θ-Euclidean k-Fuzzy Ideals of Semirings

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Abstract— In this paper, we introduce the notion θ-Euclidean k-fuzzy ideal in semirings and to study the properties of the image and pre image of a θ-Euclidean k-fuzzy ideal in a semirings under epimorphism.

Keywords—semiring, fuzzy ideal, k-fuzzy ideal, θ-Euclidean L-fuzzy ideal, θ-Euclidean fuzzy k-ideal, θ-Euclidean k-fuzzy ideal.

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I. INTRODUCTION


Ayten Koç, Erol Balkanay [7, 8] introduced a concept of θ-Euclidean L-fuzzy ideals, θ-Euclidean left subset in rings and studied the properties of ideals θ-Euclidean L-fuzzy ideals, θ-Euclidean left subset in rings. C.B Kim et al [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a k-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k-ideals in k-semirings.

The purpose of this paper is to introduce θ-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a θ-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a θ-Euclidean k-fuzzy ideal.

II. PRELIMINARIES

An algebra (S,+,·) is said to be a semiring if (S,+) and (S,·) are semigroup satisfying a(b+c)=ab+ac and (b+c)a=ba+ca, for all a, b, c ∈ S. A semiring S may have an identity 1, defined by 1.a=a=1.a and a zero 0, defined by 0+a=a=0+a and a.0=0=0.a for all a ∈ S. A non-empty subset I of S is said to be left (resp., right) ideal if x,y ∈ I and r ∈ S imply that x + y ∈ I and rx ∈ I (resp.,xr ∈ I). If I is both left and right ideal of S, we say I is a two-sided ideal, or simply ideal, of S. A left ideal I of a semiring S is said to be a left k-ideal if a ∈ I and x ∈ S and if a + x ∈ I or x + a ∈ I then x ∈ I. Right k-ideal is defined dually, and two-sided k-ideal or simply a k-ideal is both a left and a right k-ideal.

Definition 2.1 [10]: Let K and S be any sets and let f : K → S be a function. A fuzzy subset μ of K is called f-invariant if f(x) = f(y) implies μ(x) = μ(y), where x,y ∈ K.

Definition 2.2 [2]: A fuzzy subset μ of a semiring S is said to be fuzzy left (resp., right) ideal of S if

(1) μ(x+y) ≥ min {μ(x),μ(y)} and
(2) μ(xy) ≥ μ(y) (resp., μ(xy) ≥ μ(x))

for all x,y ∈ S. If μ is a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S.

Definition 2.3 [10]: A fuzzy ideal μ of a semiring S is said to be a k-fuzzy ideal of S if μ(x+y) = μ(0) and μ(y) = μ(0) imply μ(x) = μ(0), for all x,y ∈ S.

Definition 2.4 [8]: Let θ : S → [0,1] and μ : S → [0,1] be a fuzzy subsets of S. For any, 0 ≠ y ∈ S the set

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Let $S$ be a semiring and let $\theta : S \to [0,1]$ be a non-constant fuzzy subset of $S$. A fuzzy ideal $\mu : S \to [0,1]$ is called a $\theta$-Euclidean $k$-fuzzy ideal if $\mu$ satisfies the following axioms:

(i) $\mu(x + y) = \mu(0)$ and $\mu(y) = \mu(0)$ imply $\mu(x) = \mu(0)$ for all $x, y$ in $R$.

(ii) For any $x, y \in R$ with $y \neq 0$, there exists elements $q, r \in R$ such that $x = yq + r$ where either $r = 0$ or else $\max \{\mu(r), \theta(r)\} \geq \max \{\mu(y), \theta(y)\}$.

Example 3.2: Let $S$ be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy subset defined by

$$\mu(a) = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even}, \\ 1 & \text{if } a \text{ is odd}. \end{cases}$$

Let $\theta : S \to [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a = 3, 5, 7, ..., \\ 1 & \text{otherwise}. \end{cases}$$

Clearly $\mu$ is a $k$-fuzzy ideal of $S$, also $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S$.

Example 3.3: Let $S$ be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy set defined by

$$\mu(a) = \begin{cases} 0 & \text{if } a = 0, \\ \frac{1}{3} & \text{if } a \text{ is non-zero even}, \\ 1 & \text{if } a \text{ is odd}. \end{cases}$$

Let $\theta : S \to [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & \text{if } a = 0, \\ 1 & \text{otherwise}. \end{cases}$$

So $\mu$ is a $k$-fuzzy ideal but $\mu$ is not a $\theta_1$-Euclidean $k$-fuzzy ideal of $S$.

Theorem 3.4: Let $A$ be a non-empty subset of $S$. Let $\mu$ be a fuzzy subset of a semiring $S$ such that $\mu$ is into $\{0, 1\}$ so that $\mu$ is the characteristic function of $A$. Then $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of a semiring $S$ then $A$ is a left ideal of $S$.

Proof: The proof is easy and straightforward. □

Theorem 3.5: Let $\mu$ be a $\theta$-Euclidean $k$-fuzzy ideal of a semiring $S$. Then for $0 \neq y \in S$, (i) $\theta_{\mu_y}$ is an ideal of $S$ and (ii) $\mu_y$ is an ideal of $S$ and (iii) $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of $S$, for $t \in [0,1]$.

Proof: The proof is similar to [8, Theorem 3.3]. □

Theorem 3.6: Let $\mu$ be a fuzzy ideal of a semiring $S$. If $\mu_{\theta_y}$ and $\theta_{\mu_y}$ is the Euclidean level set of $\mu$ and $\theta$ respectively. Then $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of a semiring $S$.

Proof: Suppose $\mu$ is fuzzy ideal of semiring $S$. For $x, y \in S$, if $\mu(x + y) = \mu(0)$ and $\mu(y) = \mu(0)$, then $\mu(x + y) = \min \{\mu(x), \mu(y)\}$, since $\mu$ is fuzzy ideal of $S$. $\mu(0) \geq \min \{\mu(x), \mu(0)\}$. Thus $\mu$ is a $k$-fuzzy ideal of semiring $S$.

We have $\mu_{\theta_y}$ and $\theta_{\mu_y}$ is the Euclidean level set of $\mu$ and $\theta$ respectively. Then, for $x, y \in S$, with $0 \neq y$, there exists $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \max \{\mu(y), \theta(y)\}$ and $\theta(r) \geq \max \{\mu(y), \theta(y)\}$. Thus $\max \{\mu(r), \theta(r)\} \geq \max \{\mu(y), \theta(y)\}$. Hence $\mu$ is a $\theta$-Euclidean $k$-fuzzy ideal of a semiring $S$. □
is a $f^{-1}(\theta)$-Euclidean k-fuzzy ideal of fuzzy ideal of $S$.

**Proof:** Suppose $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$.

(i) For all $x, y \in S'$

\[ f^{-1}\mu(x + y) = \mu(f(x + y)) = \mu(f(x) + f(y)) \]

\[ \geq \min \{ \mu(f(x)), \mu(f(y)) \} \]

\[ = \min \{ f^{-1}\mu(x), f^{-1}\mu(y) \} \]

(ii) For all $x, y \in S'$

\[ f^{-1}\mu(xy) = \mu(f(xy)) = \mu(f(x)f(y)) \]

\[ \geq \max \{ \mu(f(x)), \mu(f(y)) \} \]

\[ = \max \{ f^{-1}\mu(x), f^{-1}\mu(y) \} \]

(iii) For all $x, y \in S'$, if $f^{-1}\mu(x + y) = f^{-1}\mu(0)$

and $f^{-1}\mu(y) = f^{-1}\mu(0)$ then

\[ f^{-1}\mu(x) = \mu(f(x)) = \mu(x) = \mu(0) \]

\[ = f^{-1}\mu(0) \]

(iv) We have $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$, then for any $x, y \in S$, then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where either $f(r) = 0$ or else

\[ \max \{ \mu(f(y)), \theta(f(y)) \} \geq \max \{ \mu(f(r)), \theta(f(r)) \} . \]

That is $f(x) = f(yq + r)$ where either $f(r) = 0$ or else

\[ \max \{ f^{-1}\mu(y), f^{-1}\theta(y) \} \geq \max \{ f^{-1}\mu(r), f^{-1}\theta(r) \} . \]

Thus

\[ f(x) = f(yq + r) \]

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else

\[ \max \{ f^{-1}\mu(y), f^{-1}\theta(y) \} \geq \max \{ f^{-1}\mu(r), f^{-1}\theta(r) \} . \]

Conversely, suppose $f^{-1}(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of $S$.

(i) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

\[ \mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}\mu(xy) \]

\[ \geq \min \{ f^{-1}\mu(x), f^{-1}\mu(y) \} \]

\[ = \min \{ \mu(f(x)), \mu(f(y)) \} \]

\[ = \max \{ \mu(a), \mu(b) \} . \]

(ii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$.

\[ \mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}\mu(xy) \]

\[ \geq \max \{ f^{-1}\mu(x), f^{-1}\mu(y) \} \]

\[ = \max \{ \mu(f(x)), \mu(f(y)) \} \]

\[ = \max \{ \mu(a), \mu(b) \} . \]

(iii) For any $x, y \in S$ then $a = f(x), b = f(y) \in S'$, if $\mu(ab) = \mu(0)$ and $\mu(b) = \mu(0)$ imply

\[ \mu(a) = \mu(f(x)) = f^{-1}\mu(0) = f^{-1}\mu(0) = \mu(0) \]

(iv) For any $x, y, q, r \in S$ then

\[ a = f(x) \]

\[ b = f(y), c = f(q), d = f(r) \in S' . \]

We have $f^{-1}(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of fuzzy ideal of $S$, then there exists $q, r \in S$ such that $x = yq + r$ either $r = 0$ or else

\[ \max \{ f^{-1}\mu(y), f^{-1}\theta(y) \} \geq \max \{ f^{-1}\mu(r), f^{-1}\theta(r) \} . \]

That is $f(x) = f(yq + r)$ either $f(r) = 0$ or else

\[ \max \{ f^{-1}\mu(y), \theta(f(y)) \} \geq \max \{ f^{-1}\mu(r), \theta(f(r)) \} . \]

\[ \mu(ab) = \mu(f(x)f(y)) = \mu(f(xy)) = f^{-1}\mu(xy) \]

\[ \geq \max \{ f^{-1}\mu(x), f^{-1}\mu(y) \} \]

\[ = \max \{ \mu(f(x)), \mu(f(y)) \} \]

\[ = \max \{ \mu(a), \mu(b) \} . \]

**Definition 3.8:** Let $f : S \rightarrow S'$ be an homomorphism of the semirings. Let $\mu$ be a fuzzy subset of $S$, we define a fuzzy subset $f(\mu)$ of $S'$ by

\[ f(\mu)(y) = \left\{ \begin{array}{ll} \sup \{ \mu(t) \mid t \in R, f(t) = y \} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{array} \right. \]

**Theorem 3.9:** Let $f : S \rightarrow S'$ epimorphism of semirings. Let $\mu$ be a $f$-invariant $\theta$-Euclidean k-fuzzy ideal of $S$. Then $f(\mu)$ is a $f(\theta)$-Euclidean k-fuzzy ideal of $S'$.

**Proof:** Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$. Then $x + y = f(a) + f(b) = f(a + b)$ and $xy = f(a)f(b) = f(ab)$. Since $\mu$ is $f$-invariant

Thus

\[ f(\mu)(x + y) = f(\mu)f(a + b) \]

\[ = \sup \{ \mu(t) \mid t \in S, f(t) = f(a + b) \} \]

\[ = \sup \{ \mu(t) \mid t \in S, f(t) = f(a + b) \} . \]
$k$-fuzzy ideal of $S$. Here, we mean that $(\mu' \circ f)(x) = \mu'[f(x)]$.

**Proof.** Let $\mu = \mu' \circ f, \theta = \theta' \circ f$ and also $a, b \in S$ and $\mu'$ is an $\theta'$-Euclidean $k$-fuzzy ideal of $S'$.

It was proved that $\mu$ is a fuzzy ideal of $S$ [5] and $\mu$ is a $\theta'$-Euclidean fuzzy ideal of $S$ [7]. If $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$, then

$$\mu(a) = \mu' \circ f(a) = \mu'(f(a)) = \mu'(0).$$

Since $\mu'$ is an $\theta'$-Euclidean $k$-fuzzy ideal of $S'$.

$$\mu'(f(0)) = \mu'(0) = \mu(0).$$

Hence $\mu' \circ f : S \to [0,1]$ is a $(\theta' \circ f)$-Euclidean $k$-fuzzy ideal of $S$. $\square$

**REFERENCES**


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