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θ-Euclidean k-Fuzzy Ideals of Semirings

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***Abstract*—** In this paper, we introduce the notion θ-Euclidean k–fuzzy ideal in semirings and to study the properties of the image and pre image of a θ-Euclidean k–fuzzy ideal in a semirings under epimorphism.

***Keywords*—**semiring, fuzzy ideal, k–fuzzy ideal, θ-Euclidean L-fuzzy ideal, θ-Euclidean fuzzy k–ideal, θ-Euclidean k-fuzzy ideal.

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I. INTRODUCTION

L.A. Zadeh [1] introduced the notion of a fuzzy subset μ of a set X as a function from X into the closed unit interval $[0,1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld [2].W.J. Liu [3] introduced and studied fuzzy ideals of rings. T.K. Dutta and B.K. Biswas [4] studied fuzzy ideals, fuzzy prime ideals of semirings and they defined fuzzy k–ideal and fuzzy prime k–ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k–ideals. S.I. Baik and H.S Kim [6] studied more about the fuzzy k-ideals in semirings and investigated their properties. Y.B. Jun *et.al* [5] extended the concept of L–fuzzy ideal of rings to semirings. Ayten Koç, Erol Balkanay [7, 8] introduced a concept of θ-Euclidean L–fuzzy ideals, θ-Euclidean level subset in rings and studied the properties of ideals θ-Euclidean L–fuzzy ideals, θ-Euclidean level subset in rings. C.B Kim *et al* [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a k-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k–ideals in k–semirings.

The purpose of this paper is to introduce θ-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a θ-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a θ-Euclidean k– fuzzy ideal.

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II. PRELIMINARIES

An algebra $(S;+,.)$ is said to be a semiring if $(S;+)$ and $(S;.)$ are semigroup satisfying $a.(b+c) = a.b + a.c$ and $(b+c).a = b.a + c.a$, for all $a,b,c \in S$ .A semiring S may have an identity 1, defined by $1.a = a = a.1$ and a zero 0, defined by $0 + a = a = a + 0$ and $a.0 = 0 = 0.a$ for all $a \in S$. A non – empty subset I of S is said to be left (*resp*., right) ideal if $x, y \in I$ and $r \in S$ imply that $x + y \in I$ and $rx \in I$ $(resp., xr \in I)$. If I is both left and right ideal of *S*, we say *I* is a two-sided ideal, or simply ideal, of *S*. A left ideal I of a semiring *S* is said to be a left k–ideal if $a \in I$ and $x \in S$ and if $a + x \in I$ or $x + a \in I$ then $x \in I$. Right k–ideal is defined dually, and two–sided k–ideal or simply a k–ideal is both a left and a right k–ideal.

Definition 2.1 [10]: *Let K and S be any sets and let* $f : K \to S$ *be a function. A fuzzy subset* μ *of K is called* f *–invariant if* $f(x) = f(y)$ *implies* $\mu(x) = \mu(y)$*,where* $x, y \in K$.

Definition 2.2 [2] : *A fuzzy subset* μ *of a semiring S is said to be fuzzy left (resp., right)*
ideal of S if

(i) $\mu(x+y) \geq \min\{\mu(x),\mu(y)\}$ *and*

$(ii)\mu(xy) \geq \mu(y)$ $(resp., \mu(xy) \geq \mu(x))$

for all $x, y \in S$ *. If* μ *is a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S.*

Definition 2.3 [10]: *A fuzzy ideal* μ *of a semiring S is said to be a k-fuzzy ideal of S* $\mu(x+y) = \mu(0)$ *and* $\mu(y) = \mu(0)$ *imply* $\mu(x) = \mu(0)$*, for all* $x, y \in S$*.*

Definition 2.4 [8]: *Let* $\theta : S \to [0,1]$ *and* $\mu : S \to [0,1]$ *be a fuzzy subsets of S. For any,* $0 \neq y \in S$ *the set*

$$\mu_{\theta_y} = \left\{ \begin{array}{l} x \in S \mid \text{there exists } q, r \in S \text{ such that } x = yq + r \\ \text{where either } r = 0 \text{ or else } \mu(r) \geq \max\{\mu(y), \theta(y)\} \end{array} \right\}$$

is called a $\theta-$*Euclidean level subset of* μ .

II. θ-EUCLIDEAN K-FUZZY IDEALS

Definition 3.1: *Let S be a semiring and let* $\theta : S \to [0,1]$ *be a non–constant fuzzy subset of S. A fuzzy ideal* $\mu : S \to [0,1]$ *is called a* θ*-Euclidean k-fuzzy ideal if* μ *satisfies the following axioms*

(i) $\mu(x+y) = \mu(0)$ *and* $\mu(y) = \mu(0)$ *imply* $\mu(x) = \mu(0)$, *for all* x, y *in R.*

(ii) *For any* $x, y \in R$ *with* $y \neq 0$, *there exists elements* $q, r \in R$ *such that* $x = yq + r$, *where either* $r = 0$ *or else* $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$.

Example 3.2: Let *S* be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy subset defined by

$$\mu(a) = \begin{cases} 1 & \text{if} \quad a = 0, \\ \frac{1}{3} & \text{if} \quad a \text{ is non}-\text{zero even}, \\ 0 & \text{if} \quad a \text{ is odd}. \end{cases}$$

Let $\theta : S \to [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & \text{if} \quad a = 0, \\ \frac{1}{3} & \text{if} \quad a = 3, 5, 7, ... \\ \frac{1}{|a|} & \quad \text{otherwise}. \end{cases}$$

Clearly μ is a k–fuzzy ideal of *S,* also μ is a θ-Euclidean k–fuzzy ideal of *S.*

Example 3.3: Let S be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy set defined by

$$\mu(a) = \begin{cases} 1 & \text{if} \quad a = 0, \\ \frac{1}{3} & \text{if} \quad a \text{ is non}-\text{zero even}, \\ 0 & \text{if} \quad a \text{ is odd}. \end{cases}$$

Let $\theta_1 : S \to [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & \text{if} \quad a = 0 \\ \frac{1}{|a|} & \text{otherwise}. \end{cases}$$

So μ is a k–fuzzy ideal but μ is not a θ_1-Euclidean k–fuzzy ideal of *S.*

Theorem 3.4: *Let A be a non empty subset of S. Let* μ *be a fuzzy subset of a semiring S such that* μ *is into* $\{0,1\}$ *,so that* μ *is the characteristic function of A. Then* μ *is a* θ*-Euclidean k-fuzzy ideal of a semiring S then A is a left ideal of S.*

Proof: The proof is easy and straight forward. □

Theorem 3.5: *Let* μ *be a* θ*-Euclidean k-fuzzy ideal of a semiring S. Then for* $0 \neq y \in S$ *,* (i) μ_{θ_y} *is an ideal of S* $(ii)\theta_{\mu_y}$ *is an ideal of S. and* $(iii)\mu_t$ *is a* θ*-Euclidean k-fuzzy ideal of S, for* $t \in [0,1]$.

Proof: The proof is similar to [8, Theorem 3.3]. □

Theorem 3.6: *Let* μ *be a fuzzy ideal of a semiring S. If* μ_{θ_y} *and* θ_{μ_y} *is the Euclidean level set of* μ *and* θ *respectively. Then* μ *is a* θ*-Euclidean k-fuzzy ideal of a semiring S.*

Proof: Suppose μ is fuzzy ideal of semiring S. For $x, y \in S$,if $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$,then $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$,since μ is fuzzy ideal of *S*.

$$\mu(0) \geq \min\{\mu(x), \mu(0)\}$$
$$\mu(x) = \mu(0) .$$

Thus μ is a k-fuzzy ideal of semiring *S*.

We have μ_{θ_y} and θ_{μ_y} is the Euclidean level set of μ and θ respectively. Then, for $x, y \in S$, with $0 \neq y$,there exists $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \max\{\mu(y), \theta(y)\}$ and $\theta(r) \geq \max\{\mu(y), \theta(y)\}$.

Thus $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$.

Hence μ is a θ-Euclidean k-fuzzy ideal of a semiring *S*. □

Definition 3.7 ([10]**)**: Let $f : S \to S'$ *be a homomorphism of semirings. Let* μ *be a fuzzy subset of* S' *.We define a fuzzy subset* $f^{-1}\mu$ *of S by* $f^{-1}\mu(x) = \mu(f(x))$ *,for all* $x \in S$

Theorem 3.7: *Let* $f : S \to S'$ *be an epimorphism of semirings and* μ *be a fuzzy ideal of* S' *. Then* μ *is a* θ*-Euclidean k-fuzzy ideal of* S' *if and only if* $f^{-1}(\mu)$

is a $f^{-1}(\theta)$ -Euclidean k-fuzzy ideal of fuzzy ideal of S.

Proof: Suppose μ is a θ-Euclidean k-fuzzy ideal of S'.

(i) For all $x, y \in S'$

$$f^{-1}\mu(x+y) = \mu\left(f(x+y)\right) = \mu\left(f(x)+f(y)\right) \geq \min\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\} = \min\left\{f^{-1}\mu(x), f^{-1}\mu(y)\right\}$$

(ii) For all $x, y \in S'$

$$f^{-1}\mu(xy) = \mu\left(f(xy)\right) = \mu\left(f(x)f(y)\right) \geq \max\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\} = \max\left\{f^{-1}\mu(x), f^{-1}\mu(y)\right\}$$

(iii) For all $x, y \in S'$,if $f^{-1}\mu(x+y) = f^{-1}\mu(0)$ and $f^{-1}\mu(y) = f^{-1}\mu(0)$ than

$$f^{-1}\mu(x) = \mu\left(f(x)\right) = \mu\left(x\right) = \mu\left(0\right) = \mu\left(f(0)\right) = f^{-1}\mu(0).$$

iv) We have μ is a θ-Euclidean k-fuzzy ideal of S',then for any $x, y \in S$, then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

That is $f(x) = f(yq) + f(r)$ where either $f(r) = 0$ or else $\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}$.

Thus $f(x) = f(yq+r)$ where either $f(r) = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

Conversely, suppose $f^{-1}(\mu)$ is a θ-Euclidean k-fuzzy ideal of *S*.

(i) For any $x, y \in S$ then $a = f(x)$, $b = f(y) \in S'$.

$$\mu(a+b) = \mu\left(f(x)+f(y)\right) = \mu\left(f(x+y)\right) = f^{-1}\mu(x+y) \geq \min\left\{f^{-1}\mu(x), f^{-1}\mu(y))\right\} = \min\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\} = \max\left\{\mu\left(a\right), \mu\left(b\right)\right\}.$$

(ii) For any $x, y \in S$ then $a = f(x)$, $b = f(y) \in S'$.

$$\mu(ab) = \mu\left(f(x)f(y)\right) = \mu\left(f(xy)\right) = f^{-1}\mu(xy) \geq \max\left\{f^{-1}\mu(x), f^{-1}\mu(y))\right\} = \max\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\} = \max\left\{\mu\left(a\right), \mu\left(b\right)\right\}.$$

(iii) For any $x, y \in S$ then $a = f(x)$, $b = f(y) \in S'$,if $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$ imply

$$\mu(a) = \mu(f(x)) = f^{-1}\mu(x) = f^{-1}\mu(0) = \mu(f(0)) = \mu(0)$$

(iv) For any $x, y, q, r \in S$ then $a = f(x)$, $b = f(y), c = f(q), d = f(r) \in S'$.

We have $f^{-1}(\mu)$ is a θ-Euclidean k-fuzzy ideal of fuzzy ideal of *S*, then there exists $q, r \in S$ such that $x = yq + r$ either $r = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

That is $f(x) = f(yq+r)$ either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

that is $f(x) = f(y)f(q) + f(r)$ either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

Thus there exists $c, d \in S'$ such that $a = bc + d$ either $r = 0$ or else $\max\left\{\mu(c), \theta(c)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$. □

Definition 3.8: *Let $f: S \to S'$ be an homomorphism of the semirings. Let μ be a fuzzy subset of S.we define a fuzzy subset $f(\mu)$ of S' by*

$$f(\mu)(y) = \begin{cases} \sup\left\{\mu(t) \mid t \in R, f(t) = y\right\} & \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{if } f^{-1}(y) = \phi \end{cases}$$

Theorem 3.9: *Let $f: S \to S'$ epimorphism of semirings. Let μ be a f-invariant θ-Euclidean k-fuzzy ideal of S. Then $f(\mu)$ is $f(\theta)$ - Euclidean k-fuzzy ideal of S'.*

Proof: Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$.Then $x + y = f(a) + f(b) = f(a+b)$ and $xy = f(a)f(b) = f(ab)$.Since μ is *f*-invariant

Thus

(i) $f(\mu)(x+y) = f(\mu)f(a+b)$

$$= \sup\left\{\mu(t) \mid t \in S, f(t) = f(a+b)\right\}$$

$$= \sup\left\{\mu(t) \mid t \in S, \mu(t) = \mu(a+b)\right\}$$
$$= \mu(a+b)$$
$$\geq \min\left\{\mu(a), \mu(b)\right\},$$

since μ is a k-fuzzy ideal of S.

$$= \min\left\{\mu\left(f^{-1}(x)\right), \mu\left(f^{-1}(y)\right)\right\}$$
$$= \min\left\{f(\mu)(x), f(\mu)(y)\right\}.$$

(ii) $f(\mu)(xy) = f(\mu)f(ab) = \mu(ab)$

$$\geq \max\left\{\mu(a), \mu(b)\right\},$$

since μ is a k-fuzzy ideal of S.

$$= \max\left\{\mu\left(f^{-1}(x)\right), \mu\left(f^{-1}(y)\right)\right\}$$
$$= \max\left\{f(\mu)(x), f(\mu)(y)\right\}.$$

(iii) If $f(\mu)(x+y) = f(\mu)(0)$ and $f(\mu)(y) = f(\mu)(0)$ imply that

$$f(\mu)(x) = f(\mu)\left(f(a)\right) = \mu(a)$$
$$= \mu(0) = \mu\left(f^{-1}(0)\right) = f(\mu)(0).$$

(iv) We have μ is *f*-invariant θ-Euclidean k-fuzzy ideal of *S*. If $a, b, c, d \in S$ then $x = f(a)$, $y = f(b), q = f(c), r = f(d)$, for all $x, y, q, r \in S'$.

Then for any $a, b \in S$ there exists elements $c, d \in S$, such that $a = bc + d$, where either $d = 0$ or else $\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$.

That is, $f(a) = f(bc+d)$,

thus $f(a) = f(b)f(c) + f(d)$,

Thus $x = yq + r$. Let $d = 0$.

Then $f(d) = f(0) = 0$. We get $r = 0$.

Finally, we have

$$\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\},$$

Since μ is *f*-invariant.

$$f(\mu)(y) = f(\mu)f(b) = \sup\left\{\mu(t) \mid t \in R, f(t) = f(b)\right\}$$
$$= \sup\left\{\mu(t) \mid t \in R, \mu(t) = \mu(b)\right\}$$
$$= \mu(b)$$

so that $\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$ then

$$\max\left\{f(\mu)(y), f(\theta)(y)\right\} \geq \max\left\{f(\mu)(r), f(\theta)(r)\right\}.$$

Hence $f(\mu)$ is a $f(\theta)$ - Euclidean k-fuzzy ideal of S'.

Theorem 3.10: *Let $f : S \to S'$ be an isomorphism of the semirings and $\mu' : S' \to [0,1]$ be a θ-Euclidean k–fuzzy ideal of S'. Then $\mu' \circ f : S \to [0,1]$ is a $(\theta' \circ f)$-Euclidean k–fuzzy ideal of S. Here, we mean that $(\mu' \circ f)(x) = \mu'[f(x)]$.*

Proof: Let $\mu = \mu' \circ f, \theta = \theta' \circ f$ and also $a, b \in S$ and μ' is an θ-Euclidean k–fuzzy ideal of S'.

$$\begin{tikzcd} S \arrow[rr, "\mu"] \arrow[dr, "f"'] & & {[0,1]} \\ & S' \arrow[ur, "\mu'"'] & \end{tikzcd}$$

It was proved that μ is a fuzzy ideal of S [5] and μ is a θ-Euclidean fuzzy ideal of S [7].

If $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$, then

$\mu(a) = \mu' \circ f(a) = \mu'(f(a)) = \mu'(0)$. Since μ' is an θ-Euclidean k–fuzzy ideal of S'.

$$= \mu'(f(0))$$
$$= \mu' \circ f(0)$$
$$= \mu(0)$$

Hence $\mu' \circ f : S \to [0,1]$ is a $(\theta' \circ f)$-Euclidean k–fuzzy ideal of *S*. □

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