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Multi-Rate Exact Discretization based on Diagonalization of a Linear System - A Multiple-Real-Eigenvalue Case

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Abstract**—**A multi-rate discrete-time model, whose response agrees exactly with that of a continuous-time original at all sampling instants for any sampling periods, is developed for a linear system, which is assumed to have multiple real eigenvalues. The sampling rates can be chosen arbitrarily and individually, so that their ratios can even be irrational. The state space model is obtained as a combination of a linear diagonal state equation and a nonlinear output equation. Unlike the usual lifted model, the order of the proposed model is the same as the number of sampling rates, which is less than or equal to the order of the original continuous-time system. The method is based on a nonlinear variable transformation, which can be considered as a generalization of linear similarity transformation, which cannot be applied to systems with multiple eigenvalues in general. An example and its simulation result show that the proposed multi-rate model gives exact responses at all sampling instants.

Keywords**—**Multi-rate discretization, linear systems, triangularization, similarity transformation, diagonalization, exponential transformation, multiple eigenvalues

I. Introduction

DISCRETIZATION techniques are useful in a variety of areas including digital signals processing, measurement, systems analysis, and digital control [1],[2]. Signals with a wide range of frequencies often calls for the use of multiple sampling periods, where all state variables are sampled at different rates depending on their frequency components. Convenient tools are readily available for simulation studies [3], where a discrete-time lifting technique [4] is used. Although they are highly useful for simulation and evaluations of multi-rate designs, they are not necessarily a simple model for analysis or implementation. While a substantial increase in the order of lifted model may be accommodated in simulations, this is usually not the case in the analyses and implementation of digital controllers. Furthermore, sampled signals form an approximation model of the original system and exactness is often not pursued. Although these may specific only for digital controllers, requirements for low order and exactness are nevertheless important. Exact discretization, where the response of the discretized model matches exactly that of the original continuous-time system, is well known for single-rate case [5]. The approach used in the present paper is to transform the given system into a diagonal form for which this discretization technique can be applied.

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Diagonalization of system matrices plays important roles in the analysis and synthesis of linear systems. For instance, such systems tend to have better numerical properties and can be achieved by similarity transformation [6]. In the control perspective, system decoupling followed by state feedback achieves arbitrary diagonalization [7]. This may be considered as a variable transformation and has inspired the use of exponential transformation in [8], where a sufficient condition for exact linearization of nonlinear systems is presented. This method is applied to exact single-rate discretization of a nonlinear system in [8]. This is applied in the present paper to exact multi-rate discretization of a linear system with multiple real eigenvalues.

The paper is organized as follows: In Section 2, a second-order system with double eigenvalues is used to explain the concept and its triangular transformation is reviewed. The triangular system is then diagonalized, where the inverse transformation that was necessary in [4] is avoided. In Section 3, the resulting system with arbitrary eigenvalues is discretized exactly using multiple (including single) sampling rates. In Section 4, an example is presented with a simulation result. Section 5 presents conclusions.

II. Diagonalization of a Linear System with Multiple Real Eigenvalues

The goal of this section is to find a variable transformation for converting a given linear system with multiple real eigenvalues into a diagonal system with arbitrary eigenvalues. This is to be carried in three steps. The first is to transform the system into a triangular system, which is always possible using a similarity transformation. The second is to convert the triangular system into a diagonal system with fixed multiple real eigenvalues. The third is to introduce a new one-to-one mapping to transform the diagonal system into another with arbitrary and possibly distinct real eigenvalues. In the following, these are explained for a second-order case for ease of exposition.

A. The System

Let the linear time-invariant system with double and real eigenvalues be given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix},\quad \begin{bmatrix}x_1(0)\\ x_2(0)\end{bmatrix}=\begin{bmatrix}x_{10}\\ x_{20}\end{bmatrix} \tag{1}$$

where x_1 and x_2 are the state variables with their initial conditions given by x_{10} and x_{20}, and a_{ij} are constant coefficients. The eigenvalues of this system are assumed to be double, so that

$$(a_{11}-a_{22})^2+4a_{12}a_{21}=0. \tag{2}$$

The eigenvalues are given, therefore, by

$$\lambda=\frac{a_{11}+a_{22}}{2}. \tag{3}$$

It is assumed that the system is not triangular, which implies that $a_{11}\neq a_{22}$, since otherwise parameters would be $a_{12}=0$ or $a_{21}=0$ or both, indicating that the system is triangular. If the given system is already triangular, the first step explained next is omitted.

B. Triangularization

The first step is to transform system (1) into a triangular system using a standard similarity transformation [6]. Let the desired triangular system be given by

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}\lambda & \varepsilon\\ 0 & \lambda\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix},\begin{bmatrix}y_1(0)\\ y_2(0)\end{bmatrix}=\begin{bmatrix}y_{10}\\ y_{20}\end{bmatrix} \tag{4}$$

where y_1 and y_2 are the state variables with their initial conditions given by y_{10} and y_{20}. Parameter ε is a non-zero constant and may be set to unity, and λ is the eigenvalue given by (3), since they should remain the same under similarity transformation. It should be noted that the lower triangular form works just as well in this section.

The similarity transform of the form given by

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix} \tag{5}$$

can always be determined as one that satisfies, non-uniquely, the following [6]:

$$\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}=\begin{bmatrix}b_{11} & b_{12}\\ b_{21} & b_{22}\end{bmatrix}\begin{bmatrix}\lambda & \varepsilon\\ 0 & \lambda\end{bmatrix}. \tag{6}$$

The similarity transformation used in the present study is given by

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}a_{12} & 2a_{12}\\ \frac{1}{2}(-a_{11}+a_{22}) & -a_{11}+a_{22}+\varepsilon\end{bmatrix}\begin{bmatrix}y_1\\ y_2\end{bmatrix}. \tag{7}$$

The initial condition can be determined as

$$\begin{bmatrix}y_{10}\\ y_{20}\end{bmatrix}=\begin{bmatrix}\frac{-a_{11}+a_{22}+\varepsilon}{\varepsilon a_{12}} & -\frac{2}{\varepsilon}\\ \frac{a_{11}-a_{22}}{2\varepsilon a_{12}} & \frac{1}{\varepsilon}\end{bmatrix}\begin{bmatrix}x_{10}\\ x_{20}\end{bmatrix} \tag{8}$$

which always exists under the present condition.

C. Diagonalization of Triangular Systems

In this subsection, the diagonalization is achieved in two steps; first to a diagonal system with fixed multiple eigenvalues and second to one with arbitrary (distinct or multiple) eigenvalues.

Double Eigenvalues

Let the diagonal system be given by

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}c & 0\\ 0 & c\end{bmatrix}\begin{bmatrix}z_1\\ z_2\end{bmatrix},\begin{bmatrix}z_1(0)\\ z_2(0)\end{bmatrix}=\begin{bmatrix}z_{10}\\ z_{20}\end{bmatrix} \tag{9}$$

where z_1 and z_2 are the state variables with their initial conditions given by z_{10} and z_{20}. In the above, the value of eigenvalue c is not, unfortunately, arbitrary and will be determined shortly. The transformation from systems (4) to (9) is to be achieved using the following transformation:

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}e^{v_1(z_1,z_2)}\\ e^{v_2(z_1,z_2)}\end{bmatrix} \tag{10}$$

where $v_1(z_1,z_2)$ and $v_2(z_1,z_2)$ are functions of z_1 and z_2. Functions $v_1(z_1,z_2)$ and $v_2(z_1,z_2)$ are to be determined explicitly in the rest of this subsection. To this end, differentiate (10), using (4), to obtain

$$\begin{bmatrix}\dot{y}_1\\ \dot{y}_2\end{bmatrix}=\begin{bmatrix}\left(\dot{z}_1\frac{\partial v_1}{\partial z_1}+\dot{z}_2\frac{\partial v_1}{\partial z_2}\right)y_1\\ \left(\dot{z}_1\frac{\partial v_2}{\partial z_1}+\dot{z}_2\frac{\partial v_2}{\partial z_2}\right)y_2\end{bmatrix}=\begin{bmatrix}\lambda y_1+\varepsilon y_2\\ \lambda y_2\end{bmatrix}. \tag{11}$$

Using (9) and (10), this can be rewritten as

$$\begin{bmatrix}cz_1\frac{\partial v_1}{\partial z_1}+cz_2\frac{\partial v_1}{\partial z_2}\\ cz_1\frac{\partial v_2}{\partial z_1}+cz_2\frac{\partial v_2}{\partial z_2}\end{bmatrix}=\begin{bmatrix}\lambda+\varepsilon e^{v_2-v_1}\\ \lambda\end{bmatrix} \tag{12}$$

which is a Lagrange partial differential equation [9]. A number of methods exist to solve this equation and a method of characteristic curves is used in the present study. The characteristic equations are given by

$$\begin{cases}\dfrac{dv_1}{\lambda+\varepsilon e^{v_2-v_1}}=\dfrac{dz_1}{cz_1}=\dfrac{dz_2}{cz_2}\\ \dfrac{dv_2}{\lambda}=\dfrac{dz_1}{cz_1}=\dfrac{dz_2}{cz_2}\end{cases} \tag{13}$$

from which the following set of equations are obtained:

$$\begin{cases}\dfrac{dv_1}{\lambda+\varepsilon e^{v_2-v_1}}=\dfrac{dz_1}{cz_1}=\dfrac{dz_2}{cz_2}\\ \quad=\dfrac{\begin{pmatrix}g_{11}(v_1,z_1,z_2)dv_1\\ +g_{12}(v_1,z_1,z_2)dz_1\\ +g_{13}(v_1,z_1,z_2)dz_2\end{pmatrix}}{\begin{pmatrix}g_{11}(v_1,z_1,z_2)(\lambda+\varepsilon e^{v_2-v_1})\\ +g_{12}(v_1,z_1,z_2)cz_1\\ +g_{13}(v_1,z_1,z_2)cz_2\end{pmatrix}}\\ \dfrac{dv_2}{\lambda}=\dfrac{dz_1}{cz_1}=\dfrac{dz_2}{cz_2}\\ \quad=\dfrac{\begin{pmatrix}g_{21}dv_2\\ +g_{22}(z_1,z_2)dz_1\\ +g_{23}(z_1,z_2)dz_2\end{pmatrix}}{\begin{pmatrix}g_{21}\lambda\\ +g_{22}(z_1,z_2)cz_1\\ +g_{23}(z_1,z_2)cz_2\end{pmatrix}}\end{cases}. \tag{14}$$

In (14), g_{11}, g_{12}, and g_{13} are functions of v_1, z_1, and z_2, g_{22} and g_{23} those of z_1 and z_2, and g_{21} a constant. These must be determined such that the denominators on the right-most sides are zero and, at the same time, the corresponding numerators are exact differentials. The necessary and sufficient conditions on the exactness are

$$\begin{cases}\begin{cases}\dfrac{\partial g_{11}(v_1,z_1,z_2)}{\partial z_1}=\dfrac{\partial g_{12}(v_1,z_1,z_2)}{\partial v_1}\\ \dfrac{\partial g_{12}(v_1,z_1,z_2)}{\partial z_2}=\dfrac{\partial g_{13}(v_1,z_1,z_2)}{\partial z_1}\\ \dfrac{\partial g_{13}(v_1,z_1,z_2)}{\partial v_1}=\dfrac{\partial g_{11}(v_1,z_1,z_2)}{\partial z_2}\end{cases},\\ \dfrac{\partial g_{22}(z_1,z_2)}{\partial z_2}=\dfrac{\partial g_{23}(z_1,z_2)}{\partial z_1}\end{cases} \tag{15}$$

at which time the following hold:

$$\begin{cases}g_{11}(v_1,z_1,z_2)dv_1+g_{12}(v_1,z_1,z_2)dz_1\\ \qquad+g_{13}(v_1,z_1,z_2)dz_2=0\\ g_{21}dv_2+g_{22}(z_1,z_2)dz_1\\ \qquad+g_{23}(z_1,z_2)dz_2=0\end{cases}. \tag{16}$$

Candidate functions g_{21}, g_{22}, and g_{23} that have been chosen to meet the two requirements are the following:

$$\begin{cases} g_{21} = -\dfrac{c}{\lambda} \\ g_{22} = \dfrac{1}{z_1 + \beta z_2}, \\ g_{23} = \dfrac{\beta}{z_1 + \beta z_2} \end{cases} \tag{17}$$

where is β an arbitrary parameter. In this case, the exact differential is given by

$$-\frac{c}{\lambda} dv_2 + \frac{1}{z_1 + \beta z_2} dz_1 + \frac{\beta}{z_1 + \beta z_2} dz_2 = 0. \tag{18}$$

This yields, upon integration,

$$-\frac{c}{\lambda} v_2 + ln\big(\kappa_1 F(z_1, z_2)\big) = h_2, \tag{19}$$

where $F(z_1, z_2)$ is defined as

$$F(z_1, z_2) = z_1 + \beta z_2, \tag{20}$$

κ_1 is the signum of $F(z_1, z_2)$, given as

$$\kappa_1 = \begin{cases} 1 \ (F(z_1, z_2) > 0) \\ -1 \ (F(z_1, z_2) < 0)' \end{cases} \tag{21}$$

and h_2 is a constant. Since this constant must satisfy the initial condition, it should be chosen as

$$h_2 = ln\left(\kappa_{10} F_0(z_{10}, z_{20}) y_{20}^{-\frac{c}{\lambda}}\right). \tag{22}$$

where $F_0(z_{10}, z_{20})$ is defined as

$$F_0(z_{10}, z_{20}) = z_{10} + \beta z_{20} \tag{23}$$

and κ_{10} is the signum of $F_0(z_{10}, z_{20})$, given by

$$\kappa_{10} = \begin{cases} 1 \ (F_0(z_{10}, z_{20}) > 0) \\ -1 \ (F_0(z_{10}, z_{20}) < 0) \end{cases}. \tag{24}$$

Function v_2 can now be obtained as

$$e^{v_2} = \left(\frac{\kappa_1 F}{\kappa_{10} F_0}\right)^{\frac{\lambda}{c}} y_{20}. \tag{25}$$

To determine the other function, e^{v_1}, in (10), substitute (25) into the first equation of (14) to obtain

$$g_{11}(v_1, z_1, z_2)\left(\lambda + \varepsilon y_{20}\left(\frac{\kappa_1 F}{\kappa_{10} F_0}\right)^{\frac{\lambda}{c}} e^{-v_1}\right) + g_{12}(v_1, z_1, z_2) c z_1 + g_{13}(v_1, z_1, z_2) c z_2 \tag{26}$$

which can be made equal to zero by choosing functions, for instance, as

$$\begin{cases} g_{11} = \dfrac{e^{v_1}}{F^2} \\ g_{12} = -\dfrac{2e^{v_1}}{F^3} - \dfrac{2\varepsilon y_{20}}{F_0^2 \lambda F} \\ g_{13} = -\dfrac{2\beta e^{v_1}}{F^3} - \dfrac{2\varepsilon y_{20} \beta}{F_0^2 \lambda F} \end{cases} \tag{27}$$

and choosing the eigenvalue of system (9) as

$$c = \frac{\lambda}{2}. \tag{28}$$

Substituting (27) into the first equation in (16), and integrating the resulting relation, one obtains

$$e^{v_1} = F^2\left(h_1 + \frac{2\varepsilon y_{20} ln(\kappa_1 F)}{\lambda F_0^2}\right), \tag{29}$$

where h_1 is a constant, which depends on initial conditions as in (22), and should be chosen as

$$h_1 = \frac{\lambda y_{10} - 2\varepsilon y_{20} ln(\kappa_{10} F_0)}{\lambda F_0^2}. \tag{30}$$

The variable transformation (10) is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_{10}\left(\dfrac{F}{F_0}\right)^2 + y_{20}\left(\dfrac{F}{F_0}\right)^2 ln\left(\dfrac{\kappa_1 F}{\kappa_{10} F_0}\right)^{\frac{2\varepsilon}{\lambda}} \\ y_{20}\left(\dfrac{F}{F_0}\right)^2 \end{bmatrix}, \tag{31}$$

where F_0 is arbitrary, as will be shown shortly, and is assumed to be nonzero. Furthermore, κ_1 and κ_{10} can be removed from (31) as follows. First, it should be noted that (31) is arranged to hold true for any initial conditions on z_1 and z_2 and thus, they can be chosen arbitrarily. Thus, let them be arbitrary parameters denoted by

$$z_{10} = \gamma_1, \ z_{20} = \gamma_2. \tag{32}$$

Second, function F does not change its sign, since its derivative can be written as

$$\dot{F} = \dot{z}_1 + \beta \dot{z}_2 = \frac{\lambda}{2} z_1 + \beta \frac{\lambda}{2} z_2 = \frac{\lambda}{2} F, \tag{33}$$

which is a first-order system whose response is monotonic. Therefore, the sign of F depends solely on that of F_0 and, thus,

$$\kappa_1 = \kappa_{10}. \tag{34}$$

The transformation between (4) and (9) is given by

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} y_2\left(\dfrac{y_{10}}{y_{20}} + ln\left(\dfrac{F}{\gamma_1 + \beta\gamma_2}\right)^{\frac{2\varepsilon}{\lambda}}\right) \\ y_{20}\left(\dfrac{F}{\gamma_1 + \beta\gamma_2}\right)^2 \end{bmatrix}. \tag{35}$$

D. Distinct Eigenvalues

The given system (1), which has double eigenvalues, was converted into a triangular system (4) with the same eigenvalues using a similarity transformation (7). System (4) was then transformed into a diagonal system (9) with double eigenvalues but their values were halved, using the method presented in [4]. System (9) is now to be related yet to another diagonal system whose eigenvalues can now be set arbitrarily, double or distinct. This is possible since once the system is diagonalized, the system is basically a collection of first-order sub-systems, each of which can then be modified to another sub-system with arbitrary eigenvalue individually.

The final diagonal system is written as

$$\begin{bmatrix} \dot{w}_1 \\ \dot{w}_2 \end{bmatrix} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}, \begin{bmatrix} w_1(0) \\ w_2(0) \end{bmatrix} = \begin{bmatrix} w_{10} \\ w_{20} \end{bmatrix}, \tag{36}$$

where w_1 and w_2 are state variables, w_{10} and w_{20} their initial values, and m_1 and m_2 are non-zero but otherwise arbitrary eigenvalues. The transformation between systems (9) and (36) can be achieved using the following:

$$\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} e^{p_1(w_1)} \\ e^{p_2(w_2)} \end{bmatrix}, \tag{37}$$

where $p_1(w_1)$ and $p_2(w_2)$ are functions of single variable, which suffices for first-order sub-systems. These functions, $p_1(w_1)$ and

$p_2(w_2)$, are derived below:

Differentiating (37) and using system (9) with relationship (31) give

$$\begin{bmatrix}\dot{z}_1\\ \dot{z}_2\end{bmatrix}=\begin{bmatrix}\dot{w}_1\dfrac{dp_1}{dw_1}z_1\\ \dot{w}_2\dfrac{dp_2}{dw_2}z_2\end{bmatrix}=\begin{bmatrix}\dfrac{\lambda}{2}z_1\\ \dfrac{\lambda}{2}z_2\end{bmatrix}, \tag{38}$$

which can be re-arranged, using (36), into

$$\begin{bmatrix}m_1w_1\dfrac{dp_1}{dw_1}\\ m_2w_2\dfrac{dp_2}{dw_2}\end{bmatrix}=\begin{bmatrix}\dfrac{\lambda}{2}\\ \dfrac{\lambda}{2}\end{bmatrix}. \tag{39}$$

This leads to the following exact differential equation:

$$\begin{cases}\dfrac{dp_1}{\frac{\lambda}{2}}-\dfrac{dw_1}{m_1w_1}=0\\ \dfrac{dp_2}{\frac{\lambda}{2}}-\dfrac{dw_2}{m_2w_2}=0\end{cases} \tag{40}$$

which yields

$$\begin{cases}\dfrac{2}{\lambda}p_1+ln|w_1|^{-\frac{1}{m_1}}=q_1\\ \dfrac{2}{\lambda}p_2+ln|w_2|^{-\frac{1}{m_2}}=q_2\end{cases} \tag{41}$$

with constants q_1 and q_2 chosen as

$$\begin{cases}ln\left(\gamma_1^{\frac{2}{\lambda}}|w_{10}|^{-\frac{1}{m_1}}\right)=q_1\\ ln\left(\gamma_2^{\frac{2}{\lambda}}|w_{20}|^{-\frac{1}{m_2}}\right)=q_2\end{cases}. \tag{42}$$

With these values, p_1 and p_2 are determined as

$$\begin{cases}p_1=ln\left(\gamma_1\left(\dfrac{|w_1|}{|w_{10}|}\right)^{\frac{\lambda}{2m_1}}\right)\\ p_2=ln\left(\gamma_2\left(\dfrac{|w_2|}{|w_{20}|}\right)^{\frac{\lambda}{2m_2}}\right)\end{cases}. \tag{43}$$

Since the states w_1 and w_2 are of first-order sub-systems in (36), their signs are the same as those of w_{10} and w_{20}, so that the desired variable transformation is finally found as

$$\begin{bmatrix}z_1\\ z_2\end{bmatrix}=\begin{bmatrix}\gamma_1\left(\dfrac{w_1}{w_{10}}\right)^{\frac{\lambda}{2m_1}}\\ \gamma_2\left(\dfrac{w_2}{w_{20}}\right)^{\frac{\lambda}{2m_2}}\end{bmatrix}. \tag{44}$$

It should be noted that the initial conditions, w_{10} and w_{20}, must be non-zero but otherwise can be chosen arbitrarily for any given z_{10} and z_{20}.

By combining (35) and (44), using (20), the transformation that relates directly the triangular system (4) to the diagonal system (36), can be determined as

$$\begin{bmatrix}y_1\\ y_2\end{bmatrix}=\begin{bmatrix}y_2\left(\dfrac{y_{10}}{y_{20}}+\dfrac{2\varepsilon}{\lambda}ln\dfrac{G}{\gamma_1+\beta\gamma_2}\right)\\ y_{20}\dfrac{G^2}{(\gamma_1+\beta\gamma_2)^2}\end{bmatrix} \tag{45}$$

where

$$G(w_1,w_2)=\gamma_1\left(\frac{w_1}{w_{10}}\right)^{\frac{\lambda}{2m_1}}+\beta\gamma_2\left(\frac{w_2}{w_{20}}\right)^{\frac{\lambda}{2m_2}}. \tag{46}$$

E. Diagonalization

The triangularization of system (1) to (4), the diagonalization with multiple eigenvalues of system (4) to (9), and that with arbitrary eigenvalues of system (9) to (36) can be combined into a single transformation. This can be obtained by substituting (45) into (7) so that y_1 and y_2 are eliminated and using the relationship (8) on initial conditions. The resulting transformation is obtained as

$$\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\frac{G^2}{(\gamma_1+\beta\gamma_2)^2}\begin{bmatrix}x_{10}+\dfrac{n}{\lambda}ln\dfrac{G}{\gamma_1+\beta\gamma_2}\\ x_{20}-\dfrac{n(a_{11}-a_{22})}{2a_{12}\lambda}ln\dfrac{G}{\gamma_1+\beta\gamma_2}\end{bmatrix} \tag{47}$$

where *G* is given by (46) and

$$n=(a_{11}-a_{22})x_{10}+2a_{12}x_{20}. \tag{48}$$

It must be pointed out that in the diagonalization conversions, the eigenvalue of the given system, λ, must be assumed nonzero, while it can be zero in the triangularization and multi-rate discretization that follows.

III. Multi-rate discretization

Using the diagonal form (36) where the eigenvalues are chosen to be identical, a multi-rate exact discrete-time model can be obtained easily, where there is no order increase and sampling rates are chosen arbitrarily. The exact discrete-time model is a model whose response matches coincide with that of the original continuous-time system at all sampling instants for any sampling period. When all state variables are sampled at the same rate, it reduces to the well-known exact model [3]. The most primitive approach to the analysis of a multi-rate n-th order system, where each state is sampled at distinctive rate, is to prepare an n-th order single-rate model for each sampling rate and, thus, use the total of $n\times n$ numbers of state variables. In contrast, in system (36), the eigenvalues and initial stats are chosen to be identical as

$$m_1=m_2=m,\qquad w_{10}=w_{20}=w_0, \tag{49}$$

so that the states become identical as

$$w_1=w_2=w. \tag{50}$$

Thus, the diagonal system is practically a first-order system as

$$\dot{w}=mw. \tag{51}$$

The exact discrete-time models of this linear system sampled with the periods of T_1 and T_2, which are then collected as diagonal elements, yield the following exact state equation:

$$\begin{bmatrix}\delta_{T_1}w_{1,k_1}\\ \delta_{T_2}w_{2,k_2}\end{bmatrix}=\begin{bmatrix}\dfrac{e^{mT_1}-1}{T_1} & 0\\ 0 & \dfrac{e^{mT_2}-1}{T_2}\end{bmatrix}\begin{bmatrix}w_{1,k_1}\\ w_{2,k_2}\end{bmatrix}, \tag{52}$$

where w_{i,k_i} implies $w_i(k_i,T_i) = w_i(t)|_{t=k_iT_i}$ and

$$\delta_{T_i} = \frac{q_{T_i} - 1}{T_i}, \qquad q_{T_i} w_{ik_i} = w_{i,k_i+1}. \tag{53}$$

The states w_{k_i}, which are updated element-wise using (52) at different rates, can be substituted into the following nonlinear output equation to recover the original states element-wise:

$$\begin{bmatrix} x_{1,k_1} \\ x_{2,k_2} \end{bmatrix} = \begin{bmatrix} \left(\frac{w_{1,k_1}}{w_0}\right)^{\frac{\lambda}{m}} \left(x_{10} + n ln\left(\frac{w_{1,k_1}}{w_0}\right)^{\frac{1}{2m}}\right) \\ \left(\frac{w_{2,k_2}}{w_0}\right)^{\frac{\lambda}{m}} \left(x_{20} - \frac{n(a_{11} - a_{22})}{2a_{12}} ln\left(\frac{w_{2,k_2}}{w_0}\right)^{\frac{1}{2m}}\right) \end{bmatrix}. \tag{54}$$

It should be noted that parameters β, γ_1, and γ_2 in (47) have disappeared in (54). The exact multi-rate model is given as a set of linear state equation (52) with a nonlinear output equation (54).

IV. SIMULATIONS

The system used for the simulation is the second-order system given in the following state space form:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \tag{55}$$

whose eigenvalues are identical at $\lambda = -2$. Its initial conditions are chosen arbitrarily as $x_{10} = 1$ and $x_{20} = 2$. The exact discrete-time model (52) is chosen to have the eigenvalue of $m = -1$ and sampled using the period of $T_1 = 0.3$ and $T_2 = 0.5$ seconds. The resulting exact discrete-time model is obtained as the following linear state equation with the non-linear output equation:

$$\begin{bmatrix} \delta_{0.3} w_{k_1} \\ \delta_{0.5} w_{k_2} \end{bmatrix} = \begin{bmatrix} \frac{e^{-0.3} - 1}{0.3} & 0 \\ 0 & \frac{e^{-0.5} - 1}{0.5} \end{bmatrix} \begin{bmatrix} w_{k_1} \\ w_{k_2} \end{bmatrix}$$

$$\cong \begin{bmatrix} -0.8639 & 0 \\ 0 & -0.7869 \end{bmatrix} \begin{bmatrix} w_{k_1} \\ w_{k_2} \end{bmatrix} \tag{56}$$

and

$$\begin{bmatrix} x_{1,k_1} \\ x_{2,k_2} \end{bmatrix} = \begin{bmatrix} (w_{1,k_1})^2 (1 - 3 ln w_{1,k_1}) \\ (w_{2,k_2})^2 (2 + 3 ln w_{2,k_2}) \end{bmatrix}, \tag{57}$$

where $w_0 = 1$. Figures 1 and 2 show the state responses of the original continuous-time system in solid lines. The sequences obtained by the exact discrete-time model are held constant using the zero-order-hold [4] and are shown in broken stair-case lines. It can be seen from these plots that the exact model gives the responses that match exactly with the original system at all sampling instants. It should be emphasized that the sampling periods can be arbitrarily chosen, including those whose ratios are irrational. Furthermore, the order of the model remains the same at 2.

V. CONCLUSIONS

By applying an exact linearization technique proposed in a previous study [8], the diagonalization of systems with multiple-eigenvalues via a nonlinear variable transformation has been made possible. Such diagonalization is not possible using the standard similarity transformation, which is based on linear transformations. The developed technique has then been applied to the derivation of an exact multi-rate discrete-time model, where its sampling rates can be chosen arbitrarily and its order does not increase.Triangular systems used in the present study can be replaced with Jordan blocks and also be extended to cover the distinct eigenvalue portions of the system. Combining these two cases, a general real eigenvalue case may be covered. The technique should be further extended to systems with complex-conjugate eigenvalues. The linearization of nonlinear systems into a diagonal system may then be possible based on [8].Another highly important avenue to pursue is to apply the developed multi-rate exact model for the development of multi-rate digital controller design methods. An example is an extension of a single-rate digital redesign method, which can guarantee closed-loop stability for any sampling periods [10], to the multi-rate version.

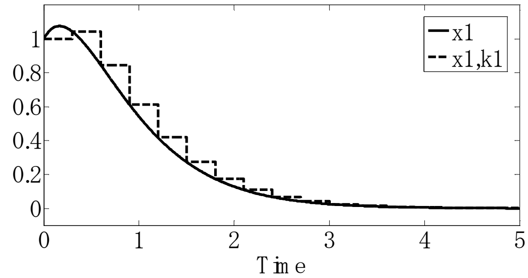

Fig. 1 State response x_1 and its exact model $x_{1,k1}$ for $T_1 = 0.3$ seconds

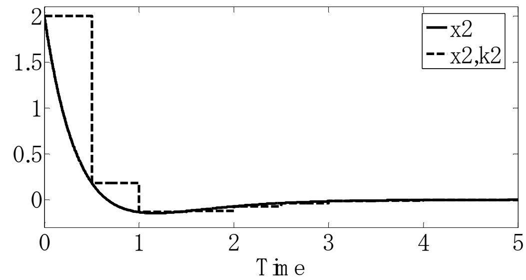

Fig. 2 State response x_2 and its exact model $x_{2,k2}$ for $T_2 = 0.5$ seconds

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