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Accurate Calculation of Free Frequencies of Beams and Rectangular Plates

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***Abstract*—**An accurate procedure to determine free vibrations of beams and plates is presented.

The natural frequencies are exact solutions of governing vibration equations witch load to a nonlinear homogeny system.

The bilinear and linear structures considered simulate a bridge. The dynamic behavior of this one is analyzed by using the theory of the orthotropic plate simply supported on two sides and free on the two others. The plate can be excited by a convoy of constant or harmonic loads. The determination of the dynamic response of the structures considered requires knowledge of the free frequencies and the shape modes of vibrations. Our work is in this context. Indeed, we are interested to develop a self-consistent calculation of the Eigen frequencies.

The formulation is based on the determination of the solution of the differential equations of vibrations. The boundary conditions corresponding to the shape modes permit to lead to a homogeneous system. Determination of the noncommonplace solutions of this system led to a nonlinear problem in Eigen frequencies.

We thus, develop a computer code for the determination of the eigenvalues. It is based on a method of bisection with interpolation whose precision reaches 10^{-12}. Moreover, to determine the corresponding modes, the calculation algorithm that we develop uses the method of Gauss with a partial optimization of the "pivots" combined with an inverse power procedure. The Eigen frequencies of a plate simply supported along two opposite sides while considering the two other free sides are thus analyzed. The results could be generalized with the case of a beam by regarding it as a plate with low width.

We give, in this paper, some examples of treated cases. The comparison with results presented in the literature is completely satisfactory.

***Keywords*—**Free frequencies, beams, rectangular plates.

I. Introduction

MANY researchers have studied the vibrations of plates and their displacement because of its importance in engineering applications. Indeed, rectangular plates are commonly used as structural components in many branches of modern technology namely mechanical, aerospace, electronic, optical, marine and structural engineering. So, there is a particular need for access to highly accurate eigenvalues for plates and beams. For example, Wu and Dai [1] used the transfer matrix method to determine the natural frequencies and mode shapes of a multi-span of beams. They applied the technique of mode superposition to study the dynamic performances of the considered beam subjected to moving loads. Moussu and Nivoiti [2] have determined an elastic constant of orthotropic plates by modal analysis. Later, D.J Gorman [3] uses a computed method to determine Eigen values for a completely free orthotropic plate by using a superposition method. He also [4], use the superposition method to obtain accurate analytical type solutions for the free in-plane vibration of rectangular plates with uniform, symmetrically distributed elastic edge supports acting normal to the boundaries.

In all this above work, the authors have determined initially the free frequencies in order to predict the dynamic behaviour of the studied structures.

In addition, an excellent reference source concerning vibration of such plates may be found in the work of Leissa [5, 6]. We can find exact characteristic equations for rectangular thin plates having two opposite sides simply supported. However, the analysis of thick plates has been presented by Lim and all [7].

In this paper, this dynamic behaviour is analysed using the orthotropic plate theory and modal superposition. So, we present an accurate method to calculate the free vibrations. The strategy presented is based on the bisection method with interpolation to determine the eigenfrequencies. However, to determine the corresponding modes, we use a Gauss method.

We present numerical examples for a beam model and a plate one. This method can also be applied to the case of a bridge under moving loads considered as an orthotropic rectangular plate.

II. Formulation of the Problem

A schematic of a two dimensional plate is shown in Fig. 1. It's a rectangular plate with its left and right edges simply supported and the other two opposite edges free. If, it's also solicited by an external load F, the governing equations of motion of this orthotropic plate can be written, according to Huffington and Hoppman [8] as follows:

$$D_X \frac{\partial 4w}{\partial x4} + 2D_{xy} \frac{\partial 4w}{\partial x2 \partial y2} + D_Y \frac{\partial 4w}{\partial y4} + C\frac{\partial w}{\partial t} + \rho h \frac{\partial 2w}{\partial t2} = F(x,y,t) \qquad (1)$$

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where : $D_x = \dfrac{E_x h^3}{12(1-\upsilon_{xy}\upsilon_{yx})}$, $D_y = \dfrac{E_y h^3}{12(1-\upsilon_{xy}\upsilon_{yx})}$,

$$D_{xy} = (D_{xy} + 2D_k) \quad , \quad D_k = \frac{G_{xy} h_3}{12} \quad , \qquad (2)$$

D_X , D_Y , D_{XY} : flexural rigidities of the plate in the x , y and xy direction .
D_K : twisting rigidity of the plate.
G_{XY} : Shear modulus.
ρ : mass density of plate material.
h : thickness of the plate .
w(x,y,t) : displacement of plate in the z direction.

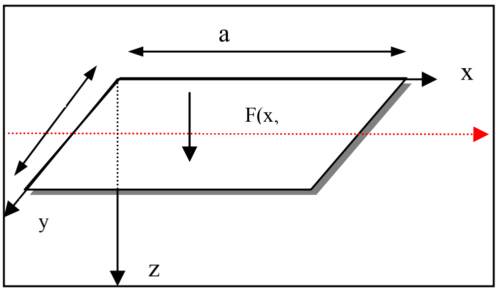


Fig. 1 Considered plate

Let us note that the side effects (shearing and rotational inertia) are neglected. The resolution of the differential equation governing the movement is obtained by using the modal superposition method and the integral of convolution, by the separation of the temporal and space variables [9]. Thus, the dynamic response to the point (x,y) of the plate and at the moment t is expressed in the form of series according to the following expression [10]:

$$W(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{m,n}(x,y).q_{m,n}(t) \qquad (3)$$

Where: $U_{m,n}(x,y) = Y_{m,n}(y).\sin(\theta_m.x)$, $q_{m,n}(t) = \sin(\omega_{m,n}.t)$

and $\theta_m = m.\pi/a$

$U_{m,n}(x,y)$ is the mode shape, $\omega_{m,n}$ is the natural frequency witch correspond to the mth mode in the x direction and the nth mode in the y one.

Substituting equation (03) in equation (01), we obtain:

$$D_y.Y_{mn}^{(4)}(y) - 2.D_{xy}\left(\frac{m\pi}{a}\right)^2.Y_{mn}^{(2)}(y) + \left[D_x\left(\frac{m\pi}{a}\right)^4 - \rho.h.\omega_{mn}^2\right]Y_{mn}(y) = 0 \qquad (4)$$

Moreover, we take into account the boundary conditions on the edges of the structure at y=0 and y=b. Those stipulate that the shearing action, the bending moment as well as the torque one are null at these ends. Then, we can write the system of equations (5):

$$\begin{cases} \dfrac{\partial^2 w}{\partial y^2} + \nu_{xy}.\dfrac{\partial^2 w}{\partial y^2} = 0, \quad -D_{xy}.\dfrac{\partial^3 w}{\partial x^2.\partial y} - D_y.\dfrac{\partial^3 w}{\partial y^3} = 0 \\ 2.D_k\dfrac{\partial^3 w}{\partial x^2.\partial y} = 0 \ , \ -D_{xy}.\dfrac{\partial^3 w}{\partial x^2.\partial y} - D_y.\dfrac{\partial^3 w}{\partial y^3} - 2.D_k\dfrac{\partial^3 w}{\partial x^2 \partial y} = 0 \end{cases} \qquad (5)$$

The solutions of the equation (4) depend on the properties of the plate considered. We can obtain the three following cases according to the value of the rigidity coefficient of the plate:

1. $$Y_{mn}(y) = X_{1mn}.\sin(r_{2mn}y) + X_{2mn}.\cos(r_{2mn}y) + X_{3mn}.\sinh(r_{1mn}y) + X_{4mn}.\cosh(r_{1mn}y) \qquad (6)$$

if $[D_x < D_1]$ where $D_1 = \rho.h.\omega_{mn}^2.\theta_m^{-4}$

2. $$Y_{mn}(y) = X_{1mn}.\sinh(r_{1mn}y) + X_{2mn}.\cosh(r_{1mn}y) + X_{3mn}.\sinh(r_{3mn}y) + X_{4mn}.\cosh(r_{3mn}y) \qquad (7)$$

if $\left[D_{xy}^2/D_y + D_1 > D_x > D_1\right]$.

3. $$Y_{mn}(y) = \cosh(r_{4mn}y).(X_{1mn}\cos(r_{5mn}y) + X_{2mn}\sin(r_{5mn}y)) + \sinh(r_{4mn}y).(X_{3mn}\cos(r_{5mn}y) + X_{4mn}\sin(r_{5mn}y)) \qquad (8)$$

if $\left[D_x > D_{xy}^2/D_y + D_1\right]$.

The parameters r_{imn} depend on the physical properties of the plate considered and the modes of vibrations retained [9].

$$r_{1mn} = \theta_m.\sqrt{\frac{D_{xy} + \sqrt{D_{xy}^2 + D_y.D_1 - D_x.D_y}}{D_y}} \ ;$$

$$r_{2mn} = \theta_m.\sqrt{\frac{-D_{xy} + \sqrt{D_{xy}^2 + D_y.D_1 - D_x.D_y}}{D_y}}$$

$$r_{3mn} = \theta_m.\sqrt{\frac{D_{xy} - \sqrt{D_{xy}^2 + D_y.D_1 - D_x.D_y}}{D_y}} \ ;$$

$$r_{4mn} = \theta_m.\sqrt{\frac{1}{2}\left(\frac{D_{xy}}{D_y} + \sqrt{\frac{D_x}{D_y} - \frac{D_1}{D_y}}\right)} \ ; \qquad (9)$$

By introducing the form of the shape modes in the four boundary conditions and by considering the different cases, we obtain three homogeneous linear systems: **[M].[X] = 0**.

X is a vector such as: $[X] = [X_{1mn} \ X_{2mn} \ X_{3mn} \ X_{4mn}]^T$ and M is a matrix wich coefficients m_{ij} depend on the boundary conditions. Thus, in each of the three cases defined previously according to the coefficient of rigidity, we replace the value of $Y_{mn}(y)$ by the suitable form (06, 07 or 08) in the equations (05). We obtain the value of the matrix M in the three cases considered [1] [11]:

1. Case of : $[D_x < D_1]$

[1]to simplify the writing of the matrices, we will note θ the value θ_m, and ri the values r_{imn}

$$M=\begin{bmatrix} 0 & -r_2^2-\upsilon_{xy}.\theta^2. & 0 & r_1^2-\upsilon_{xy}.\theta^2 \\ m_{12}\sin(r_2.b) & m_{12}\cos(r_2.b) & m_{14}.sh(r_1.b) & m_{14}.ch(r_1.b) \\ D_{xy}.r_2.\theta^2+D_y.r_2^3+ +2.D_k.\theta^2.r_2 & 0 & D_{xy}.\theta^2.r_1-D_y.r_1^3+ +2.D_k.r_1.\theta^2 & 0 \\ m_{31}.\cos(r_2.b) & -m_{31}.\sin(r_2.b) & m_{33}.ch(r_1.b) & m_{33}.sh(r_1.b) \end{bmatrix}$$

2. Case of : $\left[D_{xy}^2/D_y + D_1 > D_x > D_1 \right]$

$$M=\begin{bmatrix} 0 & r_1^2-\upsilon_{xy}.\theta^2\ . & 0 & r_3^2-\upsilon_{xy}.\theta^2 \\ m_{12}.sh(r_1.b) & m_{12}.ch(r_2.b) & m_{14}.sh(r_3.b) & m_{14}.ch(r_3.b) \\ -D_y.r_1^3+ +\left(D_{xy}+2.D_k\right)\theta^2.r_1 & 0 & -D_y.r_3^3+ +\left(D_{xy}+2.D_k\right)r_3.\theta^2 & 0 \\ m_{31}.ch(r_1.b) & m_{31}.sh(r_1.b) & m_{33}.ch(r_3.b) & m_{33}.sh(r_3.b) \end{bmatrix}$$

3. Case of : $\left[D_x > D_{xy}^2/D_y + D_1 \right]$

$$M=\begin{bmatrix} r_4^2-r_5^2-\upsilon_{xy}.\theta^2 & 0 & 0 & 2.r_4.r_5 \\ m_{11}ch(r_4.b)\cos(r_5.b)- 2.r_4.r_5.sh(r_4.b)\sin(r_5.b) & m_{11}ch(r_4 b)\sin(r_5 b)+ 2.r_4 r_5 sh(r_4.b)\cos(r_5.b) & m_{11}sh(r_4 b)\cos(r_5 b)- 2.r_4.r_5.ch(r_4.b)\sin(r_5.b) & m_{11}sh(r_4.b)\sin(r_5.b)+ 2.r_4.r_5.ch(r_4.b)\cos(r_5.b) \\ 0 & \left(D_{xy}+2.D_k\right)\theta^2.r_5- 3.D_y.r_4^2.r_5+D_y.r_5^3 & -D_y.r_4^3+3.D_y.r_4.r_5^2+ \left(D_{xy}.\theta^2+2.D_k.\theta^2\right)r_4 & 0 \\ m_{33}sh(r_4.b)\cos(r_5.b)- m_{32}ch(r_4.b)\sin(r_5 b) & m_{33}sh(r_4.b)\sin(r_5.b)+ m_{32}ch(r_4.b)\cos(r_5.b) & m_{33}ch(r_4.b)\cos(r_5.b)- m_{32}sh(r_4.b)\sin(r_5.b) & m_{33}ch(r_4.b)\sin(r_5.b)+ m_{32}sh(r_4.b)\cos(r_5.b) \end{bmatrix}$$

To obtain noncommonplace solutions, it is necessary that the determinant of the system will be null. Writing this determinant permits us to lead to the frequencies equation. Knowing that the parameters r_{imn} are not independent variables but are fonction of the pulsation ω (equations 09), the resolution of the frequencies equation is not easy and then requires an adequate data-processing treatment.

We seek to determine the pulsations ω checking this equation. For that, we develop a code which calculates the eigenvalues of the frequencies equation. It is based on a bissection method with interpolation wich precision reaches 10^{-12}.

This method permits to record the eigenvalues of the frequencies corresponding to the different mode of vibration. For each index m, we find an infinity of solutions m=1…, ∞ (Fig.2). Each solution is then located by a double index ω_{rs}.

A zoom of Fig. 2 shows the values of the Eigen frequencies determined by dichotomy.

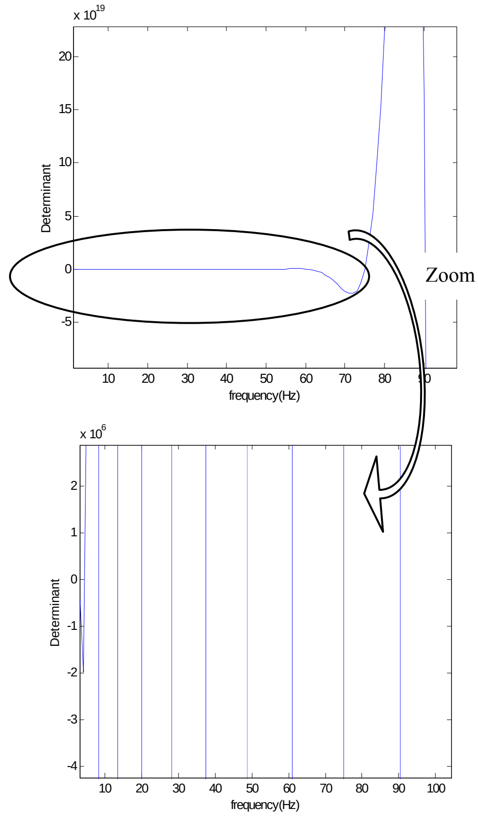


Fig. 2 Evaluation of the free frequencies

III. Results and Discussions

We consider in this study a structure with a 3.678m long and 0.1m x 0.025 m uniform cross section. The material Young's modulus is $2.1\text{x}10^9$ N/m^2. The mass density is 2300 Kg/m^3 and the Poisson ratio is 0.3.

The modes are defined by a double index m and n relating to x and y directions. The Fig. 4 indicates the natural frequency for two widths of the plate. These frequencies correspond to mode shape mainly in the y direction. The results are favourably compared with existing ones [9]. They can easily be applied to the case of a beam, by considering a very mean width of the plate. For simplicity of representation, we illustrate on the figure only the frequencies corresponding to the modes of inflection. The comparison is completely satisfactory.

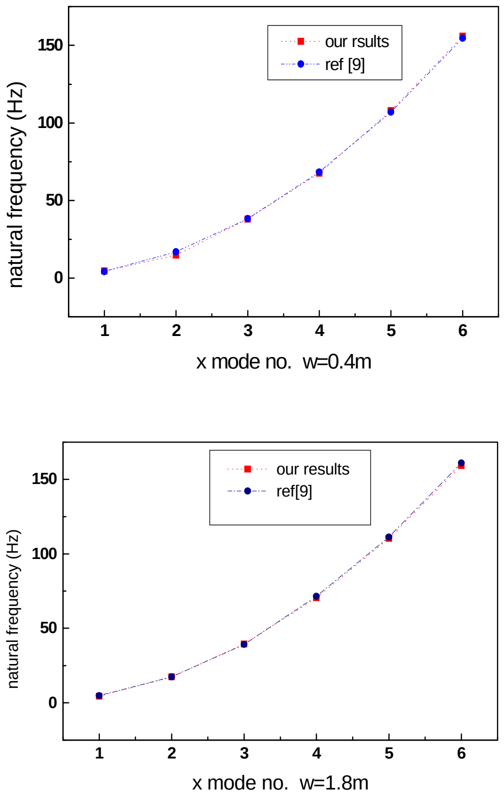


Fig. 4 Evolution of the natural frequency according to the mode

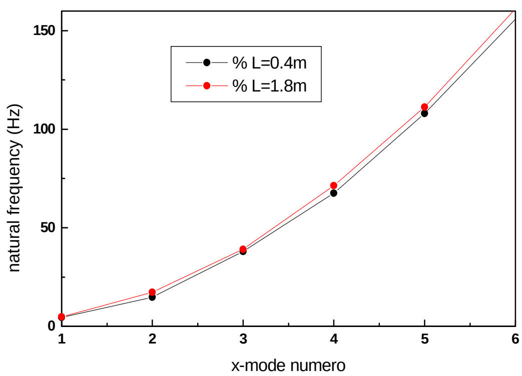


Fig. 5 Evolution of the natural frequency with both the mode and the width

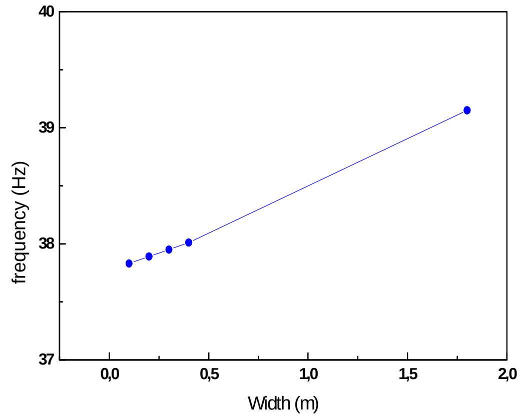
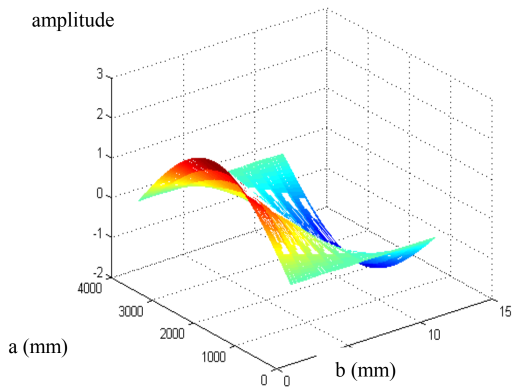


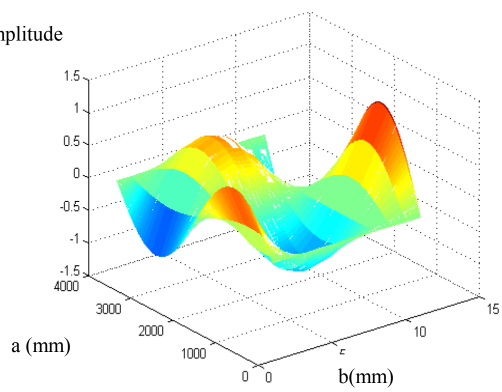
Fig. 6 Evolution of the natural frequency according the width for the third mode

We can easily observe the influence of the width of the plate on the natural frequencies, parameterised by the mode, on Fig. 5. For a given mode (Fig. 6), this one presents a monotonous growth while passing from a beam to a plate.

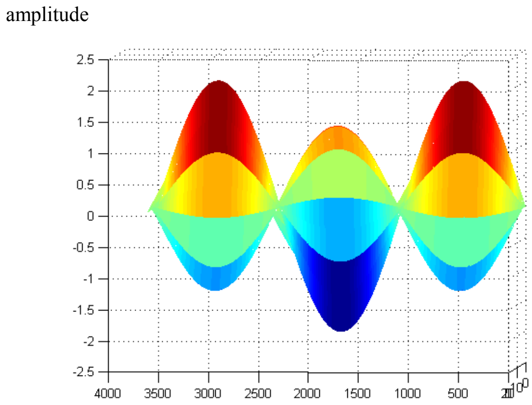
In addition, Fig. 7, presents, the principal modes of vibration obtained in the case of a beam and Fig. 8 in the case of a plate one. The mode (1,1) is that of lower own pulsation .



a) mode (1,1)

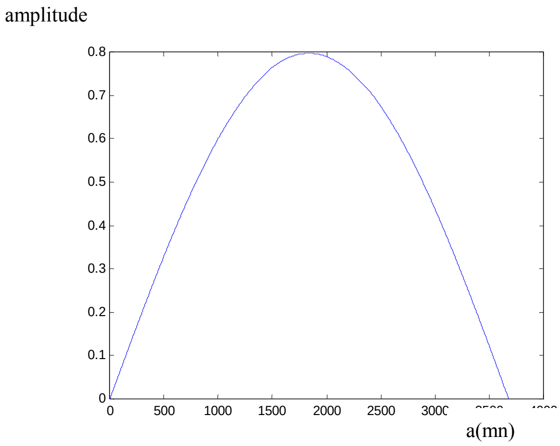


b) mode (2,1)

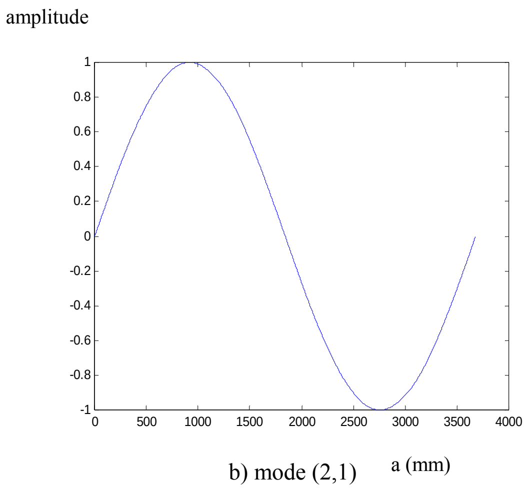


c) mode (3,1)

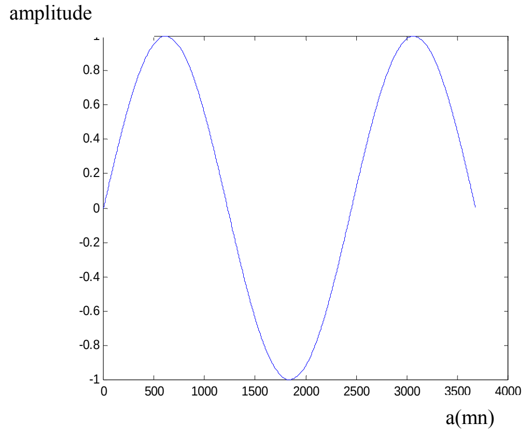
Fig. 7 a-b-c Mode shape , case of beam with 0.1m width



a) mode (1,1)



b) mode (2,1)



c) mode (3,1)

Fig. 8 a-b-c Mode shape, case of plate with 1.8m width

IV. Conclusion

A computational procedure to calculate natural frequency is presented. It's as well applicable to the case of the beams as that of the plates. The frequencies obtained are proportional to the number of mode considered. The general vibratory movement of the structure considered is the sum of all the modal movements.

The results which we present are compared with those present in the specialized literature and this comparison is completely satisfactory.

The calculation thereafter of the integral of Duhamel, in order to determine the component of variable displacement with time, will in the future permitt to determine the total displacement of the structure. As an application of this work, we will consider a study of moving loads on a bridge deck modelled as an orthotropic rectangular plate.

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