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Application of the Hybrid Methods to Solving Volterra Integro-Differential Equations

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***Abstract*—**Beginning from the creator of integro-differential equations Volterra, many scientists have investigated these equations. Classic method for solving integro-differential equations is the quadratures method that is successfully applied up today. Unlike these methods, Makroglou applied hybrid methods that are modified and generalized in this paper and applied to the numerical solution of Volterra integro-differential equations. The way for defining the coefficients of the suggested method is also given.

***Keywords*—**Integro-differential equations, initial value problem, hybrid methods, predictor-corrector method

I. INTRODUCTION

VULTURE's work in investigation of some problems of elasticity theory ended with construction of an integro-differential equation. Volterra published his first paper devoted to the solution of integro-differential equations in 1909 (see [1], p.22). We have to face the solution of integro-differential equations while solving many problems of physics, geology, biology and etc. Volterra has applied the theory of integro-differential equations to some problems of biology as to life of some populations in appropriate media, spreading of grippe, and seasonal diseases, beast of prey and victim and etc. (see [1, p.23-31],[2, p.19]).

Consider the following integro-differential equation

$$y'(x) = f(x,y) + \int_{x_0}^{x} K(x,s,y(s))ds, \quad x_0 \le x \le X. \tag{1}$$

Let's admit that the continuous functions $f(x,y)$ and $K(x,s,y)$ are defined on their respective domain

$$G = \{x_0 \le x \le X,\ |y| \le a\}$$

and

$$\overline{G} = \{x_0 \le s \le x + \varepsilon \le X + \varepsilon,\ |y| \le a\}$$

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(ε tend to zero as $h \to 0$) satisfies a Lipchitz condition in the variable y, and the solution of the equation (1) at the point x_0 satisfies the following condition:

$$y(x_0) = y_0. \tag{2}$$

Suppose that the initial-value problem (1) - (2) has a unique solution defined on some segment $[x_0, X]$. The main goal of this report is to obtain a numerical solution of initial-value problem (1) - (2). Therefore the segment $[x_0, X]$ is divided into N equal parts by the positive and constant step size h. The mesh points are defined as $x_m = x_0 + mh \ \ (m = 0,1,2,...,N)$. Denote by the y_m, y'_m approximate and $y(x_n)$, $y'(x_n)$ exact values of the function $y(x)$ and it's derivative at the point x_m $(m = 0,1,2,..)$. To the numerical solution of problem (1)-(2), Makroglou applied the following hybrid method (see [3]):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\beta_\nu f_{n+\nu}, \tag{3}$$

that the scientists applied to the numerical solution of the following initial value problem (see [4,p.19], [5])

$$y' = f(x,y), \quad y(x_0) = y_0. \tag{4}$$

In [6], the hybrid method is constructed as follows:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i} + h\beta_\nu f_{n+k-\nu}. \tag{5}$$

Unlike the mentioned papers, here we suggest the following method:

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i f_{n+i+l_i}, \quad -1 < l_i < 1 \ (i = 0,1,2,...,k), \tag{6}$$

that generalizes the known hybrid methods. As a term, the hybrid method was used by Gear (see [7]) and constructed in the joint of the methods of Runge-Kutta and Adams. The idea of construction of such methods of Runge-Kutta and Adams as increase of orders, extension of stability domain, decreasing the amount of calculations and etc. consequently, is to construct in some since effective methods. As it is seen from what has been stated above, the scientists are engaged in investigation of hybrid methods for a long time.

However, construction of effective methods with improved properties is urgent up to now. Therefore, construction of high accuracy hybrid methods is topical. Method (6) may be applied to the numerical solution of equation (1) in different forms. For instance, at first method (6) may be applied to calculate the integral participating in the right hand side of equation (1), then the obtained differential equation may be solved by the known method, or by means of equation (1) to reduce to the system consisting of differential and integral equations (see[3]):

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$$\begin{cases} y' = f(x,y) + v(x), \\ v(x) = \int_0^x K(x,s,y(s))ds. \end{cases} \quad (7)$$

Many authors, while using system (7) for calculating the values of the integral, suggested the quadratures method that may be written in the following form

$$v_{n+j} = h\sum_{i=0}^{n+j} w_{n+j,i} K(x_{n+j}, x_i, y_i). \quad (8)$$

As it follows from formula (8), while passing from the current point to the next one, the volume of calculations increases, and the values of the function, that were determined at the previous points are not taken into account in this formula. Allowing for what has been said for calculating the values of the integral, here we suggest a method that in some sense is free from the indicated shortage of the quadrature methods.

II. Application of the Hybrid Method to the Solution of the System Composed of Differential and Integral Equations

At first consider calculation of the values of the integral. To this end, for determining v_{n+k} approximate value of the function $v(x)$ at the points x_{n+k} we behave as follows. It is obvious that

$$v(x_{n+k}) = \int_{x_0}^{x_{n+k}} k(x_{n+k}, s, y(s))ds. \quad (9)$$

Then we can write the following one:

$$v(x_{n+k}) - v(x_{n+k-1}) =$$
$$= h\int_{x_0}^{x_{n+k-1}} (K(x_{n+k}, s, y(s)) - K(x_{n+k-1}, s, y(s))ds +$$
$$+ \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds$$

or

$$v(x_{n+k}) - v(x_{n+k-1}) =$$
$$= h\int_{x_0}^{x_{n+k-1}} (K'_x(\xi_{n+k}, s, y(s))ds + \quad (10),$$
$$+ \int_{x_{n+k-1}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds,$$

where $x_{n+k-1} < \xi_{n+k} < x_{n+k}$.

Suppose that the solution of equation (1) is found by any method and after taking it into account in the second equation of system (7), we get an identity. Then we can write that

$$v'(x) = K(x, x, y(x)) + \int_{x_0}^{x} K'_x(x, s, y(s))ds. \quad (11)$$

Change the variable x by ξ_{n+k}. Then we have:

$$\int_{x_0}^{x_{n+k-1}} K'_x(\xi_{n+k}, s, y(s))ds =$$
$$= v'(\xi_{n+k}) - K(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k})) -$$
$$- \int_{x_{n+k-1}}^{\xi_{n+k}} K'(\xi_{n+k}, s, y(s))ds.$$

Thus, we get

$$v(x_{n+k}) - v(x_{n+k-1}) = hv'(\xi_{n+k}) -$$
$$- hK(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k})) +$$
$$+ \int_{x_{n+k-1}}^{\xi_{n+k}} K(x_{n+k-1}, s, y(s))ds + \quad (12)$$
$$+ \int_{\xi_{n+k}}^{x_{n+k}} K(x_{n+k}, s, y(s))ds.$$

Here, for determining the derivative $v'(\xi_{n+k})$ we use finite-difference schemes, for calculating $hK(\xi_{n+k}, \xi_{n+k}, y(\xi_{n+k}))$ we use Lagrange interpolation formulae. After applying the quadratures method to the integrals participating in (12), we can write (see [8]):

$$\sum_{i=0}^{k} \alpha_i v_{n+i} = h\sum_{i=0}^{k}\sum_{j=0}^{k} \beta_i^{(j)} K(x_{n+j}, x_{n+i}, y_{n+i}). \quad (13)$$

Here the coefficients $\alpha_i, \beta_i^{(j)}$ $(i, j = 0,1,...,k)$ are some real members, moreover $\alpha_k \neq 0$. If we suppose that the function $K(x,s,y)$ is independent of x and $K(x,s,y) \equiv F(s,y)$ we can write method (13) in the form:

$$\sum_{i=0}^{k} \alpha_i v_{n+i} = h\sum_{i=0}^{k} \beta_i F_{n+i}. \quad (14)$$

This is the known k - step method with constant coefficient. Consequently to the solving equations in system (7) we may apply method (6). Therefore, consider the investigation of method (6) and suppose that it's coefficient satisfies the following condition suggested in [9]:

A. The coefficients α_i, β_i $(i = 0,1,...,k)$ are some real numbers, moreover $\alpha_k \neq 0$.

B. Characteristically polynomials

$$\rho(\lambda) \equiv \sum_{i=0}^{k} \alpha_i \lambda^i, \quad \delta(\lambda) \equiv \sum_{i=0}^{k} \beta_i \lambda^i$$

have no common factors differ from a constant.

C. $p \geq 1$ and $\delta(1) \neq 0$.

Here p is the degree of method (14) and for sufficiently smooth function $y(x)$ that is determined from the following asymptotic equality:

$$\sum_{i=0}^{k} (\alpha_i y(x+ih) - h\beta_i y'(x+ih)) = O(h^{p+1}), \quad h \to 0. \quad (15)$$

As it was noted above, one of the main questions in investigation of the considered methods depends on the values of the method's coefficients. Therefore consider determination of the coefficients of method (6).

Taking into account the differential equation of problem (4) in method (6) we can write:

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$$\sum_{i=0}^{k}\alpha_i y_{n+i} = h\sum_{i=0}^{k}\beta_i y'_{n+\gamma_i} \quad (16)$$

where $\gamma_i = i + l_i \quad (i = 0,1,...,k)$.

Use the following expansions

$$y(x+ih) = y(x) + ihy'(x) + \frac{(ih)^2}{2!}y''(x) + ... + \frac{(ih)^p}{p!}y^{(p)}(x) + O(h^{p+1}),$$

$$y'(x+\gamma_i h) = y'(x) + \gamma_i h y''(x) + \frac{(\gamma_i h)^2}{2!}y''(x) + ... + \frac{(\gamma_i h)^{p-1}}{(p-1)!}y^{(p)}(x) + O(h^p),$$

(where $x = x_0 + nh$ is a fixed point) in relation (15) we get that the following conditions are necessary and sufficient conditions for asymptotic equality (15) be fulfilled:

$$\sum_{i=0}^{k}\alpha_i = 0, \ \sum_{i=0}^{k} i\alpha_i = \sum_{i=0}^{k}\beta_i,$$

$$\sum_{i=0}^{k}\frac{i^j}{j!}\alpha_i = \sum_{i=0}^{k}\frac{\gamma_i^{j-1}}{(j-1)!}\beta_i \quad (j = 2,3,...,p). \quad (17)$$

Thus, for determining the variables $\alpha_i, \beta_i, \gamma_i \quad (i = 0,1,2,...,k)$ we get a homogeneous system of nonlinear algebraic equations. If in system (17) we put $\gamma_i = i \quad (i = 0,1,...,k)$, then from it we get the known homogeneous system of linear algebraic equations investigated by many authors beginning with Dahlquist (see [9]).

In homogeneous system of equations, the amount of unknowns equals $3k+3$, but the amount of equations equal $p+1$ and that always has a zero solution. Since this solution doesn't suit, we investigate a non-trivial solution of system (17). It is clear that a non-trivial solution should be sought in the case when

$$p + 1 < 3k + 3.$$

Hence we get that

$$p \le 3k + 1.$$

At that time, for $\gamma_i = i \ (i = 0,1,2,...,k)$ it holds $p \le 2k$ (see [9]). Consequently, investigation of the solution of system (17) in this case $l_m \ne 0 \ (0 \le m \le k)$ has both theoretical and practical interest.

Consider the special case and assume $k = 1$. Then, for determining the unknowns $\alpha_i, \beta_i, \gamma_i \quad (i = 0,1,2,...,k)$ we get the following system of algebraic equations:

$$\begin{aligned} &\alpha_0 + \alpha_1 = 0, \\ &\beta_0 + \beta_1 = \alpha_1, \\ &l\beta_0 + \gamma\beta_1 = \frac{1}{2}\alpha_1, \\ &l^2\beta_0 + \gamma^2\beta_1 = \frac{1}{3}\alpha_1, \\ &l^3\beta_0 + \gamma^2\beta_1 = \frac{1}{4}\alpha_1 \end{aligned} \quad (18)$$

where $l = \gamma_0$ and $\gamma = \gamma_1$.

Without loss of generality, we can assume $\alpha_1 = 1$. Then for determining unknowns of β_0, β_1, l and γ we receive the no homogeneous system of nonlinear algebraic equations the last two of them are nonlinear. Consequently, if we assume $l_0 = l_1 = 0$, system (18) becomes a system of linear-algebraic equations.

For investigating system (18), consider a simpler case when the method is explicit. Usually a multistep method is called explicit if in its nonlinear part the quantity $f_{n+k} = f(x_{n+k}, y_{n+k})$ (here y_{n+k} is a desired quantity, in the given case y_{n+1}, since $k = 1$) doesn't participate. In this case from (18) we get $\beta_0 = 1, l = 1/2$, and the method has the following form:

$$y_{n+1} = y_n + hf_{n+1/2}. \quad (19)$$

That is the known method of midpoint, and has the degree $p = 2$, stable and explicit multistep methods which obtained from (14) on $k = 2$ and $h := 2h$. Notice that the concept to apply explicit method to method (6) with classic definition doesn't correspond to real processes since under explicit methods usually we understand recurrent relations.

For example, we consider method (19) for whose use the values of the desired function $y(x)$ at the point $x = x_n + h/2$ should be known. Thus, we get that it is not correct to refer method (19) to the set of explicit methods, but to refer it to the set of implicit methods is also not correct, since there desired function y_{n+1} doesn't participate in the nonlinear part of method (19). In this connection, the name of method (19) explicit is formal character in some sense. Now, consider the following method

$$y_{n+1} = y_n + h(\beta_0 f_{n+1/3} + \beta_1 f_{n+2/3}).$$

Usually, this method is called the implicit method. However, a term of the form f_{n+1} doesn't participate in its right hand side. But the method

$$y_{n+1} = y_n + h(3f_{n+1/3} + f_{n+1})/4. \quad (20)$$

obtained in [10] is implicit, A-stable and has degree $p = 3$.

From system (17) we get

$$\beta_0 = (\gamma/3 - 1/4)/l^2(\gamma - l),$$

$$\beta_1 = (1/4 - l/3)/\gamma^2(\gamma - l).$$

Hence it follows that $l \ne 0, \gamma \ne 0, l \ne \gamma$.

We can say that this condition is reduced to the system with another properties.

Allowing for the values of quantities β_0 and β_1, from system (17) we have:

$$4(l+\gamma)^2 - 4l\gamma - 3(l+\gamma) - 12l^2\gamma^2 = 0,$$

$$4(l+\gamma) = 6l\gamma + 3.$$

If we denote $t = l + \gamma$, then for determining the values of the quantity t we get

$$4t^2 - 7t - 3 = 0.$$

Hence $t_1 = 1, t_2 = 3/4$.

Notice that for $t = 3/4$ we get $l\gamma = 0$. This contradicts the natural conditions indicated above. For $t = 1$ we get $l_1 = (3+\sqrt{3})/6, \ l_2 = (3-\sqrt{3})/6$. Thus, we get that there exist two methods of type (6) possessing degree $p = 4$. However, these methods coincide and have the following form:

$$y_{n+1} = y_n + h(f(x_n + h(3+\sqrt{3})/6, y_{n+(3+\sqrt{3})/6}) + f(x_n + h(3-\sqrt{3})/6, y_{n+(3-\sqrt{3})/6}))/2. \quad (21)$$

Thus, we get that the method of type (6) with maximal degree, is unique for $k = 1$. Now, apply method (21) for calculating the integral of variable bounders. Then we have:

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$$v_{n+1} = v_n + h(K(x_n + hl, x_n + hl, y_{n+l}) + + K(x_n + \gamma h, x_n + \gamma h, y_{n+\gamma}) + + K(x_{n+h}, x_{n+lh}, y_{n+l}) + + K(x_{n+h}, x_n + \gamma h, y_{n+\gamma}))/4 \quad (22)$$

where $l = (3-\sqrt{3})/6$ and $l = (3+\sqrt{3})/6$.

Now, apply method (21) to solving the differential equation from system (7). Then we have:

$$y_{n+1} = y_n + h(f(x_n + lh, y_{n+l}) + + f(x_n + \gamma h, y_{n+\gamma}))/2 + h(v_{n+l} + v_{\gamma+h})/2 \quad (23)$$

Thus, for solving the system of equations (7) we get a system of methods composed from the methods (23) and (22) for whose use only the quantities y_n and v_n are assumed to be known. For finding the numerical solutions of equation (1) by the methods (22) and (23), the quantities y_{n+l} and $y_{n+\gamma}$ should be known. Since these methods have degree $p = 4$, the methods suitable for determining y_{n+l} and $y_{n+\gamma}$ should have the degree at least $p > 3$. Here, to this end we suggest method (20) that while applying to determination of the quantities y_{n+l} and $y_{n+\gamma}$ has the following form:

$$y_{n+l} = y_n + lh(3f(x_n + lh/3, y_{n+l/3}) + + f(x_n + lh, y_{n+l}))/4 + lh(3v_{n+l/3} + v_{n+l})/4, \quad (24)$$

$$y_{n+\gamma} = y_n + \gamma h(3f(x_n + \gamma h/3, y_{n+\gamma/3}) + + f(x_n + \gamma h, y_{n+\gamma}))/4 + \gamma h(3v_{n+\gamma/3} + v_{n+\gamma})/4. \quad (25)$$

The predictor-corrector type method composed of Euler method and method of trapezoid are suggested to use for calculating the quantities $y_{n+l/3}, y_{n+l}, y_{n+\gamma/3}$ and $y_{n+\gamma}$.

The Euler method for calculating y_{n+l} is written in the following form:

$$\breve{y}_{n+l} = y_n + hlf_n + hlv_n \quad (26)$$

The method of trapezoid for calculating y_{n+l} is written in the form:

$$\bar{y}_{n+l} = y_n + hl(f(x_n, y_n) + + f(x_n + hl, \breve{y}_{n+l}))/2 + hl(v_n + \breve{v}_{n+l})/2. \quad (27)$$

For calculating the accuracy of the quantity $\bar{y}_{n+l}$ calculated by the method of trapezoid, we use the method (20) suggested above. Then we have:

$$\hat{y}_{n+l} = y_n + hl(3f_{n+1/3} + + f(x_n + lh, \bar{y}_{n+l}))/4 + h(3v_{n+1/3} + \bar{v}_{n+l})/4. \quad (28)$$

As it is seen from the formula given above for calculating the values of the quantity y_{n+l}, the values of the quantity v_{n+l} also should be determined. To this end, we suggest to use the similar method. The predictor-corrector method with using the Euler and trapezoid methods in one variant has the following form:

$$\breve{v}_{n+l} = v_n + hl(K(x_{n+l}, x_n, y_n) + + K(x_n, x_n, y_n))/2, \quad (29)$$

$$\bar{v}_{n+l} = v_n + hl(K(x_{n+l}, x_n, y_n) + K(x_n, x_n, y_n) + 2K(x_{n+l}, x_{n+l}, \breve{y}_{n+l}))/4. \quad (30)$$

Now determine the quantity $\hat{v}_{n+l}$ by means of the following scheme:

$$\hat{v}_{n+l} = v_n + hl(3K(x_n + h/3, x_n + h/3, y_{n+1/3}) + K(x_n + lh, x_n + lh, \hat{y}_{n+l}))/4. \quad (31)$$

Thus we stated a scheme for determining the approximate value of the quantity y_{n+l}. Notice that here we suggest the methods (26)-(31) for determining $\hat{y}_{n+\gamma}$ after changing the parameter l by γ. Then assuming that the approximate values of quantities y_{n+l} and $y_{n+\gamma}$ are known, for determining y_{n+1} we can use the following method:

$$y_{n+1} = y_n + h(f(x_n + lh, \hat{y}_{n+l}) + + f(x_n + \gamma h, \hat{y}_{n+\gamma}))/2 + h(\hat{v}_{n+l} + \hat{v}_{n+\gamma})/2. \quad (32)$$

On the base of the schemes above we compose an algorithm for solving problem (1)-(2). While composing an algorithm the scheme from [11] was.

To approximate the solution of the initial-value problem

$$y' = f(x, y) + \int_{x_0}^{x} K(x, s, y(s)ds,$$

$$x_0 \le x \le X; \; y(x_0) = y_0$$

At $(N+1)$ equally spaced numbers in the interval $[x_0, X]$:

INPUT endpoints x_0, X; integer N; initial condition y_0.

OUTPUT approximation y_{n+1} to $y(x_{n+1})$ at the N-values of x.

Step1 Set $h = (x - x_0)/N;$

$x = x_0;$

$y(x) = y_0;$

OUTPUT (x,y).

Step2 For $n = 1,2,..., N$ do step3-11

Step3 Set $x_n = x_0 + nh$

Step4 Set $\breve{y}_{n+l}, \breve{v}_{n+l}$ (Compute by formulas (26) and (29))

Step5 Set $\bar{y}_{n+l}, \bar{v}_{n+l}$ (Compute by formulas (27) and (30))

Step6 Set $\hat{y}_{n+l}, \hat{v}_{n+l}$ (Compute by formulas (28) and (31))

Step7 Set $\breve{y}_{n+j}, \breve{v}_{n+j}$ (Compute by formulas (26) and (29) for $l := j$)

Step8 Set $\bar{y}_{n+j}, \bar{v}_{n+j}$ (Compute by formulas (27) and (30) for $l := j$)

Step9 Set $\hat{y}_{n+j}, \hat{v}_{n+j}$ (Compute by formulas (28) and (31) for $l := j$)

Step10 Set y_{n+1} (Compute by formula (32))

Step11 Output $(x_{n+1}; y_{n+1})$

Step12 STOP.

III. Conclusion

From theoretical points of view, it is easy to see that while solving an integro-differential equation, the integral may be changed by some integral sum in the equation not carrying it over

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to the system of equations. But in this case, theoretically we get nothing however we complicate application of the constructed method. Therefore here we suggest another way that was suggested above. Here for determining the solution of equations from system (7), we used the same method. However while solving system (7) to each equation of this system we can apply different methods with variants properties (see [12]). The advantage of the suggested method is that this method in an one-step method, has sufficiently high accuracy and provides constancy of the amount of calculations at each step.

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