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Preconditioned Mixed-Type Splitting Iterative Method For Z-Matrices

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Abstract—In this paper, we present the preconditioned mixed-type splitting iterative method for solving the linear systems, $Ax = b$, where A is a Z-matrix. And we give some comparison theorems to show that the convergence rate of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method. Finally, we give a numerical example to illustrate our results.

Keywords—Z-matrix, mixed-type splitting iterative method, precondition, comparison theorem, linear system.

I. Introduction

FOR solving linear system

$$Ax = b, \tag{1}$$

where A is an $n \times n$ square matrix, x and b are n-dimensional vectors, the basic iterative method is

$$Mx^{k+1} = Nx^k + b,\ k = 0,\ 1,\ \ldots \tag{2}$$

when $A = M - N$ and M is nonsingular. Thus (2) can be written as

$$x^{k+1} = Tx^k + c,\ k = 0,\ 1,\ \ldots,$$

where $T = M^{-1}N$, $c = M^{-1}b$.

Assuming A has unit diagonal entries and let $A = I - L - U$, where I is the identity matrix, $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively.

Multiplying both sides by P, we can transform the original system (1) into the preconditioned form

$$PAx = Pb.$$

Then, we can define the basic iterative scheme:

$$M_p x^{k+1} = N_p x^k + Pb, k = 0, 1, \ldots,$$

where $PA = M_p - N_p$ and $M_p$is nonsingular. Thus the equation above can also be written as

$$x^{k+1} = Tx^k + c, k = 0, 1, \ldots,$$

where $T = M_p^{-1}N_p, c = M_p^{-1}Pb$.

In paper [1], Guang-Hui Cheng et al. presented the mixed-type splitting iterative method

$$(D+D_1+L_1-L)x^{k+1} = (D_1+L_1+U)x^k+b, \quad k = 0, 1, 2\cdots,$$

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whose iterative matrix is

$$T = (D + D_1 + L_1 - L)^{-1}(D_1 + L_1 + U) \tag{3}$$

where D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and$0 \le L_1 \le L$.

And in paper [2], Ji-Cheng Li et al. proved the Gauss-Seidel iterative method with P_β as its preconditioner is convergent, where $P_\beta = I + S_\beta =$

$$\begin{pmatrix} 1 & & & & \\ -\beta_1 a_{21} & 1 & & & \\ & -\beta_2 a_{32} & \ddots & & \\ & & \ddots & & \\ & & \ddots & \ddots & \\ & & & -\beta_{n-1} a_{n,n-1} & 1 \end{pmatrix} \tag{4}$$

and $\beta_i (i = 1, 2, \ldots n-1)$ are nonngative real numbers.

In this paper, we will establish the preconditioned mixed-type splitting iterative method with the preconditioner P_β for solving linear systems, where $P_\beta = I + S_\beta$ is given by (4). And we obtain some comparison results which show that the proper choice of the auxiliary matrices can lead to the rate of convergence of the preconditioned mixed-type splitting iterative method is faster than that of the mixed-type splitting iterative method.

II. Preconditioned Mixed-type Splitting Iterative Method

For the linear system (1), we consider its preconditioned form

$$P_\beta Ax = P_\beta b$$

with the preconditioner $P_\beta = I + S_\beta$.

Let

$$P_\beta A = A_\beta, P_\beta b = b_\beta,$$

then we get

$$A_\beta x = b_\beta.$$

We apply the mixed-type splitting iterative method to it and get the corresponding preconditioned mixed-type splitting iterative method

$$(D_\beta+D_1+L_1-L_\beta)x^{k+1} = (D_1+L_1+U_\beta)x^k+b, k = 0, 1, 2\cdots$$

with the iterative matrix

$$\tilde{T} = (D_\beta + D_1 + L_1 - L_\beta)^{-1}(D_1 + L_1 + U_\beta), \tag{5}$$

where D_β,$-L_\beta$,$-U_\beta$ are the diagonal, strictly lower and strictly upper triangular matrices obtained from A_β and D_1 is an auxiliary nonnegative diagonal matrix, L_1 is an auxiliary strictly lower triangular matrix and $0 \le L_1 \le L_\beta$.

If we choose certain auxiliary matrices, we can get the classical iterative methods.

1. The PSOR method

$$D_1 = \frac{1}{r}(1-r)D, L_1 = 0$$

$$\tilde{L}_r = (D_\beta - rL_\beta)^{-1}[(1-r)D_\beta + rU_\beta] \quad (6)$$

2. The PAOR method

$$D_1 = \frac{1}{r}(1-r)D, L_1 = 0$$

$$\tilde{L}_{rw} = (D_\beta - rL_\beta)^{-1}[(1-w)D_\beta + (w-r)L_\beta + wU_\beta] \quad (7)$$

We need the following definitions and results.

Definition 2.1 ([3]). A matrix A is a Z-matrix if $a_{ij} \le 0$, for all $i, j = 1, 2, \ldots n, i \ne j$.

Definition 2.2 ([3]). A matrix A is an M-matrix if A is a nonsingular Z-matrix, and $A^{-1} \ge 0$.

Definition 2.3 ([7]). Let $M, N \in R^{n\times n}$, the splitting of $A = M - N$ is called a regular splitting if $M^{-1} \ge 0$ and $N \ge 0$.

Lemma 2.1 ([3]). Assume that $A = M - N$ is a regular splitting of A. The splitting is convergent if and only if $A^{-1} \ge 0$.

Lemma 2.2 ([3]). Assume that A is an irreducible nonnegative matrix, then

(1)A has a positive real eigenvalue equals to its spectral radius;

(2) To $\rho(A)$, there corresponds an eigenvector $x > 0$;

(3)$\rho(A)$ is a simple eigenvalue of A.

Lemma 2.3 ([4]). Assume that A is a nonnegative matrix, then

(1) If $\alpha x \le Ax$ for some nonnegative vector$x$, $x \ne 0$, then $\alpha \le \rho(A)$;

(2)If $Ax \le \beta x$ for some positive vector x, then $\rho(A) \le \beta$. Moreover, if A is irreducible and there is a positive vector x such that $0 \ne \alpha x \le Ax \le \beta x$, then

$$\alpha \le \rho(A) \le \beta.$$

Lemma 2.4 ([5]). Let $A = M - N$ be an M-splitting of A. Then $\rho(M^{-1}N) < 1$ if and only if A is a nonsingular M-matrix.

Lemma 2.5 ([6]). Let A be a Z-matrix. Then A is a nonsingular M-matrix if and only if there is a positive vector x such that $Ax \ge 0$.

III. Convergence Analysis And Comparison Theorems

In this section, we will present the main theorems.

Theorem 3.1 Let $A = I - L - U$ be an M-matrix, where $-L$ and $-U$ are strictly lower and strictly upper triangular parts of A, respectively, $D_1 \ge 0$ and $0 \le L_1 \le L_\beta$. Then, the preconditioned mixed-type splitting iterative method is convergent.

Proof. Let

$$D_\beta = I - S_1, L_\beta = L - S_\beta + S_\beta L, U_\alpha = U + S_2,$$

where S_1, S_2 are the diagonal and upper triangular parts of $S_\beta U$. Then

$$M = D_\beta + D_1 + L_1 - L_\beta,$$

$$N = D_1 + L_1 + U_\beta.$$

Since A is an M-matrix and $0 \le L_1 \le L_\beta$, we get

$$\begin{aligned} M^{-1} &= (D_\beta + D_1 + L_1 - L_\beta)^{-1} \\ &= [(D_\beta + D_1) - (L_\beta - L_1)]^{-1} \ge 0, \\ A^{-1} &\ge 0, N = D_1 + L_1 + U_\beta \ge 0. \end{aligned}$$

According to Lemma 2.1, Lemma 2.2 and Definition 2.3, we can get the conclusion that the preconditioned mixed-type splitting iterative method is convergent for M-matrix.

Corollary 3.1 The PSOR method is convergent if the coefficient matrix A is an M-matrix and $0 < r < 1$.

Corollary 3.2 The PAOR method is convergent if the coefficient matrix A is an M-matrix and $0 < r < w < 1$.

Theorem 3.2 Let A be a nonsingular Z-matrix, such that $D_1 \ge 0$, $0 \le L_1 \le L_\beta$, $\beta \in [0,1]$ and $\tilde{T}$, T are iterative matrices of (5) and (3), respectively. Then

(i) If $\rho(T) < 1$, then $\rho(\tilde{T}) < \rho(T) < 1$;

(ii)Assume that A is an irreducible matrix and $0 < a_{ii-1}a_{i-1i} < 1,\ i = 2, \ldots n$. Then

(1) If $\rho(T) > 1$, then $\rho(\tilde{T}) > \rho(T)$;

(2) If $\rho(T) = 1$, then $\rho(\tilde{T}) = \rho(T)$;

(3) If $\rho(T) < 1$, then $\rho(\tilde{T}) < \rho(T)$.

Proof. Let

$$M_\beta = D_\beta + D_1 + L_1 - L_\beta,$$

$$N_\beta = D_1 + L_1 + U_\beta,$$

$$M = I + D_1 + L_1 - L,$$

$$N = D_1 + L_1 + U,$$

$$E_\alpha = (I + S_\beta)(I + D_1 + L_1 - L),$$

$$F_\alpha = (I + S_\beta)(D_1 + L_1 + U),$$

then we get $A = M - N$, $A_\beta = M_\beta - N_\beta = E_\beta - F_\beta$.

(i) Since A is a nonsingular Z-matrix and $D_1 \ge 0$, $0 \le L_1 \le L_\beta$, we can easily know that

$$M = I + D_1 + L_1 - L$$

is a nonsingular M-matrix and the splitting

$$A = M - N = (I + D_1 + L_1 - L) - (D_1 + L_1 + U)$$

is an M-splitting. Thus, $\rho(T) < 1$ and by Lemma 2.4, we know that A is a nonsingular M-matrix. Furthermore, we know that there exist a positive vector x such that $Ax \ge 0$ according to Lemma 2.5.

So

$$A_\beta x = (I + S_\beta)Ax \ge 0.$$

According to Lemma 2.5, A_β is also a nonsingular M-matrix.

Besides, since $L_\beta = D_\beta - I + L - S_\beta + S_\beta L + S_1$, we get

$$\begin{aligned} & E_\beta - M_\beta \\ & = (I + S_\beta)(I + D_1 + L_1 - L) - (D_\beta + D_1 + L_1 - L_\beta) \\ & = (I + D_1 + L_1 - L) + S_\beta(I + D_1 + L_1 - L) \\ & \quad -(D_\beta + D_1 + L_1 - L_\beta) \\ & = I - L + S_\beta(I + D_1 + L_1 - L) - D_\beta + L_\beta \\ & = I - L + S_\beta(I + D_1 + L_1 - L) - D_\beta + D_\beta \\ & \quad -I + L - S_\beta + S_\beta L + S_1 \\ & = S_\beta(D_1 + L_1 + S_1) \geq 0 \end{aligned}$$

then

$$A_\beta^{-1} E_\beta - A_\beta^{-1} M_\beta = A_\beta^{-1}(E_\beta - M_\beta) \geq 0$$

and

$$A_\beta^{-1} E_\beta \geq A_\beta^{-1} M_\beta \geq 0.$$

Futhermor, we have

$$\rho(M_\beta^{-1} N_\beta) \leq \rho(E_\beta^{-1} F_\beta),$$

i.e.

$$\rho(\bar{T}) \leq \rho(T) < 1.$$

(ii) Assume that $A = I - L - U$ is irreducible. Since $L + U$ is a nonnegative irreducible matrix, by the definitions of $\bar{T}$ and T, we can easily know that $\bar{T}$ and T are both nonnegative irreducible matrices.

If we let $\lambda = \rho(T)$, by Lemma 2.2, there exist a positive vector $x = (x_1, x_2 \cdots x_n)^T$ such that $Tx = \lambda x$. That is equivalent to

$$(D_1 + L_1 + U)x = \lambda(I + D_1 + L_1 - L)x \qquad (8)$$

and

$$(U - \lambda D + \lambda L)x = [(\lambda - 1)D_1 + (\lambda - 1)L_1]x. \qquad (9)$$

If we let

$$S_\beta U = S_1 + S_2,$$

where S_1, S_2 are the diagonal and upper triangular parts of $S_\beta U$, then

$$\begin{aligned} A_\beta &= D_\beta - L_\beta - U_\beta \\ &= (I - S_1) - (L - S_\beta + S_\beta L) - (U + K_2) \end{aligned},$$

where

$$D_\beta = I - S_1, L_\beta = L - S_\beta + S_\beta L, U_\beta = U + S_2,$$

and

$$\begin{aligned} & \tilde{T}x - \lambda x \\ & = (D_\beta + D_1 + L_1 - L_\beta)^{-1}(D_1 + L_1 + U_\beta)x - \lambda x \\ & = (D_\beta + D_1 + L_1 - L_\beta)^{-1}[(D_1 + L_1 + U_\beta) \\ & \quad -\lambda(D_\beta + D_1 + L_1 - L_\beta)]x \\ & = (D_\beta + D_1 + L_1 - L_\beta)^{-1}[(D_1 + L_1 + U + S_2) \\ & \quad -\lambda(I - S_1 + D_1 + L_1 - L + S_\beta - S_\beta L)]x \\ & = (D_\beta + D_1 + L_1 - L_\beta)^{-1}\{[(D_1 + L_1 + U) \\ & -\lambda(I + D_1 + L_1 - L)]x + [S_2 - \lambda(-S_1 + S_\beta - S_\beta L)]x\} \\ & = (D_\beta + D_1 + L_1 - L_\beta)^{-1}[S_2 + \lambda(S_1 - S_\beta + S_\beta L)]x \\ & (D_\beta + D_1 + L_1 - L_\beta)^{-1}(S_2 + \lambda S_1 - \lambda S_\beta + S_\beta L)x \\ & = (\lambda - 1)(D_\beta + D_1 + L_1 - L_\beta)^{-1}(S_\beta D_1 + S_\beta L_1 + S_1)x. \end{aligned}$$

Since $D_1 + L_1 + I \geq 0$ and $S_\beta D_1 + S_\beta L_1 + S_1 \geq 0$, then

(1) If $\lambda > 1$, then $\tilde{T}x - \lambda x \geq 0$, i.e. $\tilde{T}x \geq \lambda x$. By Lemma 2.3, we have $\rho(\tilde{T}) > \lambda = \rho(T)$.

(2) If $\lambda = 1$, then $\tilde{T}x - \lambda x = 0$, i.e. $\tilde{T}x = \lambda x$. By Lemma 2.3, we have $\rho(\tilde{T}) = \lambda = \rho(T)$.

(3) If $\lambda < 1$, then $\tilde{T}x - \lambda x \leq 0$, i.e. $\tilde{T}x \leq \lambda x$. By Lemma 2.3, we have $\rho(\tilde{T}) < \lambda = \rho(T)$.

IV. Numerical Example

For linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & -0.1 & -0.1 & 0 & -0.2 & -0.4 \\ -0.3 & 1 & -0.2 & 0 & -0.3 & -0.2 \\ 0 & -0.2 & 1 & -0.5 & -0.1 & 0 \\ -0.1 & -0.3 & -0.1 & 1 & -0.2 & -0.1 \\ -0.2 & -0.3 & -0.2 & -0.1 & 1 & -0.1 \\ -0.3 & -0.1 & -0.1 & -0.2 & -0.1 & 1 \end{pmatrix}.$$

If we take $\beta_2 = \beta_3 = \cdots = \beta_n \in [0, 1]$ and $D_1 = \frac{1}{2}I$, $L_1 = \frac{1}{5}L$, then by Theorem 3.1 and Theorem 3.2, we can obtain the following table:

TABLE I
Spectral Radius For Different Methods

$\beta_i \quad (i = 2, \ldots n)$	$\rho(T)$	$\rho(\widetilde{T})$
0	0.2930	0.2930
0.0500	0.2930	0.2915
0.1000	0.2930	0.2902
0.1500	0.2930	0.2890
0.2000	0.2930	0.2880
0.2500	0.2930	0.2871
0.3000	0.2930	0.2864
0.3500	0.2930	0.2858
0.4000	0.2930	0.2854
0.4500	0.2930	0.2852
0.5000	0.2930	0.2852
0.5500	0.2930	0.2853
0.6000	0.2930	0.2856
0.6500	0.2930	0.2861
0.7000	0.2930	0.2867
0.7500	0.2930	0.2874
0.8000	0.2930	0.2883
0.8500	0.2930	0.2893
0.9000	0.2930	0.2904
0.9500	0.2930	0.2915
1.0000	0.2930	0.2928

From Table I, we can conclude that the preconditioned mixed-type splitting iterative method is convergent and its convergence rate is faster than that of the mixed-type splitting iterative method.

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