Existence of Periodic Solutions in a Food Chain Model with Holling–type II Functional Response

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Abstract—In this paper, a food chain model with Holling type II functional response on time scales is investigated. By using the Mawhin's continuation theorem in coincidence degree theory, sufficient conditions for existence of periodic solutions are obtained.

Keywords—Periodic solutions; food chain model; coincidence degree; time scales.

I. INTRODUCTION

$\mathbf{I}$N recent years, dynamic equations on time scales have received a lot of attention, such as [1–5] and the references therein. The theory of time scales (measure chain) was first proposed by Stefan Hilger in his PhD thesis (see [1]). This theory unifies continuous and discrete analysis and the three main features of the calculus on time scales are unification, extension and discretization. To prove a result for a dynamic equation on a time scale is not only related to the set of real numbers or set of integers but also those pertaining to more general time scales. So it is unnecessary to explore the existence of periodic solutions of some continuous and discrete population models in separate ways.

We consider the following dynamics equations on time scales,

$$
\begin{cases}
u_1^\Delta(t) = a_1(t) - b_1(t)e^{u_1(t)} - \frac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1}, \\
u_2^\Delta(t) = -a_2(t) + \frac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1} - \frac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1}, \\
u_3^\Delta(t) = -a_3(t) + \frac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1},
\end{cases}
\tag{1}
$$

where $a_1(t), a_2(t), a_3(t), b_1(t), c_1(t), c_2(t), c_3(t)$ and $c_4(t)$ are rd-continuous positive ω-periodic functions on time scales $\mathbb{T}$. Set $y_i(t) = e^{u_i(t)}$, $i = 1, 2, 3$. If $\mathbb{T} = \mathbb{R}$, then system (1) is equivalent to the following food chain model with Holling type II functional response modeled by ordinary differential equations,

$$
\begin{cases}
\dot{y}_1(t) = y_1(t)\left(a_1(t) - b_1(t)y_1(t) - \frac{c_1(t)y_2(t)}{y_1(t)+1}\right), \\
\dot{y}_2(t) = y_2(t)\left(-a_2(t) + \frac{c_2(t)y_1(t)}{y_1(t)+1} - \frac{c_3(t)y_3(t)}{y_2(t)+1}\right), \\
\dot{y}_3(t) = y_3(t)\left(-a_3(t) + \frac{c_4(t)y_2(t)}{y_2(t)+1}\right),
\end{cases}
\tag{2}
$$

where all coefficients are positive ω-periodic functions. But on the other hand, if $\mathbb{T} = \mathbb{Z}$, then system (1) can be reduced

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to the following difference equations,

$$
\begin{cases}
y_1(n+1) = y_1(n)\exp\left(a_1(n) - b_1(n)y_1(n) - \frac{c_1(n)y_2(n)}{y_1(n)+1}\right), \\
y_2(n+1) = y_2(n)\exp\left(-a_2(n) + \frac{c_2(n)y_1(n)}{y_1(n)+1} - \frac{c_3(n)y_3(n)}{y_2(n)+1}\right), \\
y_3(n+1) = y_3(n)\exp\left(-a_3(n) + \frac{c_4(n)y_2(n)}{y_2(n)+1}\right),
\end{cases}
\tag{3}
$$

where all the coefficients are positive ω-periodic sequences and system (3) can be used to describe the discrete three–level food chain model. Existence of periodic solutions for system (3) was investigated in [6] with the help of continuation theorem. However, in the proof of the main theorem, I do not think the second inequality of (2.7) in [6] can be obtained, and it seems that the estimation of $x_3(\zeta_3)$ is contradictory to the assumptions in Theorem 2.1 .

The primary aim of this paper is to unify the existence of periodic solutions of food chain model governed by ordinary differential equations and the corresponding discrete models and to extend these results to more general time scales. The approach based on the coincidence degree theory has been widely applied to deal with the existence of periodic solutions of differential equations and difference equations but rarely applied to the dynamic equations on time scales[3–5].

The remainder of the paper is organized as follows. In the following section, some preliminary results about calculus on time scales and continuation theorem are stated. The existence of periodic solution for system (1) is established in Section 3.

II. PRELIMINARIES

For convenience, we shall first present some basic definitions and lemmas about time scales, more details can be found in [2, 7, 8]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Throughout this paper, we assume that the time scale $\mathbb{T}$ is unbounded above and below, such as $\mathbb{R}$, $\mathbb{Z}$ and $\bigcup_{k\in\mathbb{Z}}[2k, 2k+1]$. The following definitions and lemmas about time scales are from [2].

Definition 2.1. The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$, the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$, and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+ = [0, +\infty)$ are defined, respectively, by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \mu(t) = \sigma(t) - t$. If $\sigma(t) = t$, then t is called right-dense (otherwise: right-scattered), and if $\rho(t) = t$, then t is called left-dense (otherwise: left-scattered).

Definition 2.2. Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}$. Then we define $f^\Delta(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t))-f(s)-f^\Delta(t)(\sigma(t)-s)| \le \varepsilon|\sigma(t)-s| \quad \text{for all } s \in U.$$

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In this case, $f^\Delta(t)$ is called the delta (or Hilger) derivative of f at t. Moreover, f is said to be delta or Hilger differentiable on $\mathbb{T}$ if $f^\Delta(t)$ exists for all $t \in \mathbb{T}$. A function $F : \mathbb{T} \to \mathbb{R}$ is called an antiderivative of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ for all $t \in \mathbb{T}$. Then we define

$$\int_r^s f(t)\Delta t = F(s) - F(r) \quad \text{for } r, s \in \mathbb{T}.$$

Definition 2.3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist(finite) at left-dense points in $\mathbb{T}$. The set of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C_{rd}(\mathbb{T})$.

Lemma 2.4. Every rd-continuous function has an antiderivative.

Lemma 2.5. If $a, b \in \mathbb{T}$, α, $\beta \in \mathbb{R}$ and $f, g \in C_{rd}(\mathbb{T})$, then
(a) $\int_a^b [\alpha f(t) + \beta g(t)]\Delta t = \alpha \int_a^b f(t)\Delta t + \beta \int_a^b g(t)\Delta t$;
(b) if $f(t) \geq 0$ for all $a \leq t < b$, then $\int_a^b f(t)\Delta t \geq 0$;
(c) if $|f(t)| \leq g(t)$ on $[a, b) := \{t \in \mathbb{T} : a \leq t < b\}$, then $|\int_a^b f(t)\Delta t| \leq \int_a^b g(t)\Delta t$.

Lemma 2.6 ([7]). Let $t_1, t_2 \in I_\omega$ and $t \in \mathbb{T}$. If $g : \mathbb{T} \to \mathbb{R} \in C_{rd}(\mathbb{T})$ is $\omega-$periodic, then

$$g(t) \leq g(t_1) + \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s$$

and

$$g(t) \geq g(t_2) - \frac{1}{2}\int_k^{k+\omega} |g^\Delta(s)|\Delta s,$$

the constant factor $1/2$ is the best possible.

For simplicity, we use the following notations throughout this paper. Let $\mathbb{T}$ be ω-periodic, that is $t \in \mathbb{T}$ implies $t + \omega \in \mathbb{T}$,

$$k = \min\{\mathbb{R}^+ \cap \mathbb{T}\}, \quad I_\omega = [k, k + \omega] \cap \mathbb{T},$$

$$\bar{g} = \frac{1}{\omega}\int_{I_\omega} g(s)\Delta s = \frac{1}{\omega}\int_k^{k+\omega} g(s)\Delta s,$$

where $g \in C_{rd}(\mathbb{T})$ is an ω-periodic real function, i.e., $g(t + \omega) = g(t)$ for all $t \in \mathbb{T}$.

Next, we introduce some concepts and a useful result from [8].

Let X, Z be normed vector spaces, $L : \text{Dom}\, L \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \ker L = \text{codim}\,\text{Im}\, L < +\infty$ and $\text{Im}\, L$ is closed in Z. If L is a Fredholm mapping of index zero and there exist continuous projections $P : X \to X$ and $Q : Z \to Z$ such that $\text{Im}\, P = \ker L$, $\text{Im}\, L = \ker Q = \text{Im}(I - Q)$, then it follows that $L|\text{Dom}\, L \cap \ker P : (I - P)X \to \text{Im}\, L$ is invertible. We denote the inverse of that map by K_P. If Ω is an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I - Q)N : \bar{\Omega} \to X$ is compact. Since $\text{Im}\, Q$ is isomorphic to $\ker L$, there exists an isomorphism $J : \text{Im}\, Q \to \ker L$.

Lemma 2.7 (Continuation Theorem). Let L be a Fredholm mapping of index zero and N be L-compact on $\bar{\Omega}$. Suppose
(a) for each $\lambda \in (0, 1)$, every solution u of $Lu = \lambda Nu$ is such that $u \notin \partial\Omega$;

(b) $QNu \neq 0$ for each $u \in \partial\Omega \cap \ker L$ and the Brouwer degree $\deg\{JQN, \Omega \cap \ker L, 0\} \neq 0$.

Then the operator equation $Lu = Nu$ has at least one solution lying in $\text{Dom}\, L \cap \bar{\Omega}$.

III. Existence of Periodic Solutions

Theorem 3.1. If the following assumptions hold,
(a) $\bar{c}_4 > \bar{a}_3$,
(b) $\bar{c}_2 > \bar{a}_2$,
(c) $\bar{a}_1 > \frac{\bar{c}_1 \bar{a}_3}{\bar{c}_4 - \bar{a}_3} e^{2\bar{a}_2 \omega}$.

Then system (1) has at least one $\omega-$periodic solution.

Proof. Let $X = Z = \{(u_1, u_2, u_3)^T \in C(\mathbb{T}, \mathbb{R}^3) : u_i(t + \omega) = u_i(t), \ i = 1, 2, 3, \forall t \in \mathbb{T}\}, \|(u_1, u_2, u_3)^T\| = \sum_{i=1}^3 \max_{t \in I_\omega} |u_i(t)|, \quad (u_1, u_2, u_3)^T \in X$ (or in Z). Then X and Z are both Banach spaces when they are endowed with the above norm $\|\cdot\|$. Let

$$N\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} = \begin{bmatrix} a_1(t) - b_1(t)e^{u_1(t)} - \frac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1} \\ -a_2(t) + \frac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1} - \frac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1} \\ -a_3(t) + \frac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1} \end{bmatrix},$$

$$L\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} u_1^\Delta \\ u_2^\Delta \\ u_3^\Delta \end{bmatrix}, \quad P\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = Q\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_k^{k+\omega} u_1(t)\Delta t \\ \frac{1}{\omega}\int_k^{k+\omega} u_2(t)\Delta t \\ \frac{1}{\omega}\int_k^{k+\omega} u_3(t)\Delta t \end{bmatrix}.$$

Obviously, $\ker L = \{(u_1, u_2, u_3)^T \in X : (u_1(t), u_2(t), u_3(t))^T = (h_1, h_2, h_3)^T \in \mathbb{R}^3, t \in \mathbb{T}\}$, $\text{Im}\, L = \{(u_1, u_2, u_3)^T \in Z : \bar{u}_1 = \bar{u}_2 = \bar{u}_3 = 0, t \in \mathbb{T}\}$, $\dim \ker L = 3 = \text{codim}\,\text{Im}\, L$. Since $\text{Im}\, L$ is closed in Z, then L is a Fredholm mapping of index zero. It is easy to show that P and Q are continuous projections such that $\text{Im}\, P = \ker L$ and $\text{Im}\, L = \ker Q = \text{Im}(I - Q)$. Furthermore, the generalized inverse (of L) $K_P : \text{Im}\, L \to \ker P \cap \text{Dom}\, L$ exists and is given by

$$K_P\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \int_k^t u_1(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_1(s)\Delta s\Delta t \\ \int_k^t u_2(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_2(s)\Delta s\Delta t \\ \int_k^t u_3(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_3(s)\Delta s\Delta t \end{bmatrix}.$$

Thus

$$QN\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{\omega}\int_k^{k+\omega}\left(a_1(t) - b_1(t)e^{u_1(t)} - \frac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1}\right)\Delta t \\ \frac{1}{\omega}\int_k^{k+\omega}\left(-a_2(t) + \frac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1} - \frac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1}\right)\Delta t \\ \frac{1}{\omega}\int_k^{k+\omega}\left(-a_3(t) + \frac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1}\right)\Delta t \end{bmatrix},$$

and

$$K_P(I - Q)N\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$= \begin{bmatrix} \int_k^t u_1(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_1(s)\Delta s\Delta t \\ \quad -\left(t - k - \frac{1}{\omega}\int_k^{k+\omega}(t-k)\Delta t\right)\bar{u}_1 \\ \int_k^t u_2(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_2(s)\Delta s\Delta t \\ \quad -\left(t - k - \frac{1}{\omega}\int_k^{k+\omega}(t-k)\Delta t\right)\bar{u}_2 \\ \int_k^t u_3(s)\Delta s - \frac{1}{\omega}\int_k^{k+\omega}\int_k^t u_3(s)\Delta s\Delta t \\ \quad -\left(t - k - \frac{1}{\omega}\int_k^{k+\omega}(t-k)\Delta t\right)\bar{u}_3 \end{bmatrix}.$$

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Clearly, QN and $K_P(I - Q)N$ are continuous. According to Arzela–Ascoli theorem, it is not difficulty to show that $K_P(I - Q)N(\bar{\Omega})$ is compact for any open bounded set $\Omega \subset X$ and $QN(\bar{\Omega})$ is bounded. Thus, N is L-compact on $\bar{\Omega}$.

Now, we shall search an appropriate open bounded subset Ω for the application of the continuation theorem, Lemma 2.7. For the operator equation $Lu = \lambda Nu$, where $\lambda \in (0,1)$, we have

$$
\begin{cases}
u_1^\Delta(t) = \lambda \left(a_1(t) - b_1(t)e^{u_1(t)} - \dfrac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1} \right), \\
u_2^\Delta(t) = \lambda \left(-a_2(t) + \dfrac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1} - \dfrac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1} \right), \qquad (4) \\
u_3^\Delta(t) = \lambda \left(-a_3(t) + \dfrac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1} \right).
\end{cases}
$$

Assume that $(u_1, u_2, u_3)^T \in X$ is a solution of system (4) for a certain $\lambda \in (0,1)$. Integrating (4) on both sides from k to $k+\omega$, we obtain

$$
\begin{cases}
\bar{a}_1\omega = \int_k^{k+\omega} b_1(t)e^{u_1(t)}\Delta t + \int_k^{k+\omega} \dfrac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1}\Delta t, \\
\int_k^{k+\omega} \dfrac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1}\Delta t = \bar{a}_2\omega + \int_k^{k+\omega} \dfrac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1}\Delta t, \qquad (5) \\
\bar{a}_3\omega = \int_k^{k+\omega} \dfrac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1}\Delta t.
\end{cases}
$$

Since $(u_1, u_2, u_3)^T \in X$, there exist $\xi_i, \eta_i \in I_\omega$, $i = 1, 2, 3$, such that

$$
u_i(\xi_i) = \min_{t \in I_\omega}\{u_i(t)\}, \quad u_i(\eta_i) = \max_{t \in I_\omega}\{u_i(t)\}. \qquad (6)
$$

From (4) and (5), we have

$$
\begin{aligned}
\int_k^{k+\omega} \left| u_1^\Delta(t) \right| \Delta t \; &< \; \bar{a}_1\omega + \int_k^{k+\omega} b_1(t)e^{u_1(t)}\Delta t \\
&\quad + \int_k^{k+\omega} \frac{c_1(t)e^{u_2(t)}}{e^{u_1(t)}+1}\Delta t \\
&= \; 2\bar{a}_1\omega,
\end{aligned}
$$

$$
\begin{aligned}
\int_k^{k+\omega} \left| u_2^\Delta(t) \right| \Delta t \; &< \; \int_k^{k+\omega} \frac{c_2(t)e^{u_1(t)}}{e^{u_1(t)}+1}\Delta t + \bar{a}_2\omega \\
&\quad + \int_k^{k+\omega} \frac{c_3(t)e^{u_3(t)}}{e^{u_2(t)}+1}\Delta t \\
&< \; 2\bar{c}_2\omega,
\end{aligned}
$$

$$
\begin{aligned}
\int_k^{k+\omega} \left| u_3^\Delta(t) \right| \Delta t \; &< \; \bar{a}_3\omega + \int_k^{k+\omega} \frac{c_4(t)e^{u_2(t)}}{e^{u_2(t)}+1}\Delta t \\
&= \; 2\bar{a}_3\omega.
\end{aligned}
$$

By the first equation of (5) and (6), $\bar{a}_1\omega \geq \bar{b}_1\omega e^{u_1(\xi_1)}$, so $u_1(\xi_1) \leq \ln(\bar{a}_1/\bar{b}_1)$. By Lemma 2.6, we have

$$
u_1(t) \leq u_1(\xi_1) + \int_k^{k+\omega} \left| u_1^\Delta(t) \right| \Delta t < \ln \frac{\bar{a}_1}{\bar{b}_1} + 2\bar{a}_1\omega := M_1.
$$

From the third equation of (5) and (6), in view of the monotonicity of $\frac{x}{x+1}$, we can obtain

$$
\bar{a}_3\omega \geq \int_k^{k+\omega} \frac{c_4(t)e^{u_2(\xi_2)}}{e^{u_2(\xi_2)}+1}\Delta t.
$$

Thus, $u_2(\xi_2) \leq \ln \frac{\bar{a}_3}{\bar{c}_4 - \bar{a}_3}$ and

$$
u_2(t) \leq u_2(\xi_2) + \int_k^{k+\omega} \left| u_2^\Delta(t) \right| \Delta t < \ln \frac{\bar{a}_3}{\bar{c}_4 - \bar{a}_3} + 2\bar{a}_2\omega := M_2.
$$

From the second equation of (5) and (6), we have

$$
\bar{a}_2\omega \leq \bar{c}_2\omega - \bar{c}_3\omega \frac{e^{u_3(\xi_3)}}{e^{M_2}+1},
$$

this reduces to

$$
u_3(\xi_3) \leq \ln \frac{(\bar{c}_2 - \bar{a}_2)(e^{M_2}+1)}{\bar{c}_3},
$$

and

$$
\begin{aligned}
u_3(t) \; &\leq \; u_3(\xi_3) + \int_k^{k+\omega} \left| u_3^\Delta(t) \right| \Delta t \\
&< \; \ln \frac{(\bar{c}_2 - \bar{a}_2)(e^{M_2}+1)}{\bar{c}_3} + 2\bar{a}_3\omega := M_3.
\end{aligned}
$$

On the other hand, by the third equation of (5) and (6), we have $u_2(\eta_2) > \ln \frac{\bar{a}_3}{\bar{c}_4}$, and

$$
u_2(t) \geq u_2(\eta_2) - \int_k^{k+\omega} \left| u_2^\Delta(t) \right| \Delta t > \ln \frac{\bar{a}_3}{\bar{c}_4} - 2\bar{c}_2\omega := M_5.
$$

By the first equation of (5) and (6), we get

$$
\bar{b}_1\omega e^{u_1(\eta_1)} \geq \bar{a}_1\omega - \bar{c}_1\omega \frac{\bar{a}_3}{\bar{c}_4 - \bar{a}_3} e^{2\bar{a}_2\omega},
$$

then

$$
u_1(\eta_1) \geq \ln \frac{\bar{a}_1 - \frac{\bar{c}_1\bar{a}_3}{\bar{c}_4 - \bar{a}_3}e^{2\bar{a}_2\omega}}{\bar{b}_1} := N_1,
$$

and

$$
u_1(t) \geq u_1(\eta_1) - \int_k^{k+\omega} \left| u_1^\Delta(t) \right| \Delta t > N_1 - 2\bar{a}_1\omega := M_4.
$$

By the second equation of (5) and (6), we have

$$
\frac{e^{u_3(\eta_3)}}{e^{M_2}+1}\bar{c}_3\omega \leq \bar{c}_2\omega,
$$

and

$$
u_3(\eta_3) \leq \ln \frac{\bar{c}_2(e^{M_2}+1)}{\bar{c}_3} := M_6.
$$

So, we have

$$
\begin{aligned}
\max_{t \in [k, k+\omega]} |u_1(t)| &\leq \max\{|M_1|, |M_4|\} := R_1, \\
\max_{t \in [k, k+\omega]} |u_2(t)| &\leq \max\{|M_2|, |M_5|\} := R_2, \\
\max_{t \in [k, k+\omega]} |u_3(t)| &\leq \max\{|M_3|, |M_6|\} := R_3.
\end{aligned}
$$

Clearly, R_1, R_2 and R_3 are independent of λ. Let $R = R_1 + R_2 + R_3 + R_0$, where R_0 is taken sufficiently large such that for for the following algebraic equations:

$$
\begin{cases}
\bar{a}_1 - \bar{b}_1 e^x - \dfrac{\bar{c}_1 e^y}{e^x+1} = 0, \\
\bar{a}_2 - \dfrac{\bar{c}_2 e^x}{e^x+1} + \dfrac{\bar{c}_3 e^z}{e^y+1} = 0, \qquad (7) \\
\bar{a}_3 - \dfrac{\bar{c}_4 e^y}{e^y+1} = 0,
\end{cases}
$$

every solution $(x^*, y^*, z^*)^T$ of (7) satisfies $\|(x^*, y^*, z^*)^T\| < R$. Now, we define $\Omega = \{(u_1, u_2, u_3)^T \in X : \|(u_1, u_2, u_3)^T\| < R\}$. Then it is clear that Ω verifies the requirement (a) of Lemma 2.7. If $(u_1, u_2, u_3)^T \in \partial\Omega \cap \ker L =$

$\partial\Omega \cap \mathbb{R}^3$, then $(u_1, u_2, u_3)^T$ is a constant vector in $\mathbb{R}^3$ with $\|(u_1, u_2, u_3)^T\| = |u_1| + |u_2| + |u_3| = R$, so we have

$$QN \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Moreover, define

$$\phi(u_1, u_2, u_3, \mu) = \begin{bmatrix} \bar{a}_1 - \bar{b}_1 e^x \\ \bar{a}_2 + \frac{\bar{c}_3 e^z}{e^y + 1} \\ -\frac{\bar{c}_4 e^y}{e^y + 1} \end{bmatrix} + \mu \begin{bmatrix} -\frac{\bar{c}_1 e^y}{e^x + 1} \\ -\frac{\bar{c}_2 e^x}{e^x + 1} \\ \bar{a}_3 \end{bmatrix},$$

where $\mu \in [0, 1]$ is a parameter. If $(u_1, u_2, u_3)^T \in \partial\Omega \cap \ker L$, then $\phi(u_1, u_2, u_3, \mu) \neq 0$. In addition, we can easily see that the algebraic equation $\phi(u_1, u_2, u_3, 0) = 0$ has a unique solution in $\mathbb{R}^3$. Thus the invariance of homotopy produces

$$\deg(JQN, \Omega \cap \ker L, 0) = \deg(QN, \Omega \cap \ker L, 0)$$
$$= \deg(\phi(u_1, u_2, u_3, 1), \Omega \cap \ker L, 0)$$
$$= \deg(\phi(u_1, u_2, u_3, 0), \Omega \cap \ker L, 0)$$
$$= -1 \neq 0.$$

By now, we have verified that Ω fulfills all requirements of Lemma 2.7; therefore, system (1) has at least one $\omega-$periodic solution in $\text{Dom} L \cap \bar{\Omega}$. The proof is complete.

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