t-\delta(t))^T E_i e(t-\delta(t))$$
$$+ \int_{t-\delta(t)}^{t} e(\theta)^T(\sum_{j=1}^{N} r_{ij} E_j)e(\theta)d\theta$$
$$\leq e(t)^T(Q_i A_i + A_i^T Q_i + 2Q_i B_i KC$$
$$+ D_i^T Q_i D_i \sum_{j=1}^{N} r_{ij} Q_j + E_i + \varepsilon_i Q_i)e(t)$$
$$+ e(t-\delta(t))^T[\varepsilon_i^{-1} H^T M^T Q_i MH$$
$$- (1-\delta_0)E_i]e(t-\delta(t))$$
$$+ \int_{t-\delta(t)}^{t} e(\theta)^T(\sum_{j=1}^{N} r_{ij} E_j)e(\theta)d\theta$$
$$\leq -\lambda_1|e(t)|^2 + \lambda_2 \int_{t-\delta(t)}^{t} |e(\theta)|^2 d\theta$$
$$\quad (22)$$

Let us define the Lyapunov functional $V_1 : C([-\tau, 0]; R^n) \times S \times R_+ \to R$ by

$$V_1(e(t), i, t) = e^{\lambda t} V(e(t), i, t)$$

By the generalized Itô formula, we have

$$EV_1(e(t), i, t) = EV_1(\xi, r(0), 0) + E \int_{0}^{t} \mathfrak{L}V_1(e(s), r(s), s)ds \quad (23)$$

and it is straightforward to see that

$$\mathfrak{L}V_1(e(t), i, t) = e^{\lambda t}[\lambda V(e(t), i, t) + \mathfrak{L}V(e(t), i, t)] \quad (24)$$

we note that

$$V(e(t), i, t) \leq \alpha|e(t)|^2 + \alpha_1 \int_{t-\delta(t)}^{t} |e(\theta)|^2 d\theta \quad (25)$$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:3, 2010

Substituting (22), (24), (25) into (23) we obtain that

$$EV_1(e(t),i,t) \leq EV_1(\xi,r(0),0) + E\int_0^t e^{\lambda s}\lambda(\alpha|e(s)|^2$$
$$+ \alpha_1\int_{s-\delta(s)}^s |e(\theta)|^2 d\theta)ds$$
$$+ E\int_0^t e^{\lambda s}(-\lambda_1|e(s)|^2$$
$$+ \lambda_2\int_{s-\delta(s)}^s |e(\theta)|^2 d\theta)ds$$
$$= (\lambda_2 + \alpha_1\lambda)E\int_0^t e^{\lambda s}(\int_{s-\delta(s)}^s |e(\theta)|^2 d\theta)ds$$
$$- (\lambda_1 - \alpha\lambda)E\int_0^t e^{\lambda s}|e(s)|^2$$
$$+ EV_1(\xi,r(0),0)$$
(26)

We compute

$$\int_0^t e^{\lambda s}(\int_{s-\delta(s)}^s |e(\theta)|^2 d\theta)ds \leq \int_0^t e^{\lambda s}(\int_{s-\tau}^s |e(\theta)|^2 d\theta)ds$$
$$\leq \int_{-\tau}^t \int_\theta^{\theta+\tau} e^{\lambda s}ds|e(\theta)|^2 d\theta$$
$$\leq \tau e^{\lambda\tau}\int_{-\tau}^t e^{\lambda\theta}|e(\theta)|^2 d\theta$$
(27)

Substituting (27) into (26), and using (21) we get

$$EV_1(e(t),i,t) \leq EV_1(\xi,r(0),0)$$
$$+ (\lambda_2 + \alpha_1\lambda)\tau e^{\lambda\tau}\int_{-\tau}^0 e^{\lambda s}|\xi(s)|^2 ds$$
(28)

On the other hand, we also note that

$$EV_1(e(t),r(t),t) = e^{\lambda t}E[e(t)^T Q_{r(t)}e(t)$$
$$+ \int_{t-\delta(t)}^t e(\theta)^T E_{r(t)}e(\theta)d\theta]$$
$$\geq e^{\lambda t}E[e(t)^T Q_{r(t)}e(t)]$$
$$\geq e^{\lambda t}\min_{1\leq i\leq N}\lambda_{\min}(Q_i)E|e(t)|^2$$
(29)

By (28), (29), we know that

$$\lim_{t\to\infty}\sup \frac{1}{t}\ln(E(|e(t;\xi)|^2)) < -\lambda < 0$$

In other words, (13) is exponentially stable in mean square and Lyapunov exponent is not greater than $-\lambda$.

If we let $\varepsilon_i = 1, \forall i \in S$ in Theorem 3.1, we obtain the following useful result.

*Corollary 3.1.* Assume that there are two constants $\lambda_1$ and $\lambda_2$ such that

$$\lambda_1 > \tau\lambda_2$$

Assume also that there are symmetric matrices $Q_i > 0$, and $E_i \geq 0(1 \leq i \leq N)$ such that

$$\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC +$$
$$D_i^TQ_iD_i + \sum_{j=1}^N r_{ij}Q_j + E_i + Q_i) \leq -\lambda_1$$
$$\lambda_{\max}(\sum_{j=1}^N r_{ij}E_j) \leq \lambda_2$$
$$E_i \geq \frac{1}{1-\delta_0}H^TM^TQ_iMH$$

for all $i \in S$. Then, for any initial data $\xi \in C_{\mathfrak{F}_0}^b([-\tau,0];R^n)$, the solution of (13) is exponentially stable in mean square and Lyapunov exponent is not greater than $-\lambda$.

*Remark3.1.* Corollary 3.1 is stated without $\varepsilon_i$, so it looks neat, but Theorem 3.1 is more general since it allows to choose different $\varepsilon_i$ for different situations in practice, for example, if we choose $\varepsilon_i = \|HM\|$, we can get corollary 3.2.

*Corollary 3.2.* Assume there are symmetric matrices $Q_i > 0$, and $E_i \geq 0(1 \leq i \leq N)$ such that

$$E_i \geq \frac{1}{1-\delta_0}\|HM\|^{-1}H^TM^TQ_iMH$$

for all $i \in S$. Suppose we can verify $\lambda_1 > \tau\lambda_2$ where

$$\lambda_1 = -\max_{1\leq i\leq N}\{\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC$$
$$+ D_i^TQ_iD_i + \|HM\|Q_i + \sum_{j=1}^N r_{ij}Q_j + E_i)\}$$
$$\lambda_2 = \max_{1\leq i\leq N}\lambda_{\max}(\sum_{j=1}^N r_{ij}E_j)$$

Then, for any initial data $\xi \in C_{\mathfrak{F}_0}^b([-\tau,0];R^n)$, the solution of (13) is exponentially stable in mean square and Lyapunov exponent is not greater than $-\lambda$. And the $\lambda$ can be determined in the same way as stated in Theorem 3.1. Let we define

$$\beta = \frac{1}{1-\delta_0}\max_{1\leq i\leq N}\{\lambda_{\max}\|HM\|^{-1}H^TM^TQ_iMH\}$$

If we choose $E_i = \beta I$ in corollary 3.2, we can easily get corollary 3.3.

*Corollary 3.3.* Assume there are symmetric matrices $Q_i > 0(1 \leq i \leq N)$ such that

$$\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC$$
$$+ D_i^TQ_iD_i + \|HM\|Q_i + \sum_{j=1}^N r_{ij}Q_j) < -\beta$$

Then the solution of (13) has the property

$$\lim_{t\to\infty}\sup \frac{1}{t}\ln(E(|e(t;\xi)|^2)) < -\lambda < 0$$

That is the solution of (13) exponentially stable in mean square and Lyapunov exponent is not greater than $-\lambda$. Where the $\lambda$

World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:4, No:3, 2010

is the unique root to

$$\lambda(\alpha + \beta\tau e^{\lambda\tau}) = \lambda_1$$

Where $\alpha$ is the same as defined in Theorem 3.1 but

$$\lambda_1 = -\max_{1 \leq i \leq N}\{\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC$$
$$+ D_i^TQ_iD_i + \|HM\|Q_i + \sum_{j=1}^{N}r_{ij}Q_j)\} - \beta$$

*Theorem 3.2.* Define the matrix

$$K = diag(-\lambda_{\max}(A_1 + A_1^T + 2B_1KC) - \|HM\| - \|D_1\|^2,$$
$$... - \lambda_{\max}(A_N + A_N^T + 2B_NKC) - \|HM\| - \|D_N\|^2)$$

and the vector $\kappa = (1 - \delta_0)\begin{bmatrix} \|HM\|^{-1} \\ \vdots \\ \|HM\|^{-1} \end{bmatrix}$

(Set $a^{-1} = \infty$ when $\alpha = 0$ as ussal.)If $K - \Gamma$ is a nonsingular $M$-matrix and

$$\kappa \gg (K - \Gamma)^{-1}\vec{1} \tag{30}$$

where $\vec{1} = (1, ...1)^T$, then the solution of (13) exponentially stable in mean square.

*Proof.* Since $K - \Gamma$ is a nonsingular $M$-matrix, by lemma 2.1, we observe that $K - \Gamma^{-1}$ exist and all the elements of $K - \Gamma^{-1}$ are nonnegative. $K - \Gamma^{-1}$ is invertible, its each row must have at least one nonzero, and hence positive element. Let

$$\vec{q} = (q_1, q_2, ..., q_N)^T = (K - \Gamma)^{-1}\vec{1}$$

then $\vec{q} \gg 0$. By (30)

$$q_i\|HM\| < 1 - \delta_0, \quad \forall \quad i \in S$$

Let $Q_i = qI$ for $i \in S$. so,

$$\lambda_{\max}(\|HM\|^{-1}H^TM^TQ_iMH) \leq q_i(\|HM\|) < 1 - \delta_0$$

therefore $\beta < 1$. On the other hand

$$\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC + D_i^TQ_iD_i$$
$$+ \|HM\|Q_i + \sum_{j=1}^{N}r_{ij}Q_j)$$
$$\leq q_i[\lambda_{\max}(A_i + A_i^T + 2B_iKC) + \|D_i\|^2I$$
$$+ \|HM\|I] + \sum_{j=1}^{N}r_{ij}q_j$$
$$= -[(K - \Gamma)\vec{q}]_i$$

where $[(K - \Gamma)\vec{q}]_i$ stands for the $i$th element of the vector $(K - \Gamma)\vec{q}$. Then

$$\lambda_{\max}(Q_iA_i + A_i^TQ_i + 2Q_iB_iKC + D_i^TQ_iD_i$$
$$+ \|HM\|Q_i + \sum_{j=1}^{N}r_{ij}Q_j) \leq -1 < -\beta$$

for all $i \in S$. Therefore the result now follows from corollary 3.3.

*Remark3.2.* In [15], the authors studied the global asymptotically synchronization results for Lur'e systems with delay feedback control. However, the stochastic term and Markovain switching were not taken into account in the models. Therefore, our developed results in this paper are more general than reported in [12].

*Remark3.3.* In [16], the author studied the exponential stability of stochastic delay interval systems with Markovian switching using Lyapunov-Krasovskii functional and nonsingular $M$-matrix methods which the same in this paper. But he studied the system is linear system, while we studied the system is nonlinear.

*Remark3.4.* In [15], the authors studied the global asymptotically synchronization results for Lur'e systems with delay feedback control. But we studied the exponential synchronization in mean square. So, our results are better than reported in [12].

## IV. AN ILLUSTRATIVE EXAMPLE

In this section we shall present one example to illustrate our theory.

Example 4.1. Let $w(t)$ be a 2-dimensional Brownian motion, let $r(t)$ be a right-continuous Markov chain taking values in $S = \{1, 2\}$ with generator $\Gamma = (r_{ij})_{2\times2}$:

$$-r_{11} = r_{12} > 0, \quad -r_{22} = r_{21} > 0$$

of course $w(t)$ and $r(t)$ are assumed to be independent. Consider the Lur'e systems with delay feedback control and Markovian switching of the form

$$\mathfrak{M}: \begin{cases} dx(t) = A(r(t))x(t)dt + D(r(t))x(t)dw(t) \\ p(t) = Hx(t) \\ x(t) = \phi(t), t \in [-\tau, 0] \end{cases}$$

$$\mathfrak{S}: \begin{cases} dy(t) = [A(r(t))y(t) + u(t)]dt + D(r(t))y(t)dw(t) \\ q(t) = Hy(t) \\ y(t) = \psi(t), t \in [-\tau, 0] \end{cases}$$

$$\mathfrak{G}: U(t) = M(p(t - \delta(t)) - q(t - \delta(t))).$$

where $\delta(t) = 0.1\sin^2 t$ and $\dot{\delta}(t) = 0.2\sin(t)\cos(t) \leq 0.1 = \delta_0$, $A_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}$, $A_2 = \begin{bmatrix} -2 & -2 \\ 0 & -1 \end{bmatrix}$, $D_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $D_2 = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$, $M = \begin{bmatrix} \frac{1}{81} & -\frac{1}{81} \\ 0 & \frac{2}{9} \end{bmatrix}$, $H = \begin{bmatrix} 9 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Then the solution of Lur'e systems with delay feedback control and Markovian switching is exponentially synchronization in mean square.

## V. CONCLUSION

The problem of master$-$slave exponential synchronization for Lure systems has been addressed by employing time-delay feedback control techniques. By using the methods of Lyapunov-Krasovskii functional and nonsingular M-matrix,

some effective criterions for achieving synchronization have been derived. Finally, an example has been given to illustrate the validity theoretical results obtained in this paper.

### REFERENCES

[1] Pecora LM. *synchronization in chaotic systems*. Phys Rev Lett 1991;64:821-24.

[2] H.M.Guo, S.M.Zhong. *Synchronization criteria of time-delay feedback control system with sector-bounded nonlinearity*. Applied Mathematics and Computation 2007;191:550-9.

[3] Z.M.Ge, J.K.Lee. *Chaos synchronization and parameter identification for gyroscope system*. Applied Mathematics and Computation 2005;163:667-82.

[4] Z.X. Liu, S.L, S.M.Zhong, M.Ye.*pth moment exponential synchronization analysis for a class of stochastic neural networks with mixed delays*. Communications in Nonlinear Science and Numerical Simulation. 2010;15:1899-1909.

[5] X.Z.Gao, S.M.Zhong, Fengyin Gao. *Exponential synchronization of neural networks with time-varying delays*. Nonlinear Analysis: Theory, Methods and Applications. 2009;71:2003-11.

[6] H.M. Guo, S.M.Zhong, F.Y.Gao. *Design of PD controller for master-slave synchronization of Lur'e systems with time-delay*. Applied Mathematics and Computation. 2009;212:86-93.

[7] H.H.Chen. *Global synchronization of chaotic systems via linear balanced feedback control*. Applied Mathematics and Computation 2007;186:923-31.

[8] B.Wang, G.J.Wen. *On the synchronization of uncertain masterC-slave chaotic systems with disturbance*. Chaos, Solitons and Fractals. 2009;41:145-51.

[9] X.C.Li, W.Xu, R.H.Li. *Chaos synchronization of the energy resource system*. Chaos, Solitons and Fractals 2009;40:642-52.

[10] C.F.Huang, K.H.Cheng, J.J.Yan. *Robust chaos synchronization of four-dimensional energy resource systems subject to unmatched uncertainties*, Commun Nonlinear Sci Numer Simulat 2009;14:2784-92.

[11] X.F.Wu, J.P.Cai, M.H.Wang. *Global chaos synchronization of the para-metrically excited Duffing oscillators by linear state error feedback control*. Chaos, Solitons and Fractals 2008;36:121-8.

[12] Yalicn ME, Suykens JAK, Vandewallw J. *Master-slave synchronization of Lur'e systems with time-delay*, Int J Bifur Chaos 2001;11:1707-22.

[13] J.D.Cao, H.X. Li, Daniel W.Ho. *Synchronization criteria of Lur'e systems with time-delay feedback control*. Chaos, Solitons and Fractals 2005;23:1285-98.

[14] J.Xiang, Y.J.Li, W.Wei. *An improved condition for masterCslave synchronization of Lure systems with time delay*. Physics Letters A 2007;362:154-8.

[15] T.Li, J.J.Yu, Z.Wang. *Delay-range-dependent synchronization ruiterion for Lur'e systems with delay feedback control*. Commun Nonlinear Sci Numer Simulat 2009;14:1796-803.

[16] X.R.Mao. *Exponential stability of stochastic delay interval systems with Markonvain switching*. IEEE Transactions on Automatic Control. 2002;47:1604-12.

**Shouming Zhong** was born in 1955 in Sichuan, China. He received B.S. degree in applied mathematics from UESTC, Chengdu, China, in 1982. From 1984 to 1986, he studied at the Department of Mathematics in Sun Yatsen University, Guangzhou, China. From 2005 to 2006, he was a visiting research associate with the Department of Mathematics in University of Waterloo, Waterloo, Canada. He is currently as a full professor with School of Applied Mathematics, UESTC. His current research interests include differential equations, neural networks, biomathematics and robust control. He has authored more than 80 papers in reputed journals such as the In International Journal of Systems Science, Applied Mathematics and Computation, Chaos, Solitons and Fractals, Dynamics of Continuous, Discrete and Impulsive Systems, Acta Automatica Sinica, Journal of Control Theory and Applications, Acta Electronica Sinica, Control and Decision, and Journal of Engineering Mathematics

**Xia Zhou**   was born in Shaanxi Province, China, in 1981. She received the B.S. degree from Shaanxi Normal University, Shaanxi, in 2003 and the M.S. degree from the University of Electronic Science and Technology of China(UESTC), Sichuan, in 2008, both in Applied Mathematics. She is currently pursuing the ph.D degree with UESTC. I research interests include chaos synchronization and the stochastic delay dynamic systems.