Calculation of Wave Function at the Origin (WFO) for the Ground State of Doubly Heavy Mesons based on the Variational Method

Maryam Momeni Feili, Mahvash Zandy Navgaran

Abstract—The wave function at the origin is an important quantity in studying many physical problems concerning heavy quarkonia. This is because that it is using for calculating spin state hyperfine splitting and also crucial to evaluating the production and decay amplitude of the heavy quarkonium. In this paper, we present the variational method by using the single-parameter wave function to estimate the WFO for the ground state of heavy mesons.

Keywords— wave function at the origin,heavy Mesons, Bound states, Variational method, Non-relativistic quark model, Potential model, trial wave function

I. INTRODUCTION

RECENTLY, the wave function at the origin for the S-wave bound state of a heavy quark-antiquark system once again attracts physicists' attentions [1,2]. In the context of the non-relativistic potential model [1,2] demonstrated the numerical results of WFO of the S-wave $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ systems.As well known, except the Coulomb and the harmonic oscillator potentials, there are few potentials which bound state problems can be analytically solved. For solving these non-analytically soluble bound state problems, one has to use approximations. Numerically solving Schrodinger equation is the most powerful method which can reach most required accuracy. But the numerical method has some defects, for instance, it cannot give analytical aexpressions for further discussion [3].In other word, there exist systems whose Hamiltonians are known, but they cannot be solved exactly or by a perturbative treatment. That is no closely related Hamiltonian that can be solved exactly or approximately by perturbation theory because the first order is not sufficiently accurate. One of the approximation method that is suitable for solving such problems is the variational method, which is also called the Rayleigh-Ritz method [8]. This method does not require knowledge of simple Hamiltonians that can be solved exactly.The variational method has more advantages. It can give an analytical expression of the wave function. In particular, if there is only a single-parameter in the trial wave function, the resultant wave function has a simple form. Then it is very convenient in the practical application and physical

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discussion.

II. THE POTENTIAL MODELS

There are many potentialmodels which can fit the experimental spectra of the heavy quarkonia with certain accuracy. We consider three functional forms for the potential that give reasonable accounts of the $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ spectra [2].

(1). Cornell potential [4] :

$$V(r) = -\frac{4}{3}\frac{\alpha_s}{r} + kr \qquad (1)$$

With

$$\alpha_s = 0/39 \; ; \; k = \frac{1}{(2/34)^2} \; (GeV)^2 \; ; \; m_c = 1/84 \; (GeV/C^2) \; ; \; m_b = 5/18 \; (GeV/C^2)$$

(2). Martin potential [5]:

$$V(r) = kr^{0.1}$$

(2)

With

$$k = 6/898 \; ; \; m_c = 1/8 \; (GeV/C^2) \; ; \; m_b = 5/174 \; (GeV/C^2)$$

(3). Logarithmic potential [6]:

$$V(r) = k\log(r) \qquad (3)$$

With

$$k = 0/733 \; ; \; m_c = 1/5 \; (GeV/C^2) \; ; \; m_b = 4/906 \; (GeV/C^2)$$

III.THE VARIATIONAL METHOD BY USING THE SINGLE-PARAMETER TRIAL WAVE FUNCTION

Within the framework of the non-relativistic potential model, the S-state wave function $\psi(r)$ of the heavy quarkonium satisfies the Schrodinger equation

$$H\psi(r) = -\frac{1}{2\mu}\nabla^2\psi(r) + V(r)\psi(r) = E\psi(r) \qquad (4)$$

Where H is the Hamiltonian of the quarkonium, $V(r)$ denotes the central potential between quark and antiquark, E represent the energy eigenvalue, and μ is the reduced mass.

To solve Eq. (4) by using the variational method, one needs to choose a suitable trial wave function $\psi(r,c)$ with N independent parameters $\{c\} = \{c_1, c_2, ..., c_N\}$ first and then to seek out a set of parameters $\{c_o\} = \{c_{io}, i = 1, 2, ..., N\}$ which minimizes the expectation value of Hamiltonian, namely

$$E(c) = \langle H \rangle = \frac{\langle \psi(c)|H|\psi(c)\rangle}{\langle \psi(c)|\psi(c)\rangle} \qquad (5)$$

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The minimum value $E(c_o)$ gives an upper limit of the ground state energy.

In this section, we choose the simplest trial wave function in which there is only one variational parameter to study the 1S state of heavy mesons. The general form of such trial wave function is written as

$$\psi_{trial}(r) = Ne^{-ar^b} \tag{6}$$

Where N is the normalization constant, a denotes the variational parameter which will be fixed by minimizing the expectation value of Hamiltonian and b is the model parameter which determines the type of the trial wave function. In practice, we select following four trial wave functions:

(1). $b=1$, namely $\psi_{trial}(r) = Ne^{-ar}$ (hydrogen wave function or exponential wave function). It is the solution of the Coulomb potential model.

(2). $b=2$, namely $\psi_{trial}(r) = Ne^{-ar^2}$ (harmonic oscillator wave function or a Gaussian wave function).

(3). $b=3/2$, namely $\psi_{trial}(r) = Ne^{-ar^{3/2}}$. This function was used by Gupta [7].

(4). $b=4/3$, namely $\psi_{trial}(r) = Ne^{-ar^{4/3}}$. This is a newly proposed trial wave function.

The normalization constant obtained from the normalization condition:

$$\int |\psi(r)|^2 d^3r = 1 \tag{7}$$

$$4\pi N^2 \int_0^\infty e^{-2ar^b} r^2 dr = 1 \rightarrow N = \left[\frac{b(2a)^{3/b}}{4\pi \Gamma\left(\frac{3}{b}\right)}\right]^{1/2} \tag{8}$$

And

$$\Psi(0) = N \tag{9}$$

$$|\Psi(0)|^2 = N^2 \quad \Rightarrow \quad |\Psi(0)|^2 = \frac{b(2a)^{\frac{3}{b}}}{4\pi \Gamma\left(\frac{3}{b}\right)} \tag{10}$$

In the case of Cornel potential(1), potential energy $\langle V(r)\rangle$ is

$$\langle V(r)\rangle = \int \psi^*(r) V(r) \psi(r) d^3r$$

$$= 4\pi N^2 \int_0^\infty r\, dr\, e^{-2ar^b} + 4\pi N^2 k \int_0^\infty e^{-2ar^b} r^3 dr$$

$$= 4\pi \frac{b(2a)^{3/b}}{4\pi \Gamma\left(\frac{3}{b}\right)}\left(-\frac{4}{3}\right)\alpha_s \frac{\Gamma\left(\frac{2}{b}\right)}{b(2a)^{2/b}} + 4\pi \frac{b(2a)^{3/b}}{4\pi \Gamma\left(\frac{3}{b}\right)} k \frac{\Gamma\left(\frac{4}{b}\right)}{b(2a)^{4/b}} \tag{11}$$

And kinetic energy $\langle T\rangle$ is

$$T = \frac{P^2}{2\mu} = -\frac{1}{2\mu}\left(\frac{2}{r}\frac{d}{dr} + \frac{d^2}{dr^2}\right) \tag{12}$$

$$\langle T\rangle = 4\pi \int_0^\infty r^2 dr \Psi^*(r)\, T\, \Psi(r)$$

$$= -\frac{1}{\mu} 4\pi\left(-N^2ab\right)\int_0^\infty r^b e^{-2ar^b} dr - \frac{1}{2\mu} 4\pi\left(-N^2ab\right)(b-1)\int r^b e^{-2ar^b} dr$$

$$-\frac{1}{2\mu} 4\pi\left(N^2a^2b^2\right)\int_0^\infty r^{2b} e^{-2ar^b} dr$$

$$= \frac{2b(b+1)(2a)^{\frac{2}{b}}\Gamma\left(\frac{1}{b}+1\right) - b^2(2a)^{\frac{2}{b}}\Gamma\left(2+\frac{1}{b}\right)}{8\mu\ \Gamma\left(\frac{3}{b}\right)} \tag{13}$$

We know that $\Gamma(1+z) = z\ \Gamma(z)$, so we can write

$$\langle T\rangle = \frac{(2a)^{\frac{2}{b}} b^2 \Gamma\left(2+\frac{1}{b}\right)}{8\mu\ \Gamma\left(\frac{3}{b}\right)} \tag{14}$$

Then we can obtain the expectation value of Hamiltonian and consequently an algebraic equation, which is used to determine a:

$$\langle H\rangle = \langle T\rangle + \langle V\rangle$$

$$= \frac{x^2 b^2 \Gamma\left(2+\frac{1}{b}\right)}{8\mu\ \Gamma\left(\frac{3}{b}\right)} + \frac{-4\alpha_s \Gamma\left(\frac{2}{b}\right) x^2 + 3K\ \Gamma\left(\frac{4}{b}\right)}{3\Gamma\left(\frac{3}{b}\right) x} \tag{15}$$

Where

$$x = (2a)^{\frac{1}{b}} \tag{16}$$

$$\frac{d}{dx}\langle H\rangle = 0 \tag{17}$$

$$\frac{3x^3 b^2 \Gamma\left(2+\frac{1}{b}\right) - 16\mu\alpha_s \Gamma\left(\frac{2}{b}\right) x^2 - 12\mu K\ \Gamma\left(\frac{4}{b}\right)}{12\mu\ \Gamma\left(\frac{3}{b}\right) x^2} = 0 \tag{18}$$

It is very easy to solve this equation, if we rewrite it in the following form:

$$A_3 x^3 + A_2 x^2 + A_o = 0 \tag{19}$$

The real solution of x can be expressed as:

$$x_{real} = -\frac{A_2}{3A_3} + \frac{2^{\frac{1}{3}} A_2^2}{3A_3 B} + \frac{B}{3A_3 2^{\frac{1}{3}}} \tag{20}$$

Where $B = \left(B_o + \sqrt{-4A_2^6 + B_o^2}\right)^{\frac{1}{3}}$ and $B_o = -2A_2^3 - 27A_o A_3^2$.

In the case of Martin potential (2), the potential energy reads:

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$$\langle V(r) \rangle = 4\pi \int_0^\infty r^2 dr \Psi^*(r) V r(\Psi) r = (4)\pi N^2 K \int_0^\infty e^{-2ar^b} r^{2.1} dr$$

$$= 4\pi \frac{b(2a)^{\frac{3}{b}}}{4\pi \ \Gamma\left(\frac{3}{b}\right)} K \frac{\Gamma\left(\frac{3.1}{b}\right)}{b(2a)^{\frac{3.1}{b}}} = \frac{K \ \Gamma\left(\frac{3.1}{b}\right)}{(2a)^{\frac{0.1}{b}} \Gamma\left(\frac{3}{b}\right)}$$

(21)

$$\langle H \rangle = \langle T \rangle + \langle V \rangle$$

$$= \frac{(2a)^{\frac{2}{b}} b^2 \Gamma\left(2 + \frac{1}{b}\right)}{8\mu \ \Gamma\left(\frac{3}{b}\right)} + \frac{K \ \Gamma\left(\frac{3.1}{b}\right)}{(2a)^{\frac{0.1}{b}} \Gamma\left(\frac{3}{b}\right)} \qquad (22)$$

Therefore, the equation for determining a is quite simple. The solution is

$$a = \frac{1}{2} \left[\frac{(0.4)\mu \ K \ \Gamma\left(\frac{3.1}{b}\right)}{b^2 \Gamma\left(2 + \frac{1}{b}\right)} \right]^{\frac{b}{2.1}} \qquad (23)$$

In the case of Logarithmic potential (3), by the similar procedure we obtain

$$a = \frac{1}{2} \left[\frac{4\mu \ K \ \Gamma\left(\frac{3}{b}\right)}{b^2 \Gamma\left(2 + \frac{1}{b}\right)} \right]^{\frac{b}{2}} \qquad (24)$$

IV. THE VARIATIONAL RESULTS OF (WFO) OF 1S- STSTE $c\bar{c}$, $b\bar{c}$ AND $b\bar{b}$ MESONS

In the case of cornell potential, the values of reduce mass for $c\bar{c}$, $b\bar{c}$ and $b\bar{b}$ are

$$\mu = \frac{m_c}{2} = 0.99 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad c\bar{c} \qquad (25)$$

$$\mu = \frac{m_b m_c}{m_b + m_c} = 1.35 \left(\frac{GeV}{c^2} \right) | \quad ; \quad \text{for} \quad b\bar{c} \qquad (26)$$

$$\mu = \frac{m_b}{2} = 2.59 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad b\bar{b} \qquad (27)$$

In the case of Martin potential, they are

$$\mu = \frac{m_c}{2} = 0.9 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad c\bar{c} \qquad (28)$$

$$\mu = \frac{m_b m_c}{m_b + m_c} = 1.33 \left(\frac{GeV}{c^2} \right) \quad ; \quad \text{for} \quad b\bar{c} \qquad (29)$$

$$\mu = \frac{m_b}{2} = 2.58 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad b\bar{b} \qquad (30)$$

And in the case of Logarithmic potential, we have

$$\mu = \frac{m_c}{2} = 0.75 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad c\bar{c} \qquad (31)$$

$$\mu = \frac{m_b m_c}{m_b + m_c} = 1.14 \left(\frac{GeV}{c^2} \right) \quad ; \quad \text{for} \quad b\bar{c} \qquad (32)$$

$$\mu = \frac{m_b}{2} = 2.45 \left(\frac{GeV}{c^2} \right) \quad ; \qquad \text{for} \quad b\bar{b} \qquad (33)$$

All the numerical results are listed in Tables 1 to 3.

TABLE I
THE VARIATIONAL RESULTS, $|\psi(0)|^2 \ (GeV)^3$, WITH A SINGLE-PARAMETER TRIAL WAVE FUNCTION FOR 1S-STATE OF $c\bar{c}$ MESON

b	Cornell Potential	Martin Potential	Logarithmic Potential
1	0.182246	0.166931	0.129747
2	0.050411	0.051150	0.039847
3/2	0.082912	0.079977	0.062490
4/3	0.103326	0.097511	0.076168

TABLE II
THE VARIATIONAL RESULTS, $|\psi(0)|^2 \ (GeV)^3$, WITH A SINGLE-PARAMETER TRIAL WAVE FUNCTION FOR 1S-STATE OF $b\bar{c}$ MESON

b	Cornell Potential	Martin Potential	Logarithmic Potential
1	0.364261	0.293480	0.246078
2	0.096576	0.089927	0.755745
3/2	0.161307	0.140607	0.118519
4/3	0.201642	0.171433	0.144460

TABLE III
THE VARIATIONAL RESULTS, $|\psi(0)|^2 \ (GeV)^3$, WITH A SINGLE-PARAMETER TRIAL WAVE FUNCTION FOR 1S-STATE OF $b\bar{b}$ MESON

b	Cornell Potential	Martin Potential	Logarithmic Potential
1	1.347961	0.188605	0.767454
2	0.345192	0.231169	0.208874
3/2	0.595784	0.381445	0.369630
4/3	0.749597	0.119027	0.450535

V. CONCLUSION

In this paper, we carefully studied the variational method, especially in determining the wave function at the origin of the quarkonium. Retaining generality as much as possible, we employ several potential models to analyze. We compared the

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numerical results obtained in terms of the variational method with those by solving Schrodinger equation numerically [2].The results shown in Tables 1 to 3 indicate that for the Cornell potential, the trial wave function with $b = \frac{4}{3}$, can give the least relative deviation of squared WFO. The value of the deviation is about 0.11 . For the Martin and Logarithmic potentials, the situations are better when $b = \frac{3}{2}$. One obtains the least values of 0.027 and 0.013 for the relative deviations of squared WFO.The accuracy of variational results can be improved when the number of the variational parameters are increased. The resultant accuracy of WFO seriously depends on the choice of the trial wave function.The trial wave function with a single variational parameter is most convenient for use. If the accuracy of 10% for WFO in the Cornell potential case is tolerable, $\psi(r) = Ne^{-ar^{4/3}}$ would be the best choice for the 1S state trial wave function. For the Martin and Logarithmic potentials, $\psi(r) = Ne^{-ar^{3/2}}$ is the most appropriate trial wave function for the 1S state, and the corresponding WFOs have quite satisfactory accuracies.

REFERENCES

[1] S. J. Collins, T. D. Imbo, B. A. King and E. C. Martell, Phys. Lett. B393, 155 (1997).
[2] E. J. Eichten and C. Quigg, Phys. Rev. D52, 1726 (1995).
[3] Y. B. Ding, X. Q. Li and P. N. Shen, arXiv: hep-ph/9808264V1 (1998).
[4] E. J. Eichten, K. Gottfried, T. Kinoshita, K. D. Lane and T. M. Yan, Phys. Rev. D21, 203 (1980).
[5] A. Martin, Phys. Lett. B93, 338 (1980).
[6] C. Quigg and J. L. Rosner,, Phys. Lett. B71, 153 (1977).
[7] S. N. Gupta, S. F. Radford and W. W. Repko, Phys. Rev. D26, 3305 (1982).
[8] N. Zettili, *Quantum Mechanics: Concepts and Applications,* Copyright©2001 by John Wiley and Sons, Ltd, England.