World Academy of Science, Engineering and Technology
International Journal of Mathematical and Computational Sciences
Vol:1, No:1, 2007

# $\theta$-Euclidean k-Fuzzy Ideals of Semirings

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***Abstract*—** In this paper, we introduce the notion $\theta$-Euclidean k–fuzzy ideal in semirings and to study the properties of the image and pre image of a $\theta$-Euclidean k–fuzzy ideal in a semirings under epimorphism.

***Keywords*—**semiring, fuzzy ideal, k–fuzzy ideal, $\theta$-Euclidean L-fuzzy ideal, $\theta$-Euclidean fuzzy k–ideal, $\theta$-Euclidean k-fuzzy ideal.

***2000 AMS Classification:*** 16Y60, 13E05, 03G25.



## I. INTRODUCTION

L.A. Zadeh [1] introduced the notion of a fuzzy subset $\mu$ of a set $X$ as a function from $X$ into the closed unit interval $[0,1]$. The concept of fuzzy subgroups was introduced by A. Rosenfeld [2].W.J. Liu [3] introduced and studied fuzzy ideals of rings. T.K. Dutta and B.K. Biswas [4] studied fuzzy ideals, fuzzy prime ideals of semirings and they defined fuzzy k–ideal and fuzzy prime k–ideals of semirings and characterized fuzzy prime k-ideals of semirings of non-negative integers and determined all its prime k–ideals. S.I. Baik and H.S Kim [6] studied more about the fuzzy k-ideals in semirings and investigated their properties. Y.B. Jun *et.al* [5] extended the concept of L–fuzzy ideal of rings to semirings. Ayten Koç, Erol Balkanay [7, 8] introduced a concept of $\theta$-Euclidean L–fuzzy ideals, $\theta$-Euclidean level subset in rings and studied the properties of ideals $\theta$-Euclidean L–fuzzy ideals, $\theta$-Euclidean level subset in rings. C.B Kim *et al* [10] introduce the k-fuzzy ideal of semirings and studied the properties of the image and pre image of a k-fuzzy ideal in semirings. C.B Kim [9] studied some isomorphism theorems and fuzzy k–ideals in k–semirings.

The purpose of this paper is to introduce $\theta$-Euclidean k-fuzzy ideals in semirings and to study the properties of the image and pre image of a $\theta$-Euclidean k-fuzzy ideal in a semiring under epimorphism. Also we prove the structural theorem for a $\theta$-Euclidean k– fuzzy ideal.

Manuscript received October 9, 2006. This work was supported in part by the Directorate General of Technical Education, Ministry of Man Power, Sultanate of Oman.

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## II. PRELIMINARIES

An algebra $(S;+,.)$ is said to be a semiring if $(S;+)$ and $(S;.)$ are semigroup satisfying $a.(b+c)=a.b+a.c$ and $(b+c).a=b.a+c.a$, for all $a,b,c\in S$ .A semiring $S$ may have an identity 1, defined by $1.a=a=a.1$ and a zero 0, defined by $0+a=a=a+0$ and $a.0=0=0.a$ for all $a\in S$ . A non – empty subset $I$ of $S$ is said to be left (*resp.*, right) ideal if $x,y\in I$ and $r\in S$ imply that $x+y\in I$ and $rx\in I$ $(resp., xr\in I)$. If I is both left and right ideal of *S*, we say *I* is a two-sided ideal, or simply ideal, of *S*. A left ideal $I$ of a semiring *S* is said to be a left k–ideal if $a\in I$ and $x\in S$ and if $a+x\in I$ or $x+a\in I$ then $x\in I$ . Right k–ideal is defined dually, and two–sided k–ideal or simply a k–ideal is both a left and a right k–ideal.

**Definition 2.1** [10]: *Let K and S be any sets and let* $f:K\to S$ *be a function. A fuzzy subset* $\mu$ *of K is called* $f$ *–invariant if* $f(x)=f(y)$ *implies* $\mu(x)=\mu(y)$*,where* $x,y\in K$ .

**Definition 2.2** [2] : *A fuzzy subset* $\mu$ *of a semiring S is said to be fuzzy left (resp., right)*
*ideal of S if*

$(i)$ $\mu(x+y)\geq \min\{\mu(x),\mu(y)\}$ *and*

$(ii)\,\mu(xy)\geq\mu(y)$ $(resp., \mu(xy)\geq\mu(x))$

*for all* $x,y\in S$ *. If* $\mu$ *is a fuzzy ideal of S if it is both fuzzy left and a fuzzy right ideal of S.*

**Definition 2.3** [10]: *A fuzzy ideal* $\mu$ *of a semiring S is said to be a k-fuzzy ideal of S* $\mu(x+y)=\mu(0)$ *and* $\mu(y)=\mu(0)$ *imply* $\mu(x)=\mu(0)$*, for all* $x,y\in S$.

**Definition 2.4** [8]: *Let* $\theta:S\to[0,1]$ *and* $\mu:S\to[0,1]$ *be a fuzzy subsets of S. For any,* $0\neq y\in S$ *the set*

$$\mu_{\theta_y} = \begin{Bmatrix} x \in S \mid there\ exists\ q, r \in S\ such\ that\ x = yq + r \\ where\ either\ r = 0\ or\ else\ \mu(r) \geq \max\{\mu(y), \theta(y)\} \end{Bmatrix}$$

*is called a $\theta -$ Euclidean level subset of $\mu$ .*

## II. $\theta$-EUCLIDEAN K-FUZZY IDEALS

**Definition 3.1**: *Let S be a semiring and let $\theta : S \to [0,1]$ be a non–constant fuzzy subset of S. A fuzzy ideal $\mu : S \to [0,1]$ is called a $\theta$-Euclidean k-fuzzy ideal if $\mu$ satisfies the following axioms*

$(i)$ $\mu(x+y) = \mu(0)$ *and* $\mu(y) = \mu(0)$ *imply* $\mu(x) = \mu(0)$, *for all* $x, y$ *in R.*

$(ii)$ *For any* $x, y \in R$ *with* $y \neq 0$, *there exists elements* $q, r \in R$ *such that* $x = yq + r$, *where either* $r = 0$ *or else* $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$.

**Example 3.2**: Let *S* be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy subset defined by

$$\mu(a) = \begin{cases} 1 & if \quad a = 0, \\ \frac{1}{3} & if \quad a\ is\ non-zero\ even, \\ 0 & if \quad a\ is\ odd. \end{cases}$$

Let $\theta : S \to [0,1]$ be a fuzzy subset defined by

$$\theta(a) = \begin{cases} 0 & if \quad a = 0, \\ \frac{1}{3} & if \quad a = 3, 5, 7, ... \\ \frac{1}{|a|} & \quad otherwise. \end{cases}$$

Clearly $\mu$ is a k–fuzzy ideal of *S,* also $\mu$ is a $\theta$-Euclidean k–fuzzy ideal of *S.*

**Example 3.3**: Let $S$ be the set of Natural Numbers including zero and $\mu : S \to [0,1]$ be a fuzzy set defined by

$$\mu(a) = \begin{cases} 1 & if \quad a = 0, \\ \frac{1}{3} & if \quad a\ is\ non-zero\ even, \\ 0 & if \quad a\ is\ odd. \end{cases}$$

Let $\theta_1 : S \to [0,1]$ be a fuzzy subset defined by

$$\theta_1(a) = \begin{cases} 0 & if \quad a = 0 \\ \frac{1}{|a|} & otherwise. \end{cases}$$

So $\mu$ is a k–fuzzy ideal but $\mu$ is not a $\theta_1$-Euclidean k–fuzzy ideal of *S*.

**Theorem 3.4**: *Let A be a non empty subset of S. Let $\mu$ be a fuzzy subset of a semiring S such that $\mu$ is into $\{0,1\}$ ,so that $\mu$ is the characteristic function of A. Then $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of a semiring S then A is a left ideal of S.*

**Proof**: The proof is easy and straight forward. □

**Theorem 3.5**: *Let $\mu$ be a $\theta$-Euclidean k-fuzzy ideal of a semiring S. Then for $0 \neq y \in S$ , $(i)$ $\mu_{\theta_y}$ is an ideal of S $(ii)\theta_{\mu_y}$ is an ideal of S. and $(iii)\mu_t$ is a $\theta$-Euclidean k-fuzzy ideal of S, for $t \in [0,1]$.*

**Proof**: The proof is similar to [8, Theorem 3.3]. □

**Theorem 3.6**: *Let $\mu$ be a fuzzy ideal of a semiring S. If $\mu_{\theta_y}$ and $\theta_{\mu_y}$ is the Euclidean level set of $\mu$ and $\theta$ respectively. Then $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of a semiring S.*

**Proof**: Suppose $\mu$ is fuzzy ideal of semiring S. For $x, y \in S$ ,if $\mu(x+y) = \mu(0)$ and $\mu(y) = \mu(0)$ ,then $\mu(x+y) \geq \min\{\mu(x), \mu(y)\}$ ,since $\mu$ is fuzzy ideal of *S*.

$$\mu(0) \geq \min\{\mu(x), \mu(0)\}$$
$$\mu(x) = \mu(0) .$$

Thus $\mu$ is a k-fuzzy ideal of semiring *S*.

We have $\mu_{\theta_y}$ and $\theta_{\mu_y}$ is the Euclidean level set of $\mu$ and $\theta$ respectively. Then, for $x, y \in S$ , with $0 \neq y$ ,there exists $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else $\mu(r) \geq \max\{\mu(y), \theta(y)\}$ and $\theta(r) \geq \max\{\mu(y), \theta(y)\}$ . Thus $\max\{\mu(r), \theta(r)\} \geq \max\{\mu(y), \theta(y)\}$.
Hence $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of a semiring *S*. □

**Definition 3.7 (**[10]): Let *$f : S \to S'$ be a homomorphism of semirings. Let $\mu$ be a fuzzy subset of $S'$ .We define a fuzzy subset $f^{-1}\mu$ of S by $f^{-1}\mu(x) = \mu(f(x))$ ,for all $x \in S$*

**Theorem 3.7**: *Let $f : S \to S'$ be an epimorphism of semirings and $\mu$ be a fuzzy ideal of $S'$ . Then $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$ if and only if $f^{-1}(\mu)$*

*is a* $f^{-1}(\theta)$ *-Euclidean k-fuzzy ideal of fuzzy ideal of S.*

**Proof**: Suppose $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$.

*(i)* For all $x, y \in S'$

$$f^{-1}\mu(x+y) = \mu\left(f(x+y)\right) = \mu\left(f(x)+f(y)\right)$$
$$\geq \min\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\}$$
$$= \min\left\{f^{-1}\mu(x), f^{-1}\mu(y)\right\}$$

*(ii)* For all $x, y \in S'$

$$f^{-1}\mu(xy) = \mu\left(f(xy)\right) = \mu\left(f(x)f(y)\right)$$
$$\geq \max\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\}$$
$$= \max\left\{f^{-1}\mu(x), f^{-1}\mu(y)\right\}$$

*(iii)* For all $x, y \in S'$,if $f^{-1}\mu(x+y) = f^{-1}\mu(0)$ and $f^{-1}\mu(y) = f^{-1}\mu(0)$ than

$$f^{-1}\mu(x) = \mu\left(f(x)\right) = \mu\left(x\right) = \mu\left(0\right)$$
$$= \mu\left(f(0)\right) = f^{-1}\mu(0).$$

*iv)* We have $\mu$ is a $\theta$-Euclidean k-fuzzy ideal of $S'$,then for any $x, y \in S$ , then $f(x), f(y) \in S'$ there exists elements $f(q), f(r) \in S'$ such that $f(x) = f(y)f(q) + f(r)$ where either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

That is $f(x) = f(yq) + f(r)$ where either $f(r) = 0$ or else $\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}$.

Thus $f(x) = f(yq+r)$ where either $f(r) = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

Hence for any $x, y \in S$ there exists elements $q, r \in S$ such that $x = yq + r$ where either $r = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

Conversely, suppose $f^{-1}(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of *S*.

*(i)* For any $x, y \in S$ then $a = f(x)$ , $b = f(y) \in S'$ .

$$\mu(a+b) = \mu\left(f(x)+f(y)\right) = \mu\left(f(x+y)\right)$$
$$= f^{-1}\mu(x+y)$$
$$\geq \min\left\{f^{-1}\mu(x), f^{-1}\mu(y))\right\}$$
$$= \min\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\}$$
$$= \max\left\{\mu\left(a\right), \mu\left(b\right)\right\}.$$

*(ii)* For any $x, y \in S$ then $a = f(x)$ , $b = f(y) \in S'$ .

$$\mu(ab) = \mu\left(f(x)f(y)\right) = \mu\left(f(xy)\right) = f^{-1}\mu(xy)$$
$$\geq \max\left\{f^{-1}\mu(x), f^{-1}\mu(y))\right\}$$
$$= \max\left\{\mu\left(f(x)\right), \mu\left(f(y)\right)\right\}$$
$$= \max\left\{\mu\left(a\right), \mu\left(b\right)\right\}.$$

*(iii)* For any $x, y \in S$ then $a = f(x)$ , $b = f(y) \in S'$ ,if $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$ imply

$$\mu(a) = \mu(f(x)) = f^{-1}\mu(x) = f^{-1}\mu(0) = \mu(f(0)) = \mu(0)$$

*(iv)* For any $x, y, q, r \in S$ then $a = f(x)$ , $b = f(y), c = f(q), d = f(r) \in S'$ .

We have $f^{-1}(\mu)$ is a $\theta$-Euclidean k-fuzzy ideal of fuzzy ideal of *S*, then there exists $q, r \in S$ such that $x = yq + r$ either $r = 0$ or else

$$\max\left\{f^{-1}\mu(y), f^{-1}\theta(y)\right\} \geq \max\left\{f^{-1}\mu(r), f^{-1}\theta(r)\right\}.$$

That is $f(x) = f(yq+r)$ either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

that is $f(x) = f(y)f(q) + f(r)$ either $f(r) = 0$ or else

$$\max\left\{\mu\left(f(y)\right), \theta\left(f(y)\right)\right\} \geq \max\left\{\mu\left(f(r)\right), \theta\left(f(r)\right)\right\}.$$

Thus there exists $c, d \in S'$ such that $a = bc + d$ either $r = 0$ or else $\max\left\{\mu(c), \theta(c)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$. □

**Definition 3.8**: *Let* $f : S \to S'$ *be an homomorphism of the semirings. Let* $\mu$ *be a fuzzy subset of* $S$ *.we define a fuzzy subset* $f(\mu)$ *of* $S'$ *by*

$$f(\mu)(y) = \begin{cases} \sup\left\{\mu(t) \mid t \in R, f(t) = y\right\} & if\ f^{-1}(y) \neq \phi \\ 0 & if\ f^{-1}(y) = \phi \end{cases}$$

**Theorem 3.9**: *Let* $f : S \to S'$ *epimorphism of semirings. Let* $\mu$ *be a f-invariant* $\theta$*-Euclidean k-fuzzy ideal of S. Then* $f(\mu)$ *is* $f(\theta)$ *- Euclidean k-fuzzy ideal of* $S'$ .

**Proof**: Suppose $x, y \in S'$ such that $x = f(a), y = f(b)$, for all $a, b \in S$ .Then $x + y = f(a) + f(b) = f(a+b)$ and $xy = f(a)f(b) = f(ab)$ .Since $\mu$ is *f*-invariant

Thus

*(i)* $f(\mu)(x+y) = f(\mu)f(a+b)$

$$= \sup\left\{\mu(t) \mid t \in S, f(t) = f(a+b)\right\}$$

$$= \sup\left\{\mu(t) \mid t \in S, \mu(t) = \mu(a+b)\right\}$$
$$= \mu(a+b)$$
$$\geq \min\left\{\mu(a), \mu(b)\right\},$$

since $\mu$ is a k-fuzzy ideal of S.

$$= \min\left\{\mu\left(f^{-1}(x)\right), \mu\left(f^{-1}(y)\right)\right\}$$
$$= \min\left\{f(\mu)(x), f(\mu)(y)\right\}.$$

*(ii)* $f(\mu)(xy) = f(\mu)f(ab) = \mu(ab)$

$$\geq \max\left\{\mu(a), \mu(b)\right\},$$

since $\mu$ is a k-fuzzy ideal of S.

$$= \max\left\{\mu\left(f^{-1}(x)\right), \mu\left(f^{-1}(y)\right)\right\}$$
$$= \max\left\{f(\mu)(x), f(\mu)(y)\right\}.$$

*(iii)* If $f(\mu)(x+y) = f(\mu)(0)$ and $f(\mu)(y) = f(\mu)(0)$ imply that

$$f(\mu)(x) = f(\mu)\left(f(a)\right) = \mu(a)$$
$$= \mu(0) = \mu\left(f^{-1}(0)\right) = f(\mu)(0).$$

*(iv)* We have $\mu$ is *f*-invariant $\theta$-Euclidean k-fuzzy ideal of *S*. If $a, b, c, d \in S$ then $x = f(a)$, $y = f(b), q = f(c), r = f(d)$ ,for all $x, y, q, r \in S'$.
Then for any $a, b \in S$ there exists elements $c, d \in S$, such that $a = bc + d$, where either $d = 0$ or else $\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$.
That is , $f(a) = f(bc+d)$,
thus $f(a) = f(b)f(c) + f(d)$,
Thus $x = yq + r$ .Let $d = 0$.
Then $f(d) = f(0) = 0$. We get $r = 0$.
Finally, we have
$\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$,

Since $\mu$ is *f*-invariant.

$$f(\mu)(y) = f(\mu)f(b) = \sup\left\{\mu(t) \mid t \in R, f(t) = f(b)\right\}$$
$$= \sup\left\{\mu(t) \mid t \in R, \mu(t) = \mu(b)\right\}$$
$$= \mu(b)$$

so that $\max\left\{\mu(b), \theta(b)\right\} \geq \max\left\{\mu(d), \theta(d)\right\}$ then
$\max\left\{f(\mu)(y), f(\theta)(y)\right\} \geq \max\left\{f(\mu)(r), f(\theta)(r)\right\}$.
Hence $f(\mu)$ is a $f(\theta)$ - Euclidean k-fuzzy ideal of $S'$.

**Theorem 3.10**: *Let $f : S \to S'$ be an isomorphism of the semirings and $\mu' : S' \to [0,1]$ be a $\theta$-Euclidean k–fuzzy ideal of $S'$. Then $\mu' \circ f : S \to [0,1]$ is a $(\theta' \circ f)$-Euclidean k–fuzzy ideal of S. Here, we mean that $(\mu' \circ f)(x) = \mu'[f(x)]$.*

**Proof**: Let $\mu = \mu' \circ f, \theta = \theta' \circ f$ and also $a, b \in S$ and $\mu'$ is an $\theta$-Euclidean k–fuzzy ideal of $S'$.

$$\begin{tikzcd} S \arrow[rr, "\mu"] \arrow[dr, "f"'] & & {[0,1]} \\ & S' \arrow[ur, "\mu'"'] & \end{tikzcd}$$

It was proved that $\mu$ is a fuzzy ideal of $S$ [5] and $\mu$ is a $\theta$-Euclidean fuzzy ideal of $S$ [7].
If $\mu(a+b) = \mu(0)$ and $\mu(b) = \mu(0)$, then
$\mu(a) = \mu' \circ f(a) = \mu'(f(a)) = \mu'(0)$. Since $\mu'$ is an $\theta$-Euclidean k–fuzzy ideal of $S'$.

$$= \mu'(f(0))$$
$$= \mu' \circ f(0)$$
$$= \mu(0)$$

Hence $\mu' \circ f : S \to [0,1]$ is a $(\theta' \circ f)$-Euclidean k–fuzzy ideal of *S*. □

## References

[1] L.A.Zadeh, Fuzzy Sets. Inform and Control **8** (1965) 338 – 353.
[2] A.Rosenfeld, Fuzzy groups J. Math. Anal. Appl **35** (1971) 512 -517.
[3] W.J. Liu , Fuzzy invariants subgroups and fuzzy ideals,Fuzzy sets and Systems. **8** (1987) 219 – 226.
[4] T.K. Dutta and B.K.Biswas, Fuzzy k – ideals of semirings, Bull. Calcutta Math. Soc.**87** (1995), 91-96.
[5] Y.B. Jun.J.Neggers and H.S.Kim, on L–fuzzy ideals in semirings I, Czech.Math.J **48** (1998) , 669 – 675.
[6] S.I.Baik and H.S Kim, on fuzzy k-ideals in semiring , Kongweon – kyungki . Math.Jour. **8** (2000) 147 – 154.
[7] Ayten Koç and Erol Balkanay, $\theta$-Euclidean L–fuzzy ideals of rings, Turk.J.Math. **26** (2002) 149 – 158.
[8] Ayten Koç and Erol Balkanay ,On $\theta$-Euclidean L–fuzzy ideals of rings, Turk.J.Math. **28** (2004) 137 – 142.
[9] C.B Kim, Isomorphism theorems and fuzzy k–ideals of k–semirings,Fuzzy Sets and Systems **112** (2000) 333–342.
[10] C.B Kim and M.A Park, k-Fuzzy ideals in semirings, Fuzzy Sets and Systems **81** (1996) 281–286.

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