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Numerical Solution of Second-Order Ordinary Differential Equations by Improved Runge-Kutta Nystrom Method

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Abstract—In this paper we developed the Improved Runge-Kutta Nystrom (IRKN) method for solving second order ordinary differential equations. The methods are two step in nature and require lower number of function evaluations per step compared with the existing Runge-Kutta Nystrom (RKN) methods. Therefore, the methods are computationally more efficient at achieving the higher order of local accuracy. Algebraic order conditions of the method are obtained and the third and fourth order method are derived with two and three stages respectively. The numerical results are given to illustrate the efficiency of the proposed method compared to the existing RKN methods.

Keywords—Improved Runge-Kutta Nystrom method, Two step method, Second-order ordinary differential equations, Order conditions

I. Introduction

CONSIDER the special second-order ordinary differential equations of the form

$$y'' = f(x,y),\ y(x_0) = y_0,\ y'(x_0) = y'_0. \tag{1}$$

Such problem often arise in science and engineering fields such as celestial mechanics, molecular dynamics , semi-discretization of wave equations and electronics. The second-order equations can be directly solved by using Runge-Kutta Nystrom (RKN) methods or multistep methods. Phohomsiri and Udwadia [1], [2] constructed the Accelerated of Runge-Kutta method for solving autonomous first order ordinary differential equations $y' = f(y)$. Rabiei and Ismail [3] proposed the third-order Improved Runge-Kutta method for solving ordinary differential equations $y' = f(x,y)$. Rabie et. al [4], [5] developed the general form of Improved Runge-Kutta method for solving ordinary differential equations. In this paper, we developed the Improved Runge-Kutta Nystrom (IRKN) method for solving special second-order equation $y'' = f(x,y)$. The third-order Improved Runge-Kutta Nystrom (IRKN3) method used only 2-stages and the fourth-order Improved Runge-Kutta Nystrom (IRKN4) method used 3-stages per step.

In section II, we constructed the IRKN method for solving second order ODE's and the order conditions of the method are obtained in section III. In section IV, the derivation of the method is given. Numerical examples to illustrate the efficiency of the methods compared with the existing RKN methods are presented in the last section.

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II. Construction of IRKN Method

Rabie et. al [4], [5] developed the general form of Improved Runge-Kutta method as follows

$$y_{n+1} = y_n + h\left(b_1k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})\right), \tag{2}$$

where

$$\begin{aligned}
k_1 &= f(x_n, y_n), \quad k_{-1} = f(x_{n-1}, y_{n-1}),\\
k_i &= f(x_n + c_ih, y_n + h\sum_{j=1}^{i-1} a_{ij}k_j),\\
k_{-i} &= f(x_{n-1} + c_ih, y_{n-1} + h\sum_{j=1}^{i-1} a_{ij}k_{-j}).
\end{aligned}$$

For $2 \leq i \leq s$. We developed the IRKN method for solving the second-order equation directly by following the approach discussed in Dormand [6] on the derivation of RKN method and based on IRK method formulas (2).

The general form of explicit IRKN method with s-stages a as follows :

$$\begin{aligned}
y_{n+1} &= y_n + \frac{3h}{2}\, y'_n - \frac{h}{2}\, y'_{n-1} + h^2\sum_{i=2}^{s} \bar{b}_i(k_i - k_{-i}),\\
y'_{n+1} &= y'_n + h(b_1k_1 - b_{-1}k_{-1} + \sum_{i=2}^{s} b_i(k_i - k_{-i})),\\
k_1 &= f(x_n, y_n), \qquad k_{-1} = f(x_{n-1}, y_{n-1}),\\
k_i &= f(x_n + c_ih, y_n + hc_iy'_n + h^2\sum_{j=1}^{i-1} a_{ij}k_j),\\
k_{-i} &= f(x_{n-1} + c_ih, y_{n-1} + hc_iy'_{n-1} + h^2\sum_{j=1}^{i-1} a_{ij}k_{-j})).
\end{aligned} \tag{3}$$

For $i = 2, \ldots, s$..

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III. Order Conditions

To find the order conditions for IRKN method we applied the Taylor's series expansion to equations (3) (see [3], [4], [5]). Here, after using the Taylor's series expansion the order conditions of method for y'_n and y_n up to order five are presented in Table I.

TABLE I
ORDER CONDITIONS OF IRKN METHOD FOR y'_n AND y.

order of method	order condition for y'	order condition for y
first order	$b_1 - b_{-1} = 1$	
second order	$b_{-1} + \sum_{i=2}^{s} b_i = \frac{1}{2}$	
third order	$\sum_{i=2}^{s} b_i c_i = \frac{5}{12}$	$\sum_{i=2}^{s} \bar{b}_i = \frac{5}{12}$
fourth order	$\sum_{i=2}^{s} b_i c_i^2 = \frac{1}{3}$	$\sum_{i=2}^{s} \bar{b}_i c_i = \frac{1}{6}$
fifth order	$\sum_{i=2}^{s} b_i c_i^3 = \frac{31}{120}$	$\sum_{i=2}^{s} \bar{b}_i c_i^2 = \frac{31}{360}$
	$\sum_{i=2}^{s} b_i a_{ij} c_j = \frac{31}{720}$	

IV. Derivation of IRKN method

In this section we derived the IRKN method of order three with two stages (IRKN3) and IRKN method of order four with three stages (IRKN4). All the derivation of our methods with details are given as follows.

A. *Third-order method with two-stages (IRKN3)*

The IRKN3 method with two-stages ($s = 2$) from formulas (3) is given by:

$$\begin{aligned} y_{n+1} &= y_n + \frac{3h}{2}\, y'_n - \frac{h}{2}\, y'_{n-1} + h^2 \bar{b}_2 (k_2 - k_{-2}), \\ y'_{n+1} &= y'_n + h(b_1 k_1 - b_{-1} k_{-1} + b_2(k_2 - k_{-2})), \end{aligned} \tag{4}$$

$$\begin{aligned} k_1 &= f(x_n, y_n), \qquad k_{-1} = f(x_{n-1} y_{n-1}), \\ k_2 &= f(x_n + c_2 h, y_n + h c_2 y'_n + h^2 a_{21} k_1), \\ k_{-2} &= f(x_{n-1} + c_2 h, y_{n-1} + h c_2 y'_{n-1} + h^2 a_{21} k_{-1}). \end{aligned}$$

To find the coefficients of IRKN3 method in equations (4), order conditions up to order three for y_n and y'_n must be satisfied. Therefore we need to satisfy the following equations:

$$b_1 - b_{-1} = 1, \quad b_{-1} + b_2 = \frac{1}{2}, \quad b_2 c_2 = \frac{5}{12}, \quad \bar{b}_2 = \frac{5}{12}.$$

We choose the value of $b_{-1} \in [-1\ 1]$ as a free parameter, here we set $b_{-1} = -\frac{1}{3}$ and found the remaining coefficients as follows:

$$c_2 = \frac{1}{2},\ a_{21} = \frac{1}{8},\ b_1 = \frac{2}{3},\ b_2 = \frac{5}{6}.$$

B. *Fourth-order method with three-stages (IRKN4)*

Consider the IRKN4 with three-stages ($s = 3$) from formulas (3), to find the coefficients of IRKN4 method the order conditions from Table I up to order four for y_n and y'_n must be satisfied. Therefore we need to satisfy the following equations:

$$b_1 - b_{-1} = 1, \quad b_{-1} + b_2 + b_3 = \frac{1}{2},$$

$$b_2 c_2 + b_3 c_3 = \frac{5}{12}, \quad b_2 c_2^2 + b_3 c_3^2 = \frac{1}{3},$$

and

$$\bar{b}_2 + \bar{b}_3 = \frac{5}{12}, \quad \bar{b}_2 c_2 + \bar{b}_3 c_3 = \frac{1}{6}.$$

We choose $c_2 = \frac{1}{4}$ and $c_3 = \frac{3}{4}$ as free parameters and obtained the remaining parameters as follows:

$$a_{21} = \frac{1}{32},\ a_{31} = 0,\ a_{32} = \frac{9}{32},$$

$$b_{-1} = \frac{1}{18},\ b_1 = \frac{19}{18},\ b_2 = \frac{-1}{6},\ b_3 = \frac{11}{18},$$

$$\bar{b}_2 = \frac{7}{24}, \bar{b}_3 = \frac{1}{8}.$$

V. Numerical Examples

In this section, we tested a standard set of second-order initial value problems to show the efficiency and accuracy of the proposed method. The exact solution $y(x)$ and $y'(x)$ are used for starting values of y_1 and y'_1 at the first step $[x_0\ x_1]$. The following problems are solved for $x \in [0\ 10]$.

problem 1 (The undamped Duffing's equations [7])

$$y'' + y + y^3 = 0.002 cos(1.01x),$$

$$y(0) = 0.200426728067, \quad y'(0) = 0.$$

exact solution computed by Galerkin method and given by:

$$y(x) = \sum_{i=0}^{4} a_{2i+1} cos[1.01(2i+1)x],$$

with $a_1 = 0.200179477536,\ a_3 = 0.246946143 \times 10^{-3},\ a_5 = 0.304014 \times 10^{-6},\ a_7 = 0.374 \times 10^{-9}$ and $a_9 < 10^{-12}$.

Problem 2 (An almost periodic Orbit problem studied by Stiefel and Bettis [8])

$$y'' + y = 0.001\ e^{ix}, \quad y(0) = 1,\ y'(0) = 0.9995i.$$

exact solution: $y(x) = (1 - 0.0005ix)e^{ix}$.

we write in equivalent form:

$$y''_1 + y_1 = 0.001 cos(x), \quad y_1(0) = 1, \quad y'_1(0) = 0,$$

$$y''_2 + y_2 = 0.001 sin(x), \quad y_2(0) = 0, \quad y'_2(0) = 0.9995.$$

exact solutions:

$$y_1(x) = cos(x) + 0.0005 x sin(x),$$

$$y_2(x) = sin(x) - 0.0005 x cos(x)$$

TABLE II
MAXIMUM GLOBAL ERROR AGAINST STEP SIZE h FOR PROBLEM 2

h	IRKN3	IRKN4	RKNV3	RKND3	RKNV4	RKN4
0.5	1.87E-2	6.81E-4	2.80E-1	1.95E-2	9.91E-3	2.06E-3
0.1	1.13E-4	5.52E-7	1.07E-2	1.49E-4	1.40E-5	3.09E-6
0.05	1.39E-5	3.34E-8	2.63E-3	1.86E-5	8.65E-7	1.92E-7
0.01	1.07E-7	5.47E-11	1.07E-4	1.48E-7	3.74E-9	3.05E-10
0.005	1.43E-8	3.84E-12	2.69E-5	1.85E-8	2.47E-10	1.91E-11

To illustrate the efficiency of new methods we compared the numerical results with existing methods. The codes have been denoted by the following.

(i) IRKN3 : The Improved Runge-Kutta Nystrom method of order three with two stages in this paper.

(ii) RKNV3 : The third order Runge-Kutta Nystrom method with zero dissipation three stages given in van der Houwen and Sommeijer [9]

(iii) RKND3 : The third order three stages Runge-Kutta Nystrom method given in Dormand.[6]

(iv) IRKN4: The Improved Runge-Kutta Nystrom method of order four with three stages in this paper.

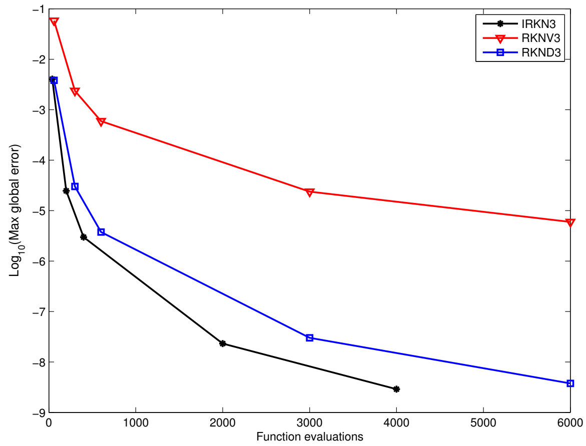
(v) RKNV4 :The fourth order Runge-Kutta Nystrom method with ten order dispersion, fifth order dissipation four stages given in Van der Huwen and Sommeijer [9]

(vi) RKN4 : The fourth order three stages classical Rnge-Kutta Nystrom method give in Garcia et al .[10]

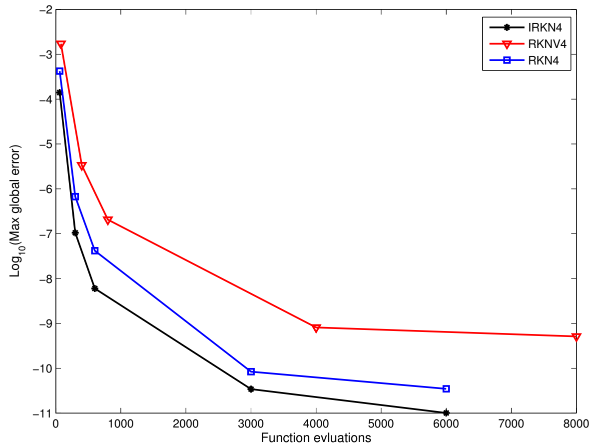
The logarithm of maximum global error versus the number of function evaluations for IRKN3 and IRKN4 methods for tested problem 1 is shown in Figure 1. In part (a) the results are shown for IRKN3 compared with the existing methods RKNV3 and RKND3 which indicate that the IRKN3 with lower number of function evaluations is more accurate. In part (b) the results are shown for the fourth order methods which indicate that the IRKN4 with three stages is more efficient compared with RKNV4 and RKN4. In Table II the maximum global error against the different values of step size h is presented for problem 2 and we observed that the new methods are more accurate compared to the given methods.

VI. CONCLUSION

In this paper we constructed the Runge-Kutta Nyström (RKN) method for solving second order ODE's. The order conditions of the IRKN methods are derived and using these order conditions, the methods are obtained for order 3 and 4 with two and three stages respectively. Numerical results are presented to illustrate the efficiency of IRKN3 and IRKN4 compared to the existing methods. The IRKN methods in terms of error accuracy and number of function evaluations are more efficient compared to RKNV3, RKND3 ,RKNV4 and RKN4 methods.



(a)



(b)

Fig. 1. The logarithm of maximum global error versus number of function evaluations for problem 1, (a) third order methods (b) fourth order methods

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