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On Strong(Weak) domination in Fuzzy graphs

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Abstract—Let G be a fuzzy graph. Then $D \subseteq V$ is said to be a strong (weak) fuzzy dominating set of G if every vertex $v \in V - D$ is strongly (weakly) dominated by some vertex u in D. We denote a strong (weak) fuzzy dominating set by sfd-set (wfd-set).

The minimum scalar cardinality of a sfd-set (wfd-set) is called the strong (weak) fuzzy domination number of G and it is denoted by $\gamma_{\mathrm{sf}}(G)\big(\gamma_{\mathrm{wf}}(G)\big)$.

In this paper we introduce the concept of strong (weak) domination in fuzzy graphs and obtain some interesting results for this new parameter in fuzzy graphs.

Keywords—Fuzzy graphs, Fuzzy domination, Strong (weak) fuzzy domination number.

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I. Basic definitions

THE concept of Strong (Weak) domination [6] in graphs was introduced by Sampathkumar and Pushpalatha.The notion of Domination in fuzzy graphs[7] was developed by A.Somasundaram and S.Somasundaram.

Fuzzy Domination number: Let $G = (V, \sigma, \mu)$ be a fuzzy graph. Then $D \subseteq V$ is said to be a fuzzy dominating set of G if for every $v \in V - D$, there exists u in D such that $\mu(u, v) = \sigma(u) \wedge \sigma(v)$. The minimum scalar cardinality of D is called the fuzzy domination number and is denoted by $\gamma_f(G)$. Note that scalar cardinality of a fuzzy subset D of V is : $|D|_f = \sum_{v \in D} \sigma(v)$.

Notation: Without loss of generality let us simply use the letter G to denote a fuzzy graph.

Neighbourhood and effective degrees of a vertex: Let G be a fuzzy graph. The neighbourhood of a vertex v in V is defined by $N(v) = \{u \in V : \mu(u, v) = \sigma(u) \wedge \sigma(v)\}$. The scalar cardinality of $N(v)$ is the neighbourhood degree of v, which is denoted by $d_N(v)$ and the effective degree of v is the sum of the weights of the edges incident on v, denoted by $d_E(v)$.

Strong(Weak) fuzzy vertex domination: Let u and v be any two vertices of a fuzzy graph G. Then u strongly dominates v (v weakly dominates u) if

(i) $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ and (ii) $d_N(u) \geq d_N(v)$.

Strong(Weak) fuzzy domination number: Let G be a fuzzy graph. Then $D \subseteq V$ is said to be a strong (weak) fuzzy dominating set of G if every vertex $v \in V - D$ is strongly (weakly) dominated by some vertex u in D. We denote a strong (weak) fuzzy dominating set by sfd-set (wfd-set).

The minimum scalar cardinality of a sfd-set (wfd-set) is called the strong (weak) fuzzy domination number of G and it is denoted by $\gamma_{\mathrm{sf}}(G)\big(\gamma_{\mathrm{wf}}(G)\big)$.

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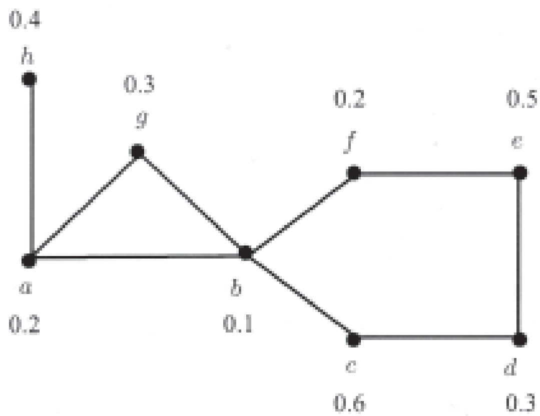

Fig.1 Fuzzy graph G

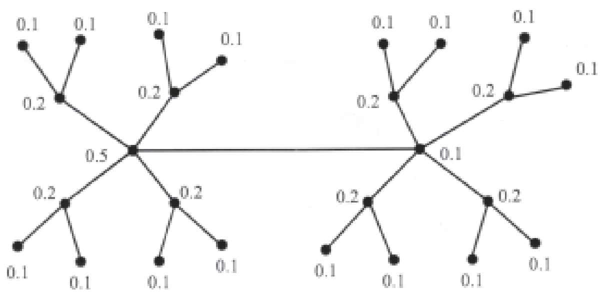

Fig.2 Fuzzy graph G

Example: For the fuzzy graph G in Fig. 1, $\gamma_{\mathrm{sf}}(G) = 0.5$ and $\gamma_{\mathrm{wf}}(G) = 1.8$, since $\{a, b, e\}$ and $\{c, f, g, h\}$ are the minimal sfd-set and wfd-set respectively.

Remark: If D is a minimal sfd-set , then $V - D$ need not be a wfd-set. For example, consider the fuzzy graph G in Fig 2.

Here all non-pendant vertices form an sfd-set D but the vertices of $V - D$ can't weakly dominate the vertices of D. Therefore $V - D$ is not a wfd-set of G.

II. Some bounds on strong (weak) fuzzy domination numbers.

In this section we present few elementary bounds on strong (weak) fuzzy domination numbers and the corresponding results.

Proposition 2.1: Let D be a minimal sfd-set of a fuzzy graph G. Then for each $v \in D$, one of the following holds:

1) No vertex in D strongly dominates v.

2) There exists $u \in V - D$ such that v is the only vertex in D which strongly dominates u.

Proposition 2.2: For a fuzzy graph G of order p,

1) $\gamma_f(G) \leq \gamma_{\mathrm{sf}}(G) \leq p - \Delta_N(G) \leq p - \Delta_E(G)$ and
2) $\gamma_f(G) \leq \gamma_{\mathrm{wf}}(G) \leq p - \delta_N(G) \leq p - \delta_E(G)$,

where $\Delta_N(G)[\Delta_E(G)]$ and $\delta_N(G)[\delta_E(G)]$ denote the maximum and minimum neighbourhood degrees (effective degrees) of G.

Proof: Since every sfd-set (wfd-set) is a fuzzy dominating set of G , $\gamma_f(G) \leq \gamma_{\mathrm{sf}}(G)$ and $\gamma_f(G) \leq \gamma_{\mathrm{wf}}(G)$. Let $u, v \in V$. If $d_N(u) = \Delta_N(G)$ and $d_N(v) = \delta_N(G)$. Then clearly V – N(u) is a sfd-set and $V - N(v)$ is a wfd-set. Therefore $\gamma_{\mathrm{sf}}(G) \leq |V - N(u)|_f$ and $\gamma_{\mathrm{wf}}(G) \leq |V - N(v)|_f$. i.e., $\gamma_{\mathrm{sf}}(G) \leq p - \Delta_N(G)$ and $\gamma_{\mathrm{wf}}(G) \leq p - \delta_N(G)$. Further since $\Delta_E(G) \leq \Delta_N(G)$ and $\delta_E(G) \leq \delta_N(G)$ we are through. ∎

Definition 2.3: A set $D \subseteq V$ of a fuzzy graph G is said to be independent if $\mu(u, v) < \sigma(u) \wedge \sigma(v)$ for all u, v in D.

Definition 2.4: $V_{\delta_N} = \{v \in V : d_N(v) = \delta_N(G)\}$ and $V_{\Delta_N} = \{v \in V : d_N(v) = \Delta_N(G)\}$.

Definition 2.5: A sfd-set (wfd-set) D of a fuzzy graph G is said to be an independent strong (weak) fuzzy dominating set of G, denoted by ISFDS (IWFDS), if it is independent. The minimum scalar cardinality of an ISFDS (IWFDS) is called the independent strong (weak) fuzzy domination number and it is denoted by $i_{\mathrm{sf}}(G)\big(i_{\mathrm{wf}}(G)\big)$.

Lemma 2.6: Let G be a fuzzy graph. If D is an IWFDS of G, then $D \cap V_{\delta_N} \neq \emptyset$.

Proof: Let $v \in V_{\delta_N}$. Since D is an IWFDS, $v \in D$ or there exists a vertex $u \in D$ such that $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for which $d_N(u) \leq d_N(v)$. If $v \in D$, then clearly $D \cap V_{\delta_N} \neq \emptyset$. On the other hand, $d_N(u) = d_N(v)$, since $d_N(v) = \delta_N(G)$. This implies that $u \in V_{\delta_N}$. Therefore $D \cap V_{\delta_N} \neq \emptyset$. ∎

Proposition 2.7: For a fuzzy graph G, $i_{\mathrm{wf}}(G) \leq p - \delta_N(G)$.

Proof: Let D be an IWFDS of G. Then by Lemma 2.6, $D \cap V_{\delta_N} \neq \emptyset$. Let $v \in D \cap V_{\delta_N}$. Since D is independent , $D \cap N(v) = \emptyset$. $\Rightarrow D \subseteq V - N(v)$
$\Rightarrow |D|_f \leq |V - N(v)|_f$
$\Rightarrow i_{\mathrm{wf}}(G) \leq |D|_f \leq p - \delta_N(G)$.
Hence proved. ∎

Proposition 2.8: Let G be a fuzzy graph with $i_{\mathrm{wf}}(G) = p - \delta_N(G)$ and let $v \in V_{\delta_N}$. Then $V - N(v)$ is independent.

Proof: Suppose that $V - N(v)$ is dependent. Applying the following algorithm we get an IWFDS of size at most $p - \delta_N(G) - \sigma(v)$ which is a contradiction to $i_{\mathrm{wf}}(G) = p - \delta_N(G)$. Hence $V - N(v)$ must be an independent set whenever $i_{\mathrm{wf}}(G) = p - \delta_N(G)$. ∎

Algorithm:
$S := N[v]$
$D := \{v\}$
`while` $S \neq V$
`begin`
`Let` $u \in \{u \in V - S$: $d_N(u)$ `is as small as possible}`
$S := S \cup N[u]$
$D := D \cup \{u\}$
`end.`

Theorem 2.9: Let G be a fuzzy graph . Then $i_{\mathrm{wf}}(G) = p - \delta_N(G)$ iff $V - N(v)$ is independent for every $v \in V_{\delta_N}$.

Proof: If $i_{\mathrm{wf}}(G) = p - \delta_N(G)$ and $v \in V_{\delta_N}$, then $V - N(v)$ is independent by Proposition 2.8. Conversely, suppose that $V - N(v)$ is independent for every $v \in V_{\delta_N}$. Let D be a minimum IWFDS of G. Then $D \cap V_{\delta_N} \neq \emptyset$. Let $v \in D \cap V_{\delta_N}$. Since $v \in D$, $D \cap N(v) = \emptyset$ and also $V - N(v)$ is independent. $\Rightarrow D = V - N(v)$.
i.e., $i_{\mathrm{wf}}(G) = p - \delta_N(G)$. ∎

We state the following results without proving them, since they are analogous to the results on $i_{\mathrm{wf}}(G)$.

Lemma 2.10: Let G be a fuzzy graph . If D is an ISFDS of G, then $D \cap V_{\Delta_N} \neq \emptyset$.

Proposition 2.11: For a fuzzy graph G, $i_{\mathrm{sf}}(G) \leq p - \Delta_N(G)$.

Proposition 2.12: Let G be a fuzzy graph with $i_{\mathrm{sf}}(G) = p - \Delta_N(G)$ and let $v \in V_{\Delta_N}$. Then $V - N(v)$ is independent.

Proposition 2.13: Let G be a fuzzy graph . Then $i_{\mathrm{sf}}(G) = p - \Delta_N(G)$ iff $V - N(v)$ is independent for every $v \in V_{\Delta_N}$.

III. Fuzzy graphs for which $\gamma_{\mathrm{wf}}(G) = p - \delta_N(G)$.

Theorem 3.1: For a fuzzy graph G, $\gamma_{\mathrm{wf}}(G) = p - \delta_N(G)$ iff one of the following holds:

1) $\delta_N(G) = p - \sigma_1$
2) $\delta_N(G) = p - (\sigma_1 + \sigma_2)$
3) $\delta_N(G) \leq p - (\sigma_1 + \sigma_2 + \sigma_3)$ and if $v \in V_{\delta_N}$, such that $\sigma(v)$ is the smallest membership grade in V_{δ_N} then $V - N(v)$ is independent and every vertex in $N(v)$ has degree $> \delta_N(G)$, where σ_1, σ_2 and σ_3 are the first three highest membership grades of vertices in V_{δ_N}.

Proof: Assume that $\gamma_{\mathrm{wf}}(G) = p - \delta_N(G)$ and $\delta_N(G) \leq p - (\sigma_1 + \sigma_2 + \sigma_3)$. Let $v \in V_{\delta_N}$ be a vertex such that $\sigma(v)$ is the smallest membership grade in V_{δ_N}. Then $V - N(v)$ is independent and $V - N(v) \subseteq V_{\delta_N}$. It is clear that each vertex in $N(v)$ is adjacent to all the vertices of $V - N(v)$. It is clear that each vertex in $N(v)$ is adjacent to all the vertices of $V - N(v)$. Suppose $u \in N(v) \cap V_{\delta_N}$. Then $\{u, v\}$ is a WFDS of G and so $\gamma_{\mathrm{wf}}(G) < p - (\sigma_1 + \sigma_2 + \sigma_3) \leq p - \delta_N(G)$, which is a contradiction. This implies that $N(v) \cap V_{\delta_N} = \emptyset$.

Conversely, suppose that one of the given conditions is true.

Case (i): If $\delta_N(G) = p - \sigma_1$, then a vertex v for which $\sigma(v) = \sigma_1$, weakly dominates all the other vertices of G. Therefore $\gamma_{\mathrm{wf}}(G) = \sigma_1 = p - \delta_N(G)$.

Case (ii): Let v_1, v_2 be any two non-adjacent vertices in V_{δ_N} for which $\sigma(v_1) = \sigma_1$ and $\sigma(v_2) = \sigma_2$. If $\delta_N(G) = p - (\sigma_1 + \sigma_2)$, then v_1 is adjacent to all the other vertices of V except v_2 and vice-versa . This implies that $\{v_1, v_2\}$ is a WFDS of G. So $\gamma_{\mathrm{wf}}(G) \leq \sigma_1 + \sigma_2$. Clearly neither v_1 nor v_2 can alone be a WFDS of G. Therefore $\gamma_{\mathrm{wf}}(G) = \sigma_1 + \sigma_2 = p - \delta_N(G)$.

Case (iii): Suppose that the condition (iii) holds. If $V - N(v)$ is independent, we have $V - N(v) \subseteq V_{\delta_N}$. Further since every vertex in $N(v)$ has neighbourhood degree $> \delta_N(G)$, every vertex in $V - N(v)$ has neighbourhood degree $\leq \delta_N(G)$.
So every vertex of $V - N(v)$ is in WFDS of G.
This implies $\gamma_{\mathrm{wf}}(G) \geq p - \delta_N(G)$; but $\gamma_{\mathrm{wf}}(G) \leq p - \delta_N(G)$.
Thus $\gamma_{\mathrm{wf}}(G) = p - \delta_N(G)$. ∎

Theorem 3.2: Let G be a connected triangle free fuzzy graph. Then $\gamma_{\mathrm{wf}}(G) = p - \delta_N(G)$ iff $G \in \{K_{\sigma_1}\} \cup \{K_{\delta_N, p-\delta_N}$ where $\delta_N(G) < \frac{p}{2}\}$.

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Proof: Suppose that $\gamma_{\rm wf}(G) = p - \delta_N(G)$. If $\delta_N(G) = 0$, then $G = K_{\sigma_1}$, where σ_1 is the highest membership grade of a vertex in V_{δ_N}. Let us assume that $\delta_N(G) > 0$.

Case (i): $\delta_N(G) = p - \sigma_1$. Since G is triangle-free, $G = K_{\sigma_1, p-\sigma_1}$.

Case(ii): $\delta_N(G) = p - (\sigma_1 + \sigma_2)$, where σ_1 and σ_2 are the first two highest membership grades of non-adjacent vertices in V_{δ_N}. Each $\{v_1, v_2\}$ is adjacent to each vertex in $V - \{v_1, v_2\}$. Further, since G is triangle free, no two vertices in $V - \{v_1, v_2\}$ are adjacent. Hence $G = K_{\delta_N, p-\delta_N}$.

Case(iii): $\delta_N(G) \leq p - (\sigma_1 + \sigma_2 + \sigma_3)$, where σ_1, σ_2 and σ_3 are the first three highest membership grades of vertices in V_{δ_N}. Let $v \in V_{\delta_N}$. Then $V - N(v)$ is independent.

$\Rightarrow$ each vertex of $V - N(v)$ is adjacent to each vertex of $N(v)$. Further, since G is triangle-free, $N(v)$ is also independent. Thus G is a complete bipartite fuzzy graph with bipartition $\big(V - N(v), N(v)\big)$. i.e., $G = K_{\delta_N, p-\delta_N}$. Since every vertex in $N(v)$ has degree $> \delta_N(G)$, $|V - N(v)|_f > \delta_N(G)$.

Therefore $p = |V - N(v)|_f + |N(v)|_f$.
$> \delta_N(G) + \delta_N(G)$.
$\Rightarrow \delta_N(G) < \frac{p}{2}$.
The converse is obvious. ■

IV. Practical application

Let G be a graph which represents the road network of a city. Let the vertices denote the junctions and the edges denote the roads connecting the junctions. The membership functions σ and μ on the vertex set and the edge set of G can be constructed from the statistical data that represents the number of vehicles passing through various junctions and the number of vehicles passing through various roads during a peak hour. Thus we get a fuzzy graph G. In this fuzzy graph a strong fuzzy dominating set D can be interpreted as the set of junctions in which traffic is heavier than the other junctions not in D.

References

[1] E.J. Cockayne and S.T. Hedetniemi,*Towards a theory of domination in graphs, Networks*,(1977), 247–261.
[2] Domke et. al,*On parameters related to strong and weak domination in graphs*,Discrete Mathematics 258 (2002), 1-11.
[3] T.W.Haynes et. al, *Fundamentals of Domination in graphs*, Marcel Dekker, New York, 1998.
[4] J.N. Mordeson and P.S. Nair, *Fuzzy graphs and Fuzzy Hypergraphs*,Physica Verlag, Heidelberg, 1998; second edition 2001.
[5] A. Rosenfeld *Fuzzy graphs*,in; L.A. Zadeh, K.S. Fu, M.Shimura(Eds.), Fuzzy sets and their Applications to Cognitive and Decision Processes, Academic Press, New York, 1975, 77-95.
[6] E. Sampathkumar, L. Pushpalatha*Strong, weak domination and domination balance in a graph*, Discrete Math. 161(1996), 235-242.
[7] A. Somasundaram and S. Somasundaram,*Domination in fuzzy graphs-I*, Pattern recognition Letters 19(1998), 787–791.

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