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Localized Non-Stability of the Semi-Infinite Elastic Orthotropic Plate

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***Abstract*—**This paper is concerned with an investigation into the localized non-stability of a thin elastic orthotropic semi-infinite plate. In this study, a semi-infinite plate, simply supported on two edges and different boundary conditions, clamped, hinged, sliding contact and free on the other edge, are considered. The mathematical model is used and a general solution is presented the conditions under which localized solutions exist are investigated.

***Keywords*—**Localized, Non-stability, Orthotropic, Semi-infinite

I. Introduction

THE existence of edge waves along the free edge of a homogeneous and isotropic semi-infinite thin plate, modeled using Kirchhoff theory , was first noted by Konenkov [1] . Konenkov established that, for isotropic plates, precisely one edge wave solution exists for all values of the two free parameters, namely the bending stiffness and Poisson's ratio. The edge wave speed is found to be proportional to and slightly less than the speed of flexural (one-dimensional) waves on a plate of infinite extent. Ambartsumyan and Belubekyan [2] considered localized bending waves along the edge of a plate using several non-classical plate theories, concluding that Timoshenko–Mindlin plates do not admit localized edge waves. One of the latest developments in the field has been the localized bending waves in an elastic orthotropic plate, by Mkrtchyan [3].

The analogy between localized vibrations of plates and plate localized non-stability was established in [4]. Further investigations on the late localized non-stability problems were done, for example [5]-[7]. In the present paper the mathematical model and differential equations is presented. The solutions are found; correspondingly, the necessary and sufficient different conditions for the existence of localized solutions are investigated. The limiting cases obtained. The results and conclusions are then reported.

II. Mathematical Modeling

A semi-infinite plate with two simply supported edges as sketched if Fig. 1, is considered. The width of the plate is b and the thickness is $2h$. The Cartesian coordinate system (x,y,z) is chosen so that the plane (xoy) is coincident with the plate middle surface, while z is the coordinate along the thickness; the x axes and y are aligned the edges. The plate in Cartesian coordinates to be defined by a domain:

$$0 \le x \le \infty \quad ,0 \le y \le b \quad ,-h \le z \le h$$

The plate is uniformly compressed along the edges $y=0$ and $y=b$ with a constant load P. The stability equation for plate middle plane normal displacement $w(x,y)$ can be expressed as [8], [9].

$$D_{11}\frac{\partial^4 w}{\partial x^4}+2\left(D_{12}+2D_{66}\right)\frac{\partial^4 w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4 w}{\partial y^4}+P\frac{\partial^2 w}{\partial y^2}=0 \quad (1)$$

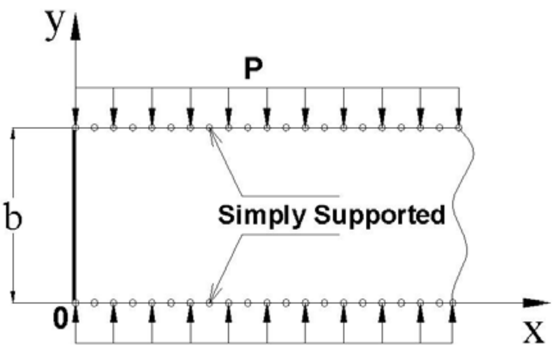

Fig.1. uniformly compressed semi-infinite plate simply supported along the edges $y=0$ and $y=b$

where D_{11}, D_{22} are the bending stiffness in the x, y direction respectively. Further D_{11}, D_{22}, D_{12} and D_{66} can be written as

$$D_{11}=\frac{h^3}{12}\frac{E_1}{1-\nu_{12}\nu_{21}},\quad D_{22}=\frac{\nu_{21}}{\nu_{12}}D_{11},$$

$$D_{12}=\nu_{12}D_{11},\quad D_{66}=\frac{h^3}{12}G_{12}$$

and

$$\nu_{21}E_1=\nu_{12}E_2$$

Here, the suffixes 1 and 2 refer to the x and y directions, respectively, so E_1 is the Young modulus in the x direction, G_{12} is the shear modulus in the x-y plane, and ν_{12} is the Poisson ratio for transverse strain in the y direction caused by

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stress in the *x* direction, with similar definition for E_2 and ν_{21}.
The boundary conditions on the simply supported edges at *y=0*, *y=b* are:

$$\begin{cases} w=0 \\ \dfrac{\partial^2 w}{\partial y^2}=0 \end{cases} \quad y=0,\ \ y=b \tag{2}$$

We consider later the edge *x=0* with different boundary conditions. One additional boundary condition is needed. If the plate is semi-infinite, the localization condition prescribes attenuation as $x \to \infty$, hence an additional constraint is

$$\lim_{x \to \infty} \ w=0 \tag{3}$$

General solution of (1) can be represented as series expansion

$$w=\sum_{n=1}^{\infty} f_n(x) \sin \lambda_n y, \ \ where \ \ \lambda_n = n\pi / b \tag{4}$$

Equations (4) and (1) yield to the following linear ordinary differential equation and the function $f_n(x)$ can be determined by solving the ordinary differential equation

$$f_n^{IV} - 2\alpha_1 \lambda_n^2 f_n^{II} + \alpha_2 \lambda_n^4 \left(1-\eta_n^2\right) f_n = 0 \tag{5}$$

$$\text{where} \ \ \alpha_1 = \frac{D_{12}+2D_{66}}{D_{11}}, \ \ \alpha_2 = \frac{D_{22}}{D_{11}}, \ \ \eta_n^2 = \frac{P}{D_{22}\lambda_n^2} \tag{6}$$

The attenuation condition of (3) implies that $f_n(x) \to 0$ as $x \to \infty$. Therefore, the general solution of (5) is in the form

$$f_n = A_n e^{-p_1 \lambda_n x} + B_n e^{-p_2 \lambda_n x} \tag{7}$$

where p_1 and p_2 are given by

$$p_{1,2} = \sqrt{\alpha_1 \pm \sqrt{\alpha_1^{\ 2} - \alpha_2 \left(1-\eta_n^{\ 2}\right)}} \tag{8}$$

Refer to (6) it is clear that

$$\alpha_1 > 0, \ \ \alpha_2 > 0 \tag{9}$$

and

$$\alpha_1 - \sqrt{\alpha_1^{\ 2} - \alpha_2 \left(1-\eta_n^{\ 2}\right)} > 0 \quad if \quad 0 \le \eta_n^2 \le 1 \tag{10}$$

The constants A_n and B_n can be obtained imposing the different boundary conditions at edge $x=0$ lead to a linear homogeneous system in A_n and B_n. The nontrivial solution is given by posing the determinant of the matrix of the coefficients to zero. That yields the equation in η_n.

The different boundary conditions at edge $x=0$ can be presented as follow

A. Clamped Edge

The boundary conditions on the clamped edge at $x=0$ are

$$w=0, \ \ \frac{\partial w}{\partial x}=0 \qquad at \ \ \ x=0$$

Substitution of (4) into above boundary conditions yields

$$f_n=0, \ \ \frac{df_n}{dx}=0 \qquad at \ \ \ x=0 \tag{11}$$

Substitution of (7) into (11), a set of simultaneous equation with regard to A_n and B_n is obtained as follow

$$\left.\begin{array}{l} A_n + B_n = 0 \\ -p_1 A_n - p_2 B_n = 0 \end{array}\right\} \Rightarrow p_2 - p_1 = 0$$

Substitution of (8) into above equation the following equation is obtained

$$\eta_n^{\ 2} = 1 + \frac{\alpha_1^{\ 2}}{\alpha_2} > 1 \tag{12}$$

Localized solution doesn't exist, because (12) doesn't satisfy condition (10).

B. Hinged Edge

The boundary conditions on the hinged edge at $x=0$ are

$$w=0, \ \ \frac{\partial^2 w}{\partial x^2}=0 \qquad at \ \ \ x=0$$

Substitution of (4) into above boundary conditions yields

$$f_n=0, \ \ \frac{d^2 f_n}{dx^2}=0 \qquad at \ \ \ x=0 \tag{13}$$

Substitution of (7) into (13), a set of simultaneous equation with regard to A_n and B_n is obtained as follow

$$\left.\begin{array}{l} A_n + B_n = 0 \\ -p_1{}^2 A_n - p_2{}^2 B_n = 0 \end{array}\right\} \Rightarrow p_2{}^2 - p_1{}^2 = 0 \qquad (14)$$

Substitution of (8) into above equation the following equation is obtained

$$\eta_n{}^2 = 1 + \frac{\alpha_1{}^2}{\alpha_2} > 1 \qquad (15)$$

Localized solution doesn't exist, because (15) doesn't satisfy condition (10).

C. Sliding Contact

The boundary conditions on the sliding contact edge at $x = 0$ are

$$\frac{\partial w}{\partial x} = 0, \ \frac{\partial^3 w}{\partial x^3} = 0 \qquad at \quad x = 0$$

Substitution of (4) into above boundary conditions yields

$$\frac{df_n}{dx} = 0, \ \frac{d^3 f_n}{dx^3} = 0 \qquad at \quad x = 0 \qquad (16)$$

Substitution of (7) into (16), a set of simultaneous equation with regard to A_n and B_n is obtained as follow

$$\left.\begin{array}{l} p_1 A_n + p_2 B_n = 0 \\ p_1{}^3 A_n + p_2{}^3 B_n = 0 \end{array}\right\} \Rightarrow p_1 p_2 (p_2{}^2 - p_1{}^2) = 0 \qquad (17)$$

There are two cases

1. C. $p_2{}^2 - p_1{}^2 = 0$

Substitution of (8) into above equation the following equation is obtained

$$\eta_n{}^2 = 1 + \frac{\alpha_1{}^2}{\alpha_2} > 1 \qquad (18)$$

Localized solution doesn't exist, because (18) doesn't satisfy condition (10).

2. C. $p_1 p_2 = 0$

Limiting case (no localization)

$$p_1 p_2 = 0 \Rightarrow p_2 = 0 \Rightarrow \eta_n{}^2 = 1 \quad , \quad p_1 = \sqrt{2\alpha_1} \qquad (19)$$

From (17) and above equations we obtain $A_n = 0$

Substitution of $A_n = 0$ into (7) the following equation is obtained

$$f_n = B_n$$

Substation of above equation into (4) the following equation is obtained

$$w = \sum_{n=1}^{\infty} B_n \sin \lambda_n y \qquad (20)$$

The equation (20) is a lost of stability by cylindrical surface.

From $\eta_n{}^2 = 1$ the minimum of *P* is obtained as follow

$$p_n)_{min} = \frac{\pi^2 D_{22}}{b^2} \qquad (21)$$

D. Free Edge

The boundary conditions at the free edge $x = 0$ are

$$\begin{array}{l} M_1 = -(D_{11} \frac{\partial^2 w}{\partial x^2} + D_{12} \frac{\partial^2 w}{\partial y^2}) = 0 \\ \tilde{N}_1 = N_1 + 2\frac{\partial H}{\partial y} = -\frac{\partial}{\partial x}\left[D_{11} \frac{\partial^2 w}{\partial x^2} + (D_{12} + 4D_{66}) \frac{\partial^2 w}{\partial y^2} \right] = 0 \end{array} \qquad (22)$$

where M_1 arising from distribution of in-plane normal stress σ_x and the twisting moment H and shear forces per unit length, N_1 arising from the shear stress in the plate and $\tilde{N}_1$ is reaction force along the edge $x = 0$.

Substation of (4) into boundary conditions (22) yields

$$\begin{array}{l} \frac{d^2 f_n}{dx^2} - \alpha_3 \lambda_n{}^2 f_n = 0 \\ \frac{d^3 f_n}{dx^3} - (\alpha_3 + 2\alpha_2) \lambda_n{}^2 \frac{df_n}{dx} = 0 \end{array} \qquad at \quad x = 0 \qquad (23)$$

Some new notations are introduced as follow

$$\alpha_3 = \frac{D_{12}}{D_{11}} \quad , \quad \alpha_4 = \frac{2D_{66}}{D_{11}} \Rightarrow \alpha_1 = \alpha_3 + \alpha_4 \qquad (24)$$

where $\alpha_2, \alpha_3, \alpha_4$ are three independent constants

By using of (6), (24) and substitution of (7) into (23), a set of simultaneous equation with regard to A_n and B_n is obtained as follow

$$\left.\begin{aligned}&(p_1^2-\alpha_3)A_n+(p_2^2-\alpha_3)B_n=0\\&p_1(p_1^2-\alpha_3-2\alpha_4)A_n+p_2(p_2^2-\alpha_3-2\alpha_4)B_n=0\end{aligned}\right\}\quad(25)$$

The condition that the determinant $\Delta=0$ yields the characteristic equation as follow

$$\begin{aligned}\Delta=&\,p_2(p_1^2-\alpha_3)(p_2^2-\alpha_3-2\alpha_4)\\&-p_1(p_2^2-\alpha_3)(p_1^2-\alpha_3-2\alpha_4)=0\end{aligned}\quad(26)$$

Instead of (26) it is possible to write

$$(p_2-p_1)M(\eta)=0\quad(27)$$

where

$$\begin{aligned}M(\eta)=&\,p_1^2p_2^2+2\alpha_4p_1p_2-\alpha_3(p_1^2+p_2^2)\\&+\alpha_3(\alpha_3+2\alpha_4)\end{aligned}\quad(28)$$

With use of equalities as follow

$$p_1^2+p_2^2=2\alpha_1\ ,\ \ \alpha_1=\alpha_3+\alpha_4\quad(29)$$

The equation (28) can be written as

$$M(\eta)=p_1^2p_2^2+2\alpha_4p_1p_2-\alpha_3^2\quad(30)$$

From (27) there are two cases as follow

1. D. $p_2-p_1=0$

Substitution of (8) into above equation the following equation is obtained

$$\eta_n^2=1+\frac{\alpha_1^2}{\alpha_2}>1\quad(31)$$

Localized solution doesn't exist, because (31) doesn't satisfy condition (10).

2. D. $M(\eta)=0$ (32)

In first limiting case $\eta_n\to1\Rightarrow p_1=\sqrt{2\alpha_1}\,,\ \ p_2=0$

From (30) the following equation is obtained

$$M(1)=-\alpha_3^2<0\quad(33)$$

In second limiting case $\eta_n\to0\Rightarrow p_1p_2=\sqrt{\alpha_2}$

From (30) the following equation is obtained

$$M(0)=\alpha_2+2\alpha_4\sqrt{\alpha_2}-\alpha_3^2\quad(34)$$

$$\alpha_2+2\alpha_4\sqrt{\alpha_2}-\alpha_3^2>0\quad(35)$$

The condition (35) is sufficient for existence of real root of (32) in the following interval

$$0<\eta_n<1$$

From (30) and (34) the following equation is obtained

$$p_1p_2=-\alpha_4\pm\sqrt{\alpha_4^2+\alpha_3^2}\quad(36)$$

From (8) the following equation is obtained

$$p_1p_2=\sqrt{\alpha_2(1-\eta_n^2)}\quad(37)$$

From (36) and (37) the following equation is obtained

$$\eta_n^2=1-\alpha_2^{-1}(2\alpha_4^2+\alpha_3^2\pm2\alpha_4\sqrt{\alpha_4^2+\alpha_3^2})\quad(38)$$

When condition (35) doesn't satisfy, there is no root or there are two roots.

III. Conclusion

In this paper localized non-stability of a thin elastic orthotropic semi-infinite plate has been analyzed. Several conclusions can be summarized in the following.

- In clamped edge conditions localized solution doesn't exist.
- In hinged edge conditions localized solution doesn't exist.
- In sliding contact conditions there are two cases, one case localized solution doesn't exit and in the other case we obtain the equation of lost of stability by cylindrical surface.
- In free edge there are two cases, one case localized solution doesn't exit and in the other case we obtain real roots.

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