Modulational Instability of Electron Plasma Waves in Finite Temperature Quantum Plasma

Swarniv Chandra, and Basudev Ghosh

Abstract—Using the quantum hydrodynamic (QHD) model for quantum plasma at finite temperature the modulational instability of electron plasma waves is investigated by deriving a nonlinear Schrodinger equation. It was found that the electron degeneracy parameter significantly affects the linear and nonlinear properties of electron plasma waves in quantum plasma.

Keywords—Amplitude Modulation, Electron Plasma Waves, Finite Temperature Model, Modulational Instability, Quantum Plasma.

I. INTRODUCTION

THE study of ultradense matter has been carried out quite extensively and intensively in the recent past. Such matter is found in metal nanostructures, neutron stars, white dwarfs and other astronomical bodies as well as in laser plasma interaction experiments. In such situations the average inter-fermionic distance is comparable or even less than the thermal de Broglie wavelength and as a result the quantum degeneracy becomes important. In such extreme conditions of density the thermal pressure of electrons may be negligible as compared to the Fermi degeneracy pressure which arises due to the Pauli exclusion principle. However in this case the quantum effect can't be neglected and proper mathematical modelling becomes necessary. Such quantum effects are generally studied with the help of two well known formulations, the Wigner-poisson and the Schrodinger Poisson formulation. The former one studies the quantum kinetic behavior of plasmas while the latter describes the hydrodynamic behaviour of plasma waves. The quantum hydrodynamic (QHD) model is derived by taking the velocity space moments of the Wigner equations. The QHD model modifies the clasical fluid model for plasmas with the inclusion of a quantum correction term generally known as the Bohm potential. The model also incorporates the quantum statistical effect through the equation of state. The model has been widely used to study quantum behaviour of plasma. A survey of the available literature [1-17] shows that most of the works done in quantum plasma in order to study the nonlinear behaviour of different plasma waves uses ultra low temperature approximation. But in most practical cases the temperature is not zero but finite. In this paper we have used the model developed by Eliasson and Shukla [18] to study the modulational instability of electron plasma waves including finite temperature effects.

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II. THE FINITE TEMPERATURE MODEL

Based on the 3D euilibrium Fermi-dirac distribution for electrons at an arbitrary temperature Eliasson and Shukla [18] derived a set of fluid equations which are valid both in the large and low temperature limits. When a plane longitudinal electron plasma wave propagates in a collisionless quantum plasma, it leads to adiabatic compression thereby causing a temperature anisotropy in the electron distribution. In quantum plasma the classical compressibility of the electron phase fluid is violated due to quantum mechanical tunnelling. However to a first approximation it can be assumed that the electron phase fluid is incompressible. Further the chemical potential (μ) remains constant during the nonequilibrium dynamics of the plasma. Under such assumptions the nonequilibrium particle density is given by:

$$
\begin{aligned}
n_0 &= \frac{1}{2\pi^2}\left(\frac{2m}{\hbar^2}\right)^{3/2}\int_0^\infty \frac{E^{1/2}dE}{\exp\left[\beta\left(E-\mu\right)+1\right]} \\
&= -\frac{1}{2\pi^2\beta^{3/2}}\left(\frac{2m}{\hbar^2}\right)^{3/2}\Gamma\left(\frac{3}{2}\right)Li_{3/2}\left[-\exp\left(\beta\mu\right)\right]
\end{aligned}
\tag{1}
$$

$Li_\gamma(y)$ is the polylogarithm function. In the ultracold limit i.e $T\to 0$, we have $\beta\to\infty$ and $\mu\to E_F$. The temperature anisotropy is given by: $\eta_{ex}(x,t)= T_{e0}/T_{ex}(x,t)=[n_0/n_e(x,t)]$
The fermi energy is given by:

$$
E_F = \left(3\pi^2 n_0\right)^{2/3}\left(\hbar^2/2m\right)
\tag{2}
$$

Now using the zeroth and first moments of the Wigner equation with the Fermi-Dirac distribution function and assuming that the Bohm potential is independent of the thermal fluctuations in a finite temperature plasma one can derive the continuity and momentum equation in the following form:

$$
\frac{\partial n_e}{\partial t}+\frac{\partial(n_e v_{ex})}{\partial x}=0
\tag{3}
$$

$$
\frac{\partial v_{ex}}{\partial t}+v_{ex}\frac{\partial v_{ex}}{\partial x}=\frac{e}{m_e}\frac{\partial\phi}{\partial x}
$$

$$
-\frac{n_0 V_{Te}^2}{n_e}G\frac{\partial\left(n_e/n_0\right)^3}{\partial x}+\frac{\hbar^2}{2m_e^2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_e}}\frac{\partial^2\sqrt{n_e}}{\partial x^2}\right]
\tag{4}
$$

where n_e and v_{ex} are respectively the particle density and fluid velocity of electron; ϕ is the electrostatic wave potential and

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$v_{Te} = \sqrt{k_B T_{Te}/m_e}$ is the themal speed. G is the ratio of two polylogarithm functions given by: $G = \dfrac{Li_{5/2}(-\exp[\beta\mu])}{Li_{3/2}(-\exp[\beta\mu])}$

The system is closed by the Poisson equation,

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e (n_e - n_i) \tag{5}$$

We now introduce the following normalization:

$x \to x\omega_{pe}/V_{Fe}, t \to t\omega_{pe}, \phi \to e\phi/2k_B T_{Fe}, n_j \to n_j/n_0$ and $u_j \to u_j/V_{Fe}$. Here $\omega_{pe} = \sqrt{4\pi n_0 e^2/m_e}$ the electron plasma oscillation frequency and $v_{Fe} = \sqrt{2k_B T_{Fe}/m_e}$ is the Fermi speed of electrons. Using the above normalization Eqs. (3)- (5) can be recast as:

$$\frac{\partial n_e}{\partial t} + \frac{\partial (n_e v_{ex})}{\partial x} = 0$$

$$\left(\frac{\partial}{\partial t} + v_{ex}\frac{\partial}{\partial x}\right)v_{ex} = \frac{\partial \phi}{\partial x} - 3G\alpha^2 n_e \frac{\partial n_e}{\partial x} + \frac{H^2}{2}\frac{\partial}{\partial x}\left[\frac{1}{\sqrt{n_e}}\frac{\partial^2 \sqrt{n_e}}{\partial x^2}\right] \tag{6}$$

$$\frac{\partial^2 \phi}{\partial x^2} = (n_e - n_i)$$

where $H = \hbar\omega_{pe}/2 k_B T_{Fe}$ is a nondimensional quantum parameter proportional to the quantum diffraction and $\alpha = (V_{Te}/V_{Fe})$. The parameter H is proportional to the ratio between the plasma energy $\hbar\omega_{pe}$ (energy of an elementary excitation associated with an electron plasma wave) and the Fermi energy $k_B T_{Fe}$.

III. DERIVATION OF THE NONLINEAR SCHRODINGER EQUATION

For electron plasma waves we can assume that the ions are immobile. Following standard procedure we make the following Fourier expansions for the field variables:

$$\begin{bmatrix} n_e \\ v_{ex} \\ \phi \end{bmatrix} = \begin{bmatrix} 1 \\ u_0 \\ 0 \end{bmatrix} + \varepsilon^2 \begin{bmatrix} n_{e0} \\ v_{e0} \\ \phi_0 \end{bmatrix} + \sum_{s=1}^{\infty}\varepsilon^s \left\{ \begin{bmatrix} n_{es} \\ v_{es} \\ \phi_s \end{bmatrix}.e^{is\psi} + \begin{bmatrix} n_{es}^* \\ v_{es}^* \\ \phi_s^* \end{bmatrix}.e^{-is\psi} \right\} \tag{7}$$

In which $\psi=kx-\omega t$ (ω, k being the normalized wave wavenumber respectively), the field quantities n_{eo}, u_{eo}, ϕ_0, n_{es}, u_{es}, ϕ_s, n_{io} and u_{io} are assumed to vary slowly with x and t. i.e. they are supposed to be functions of $\xi = \varepsilon(x-c_g t)$ and $\tau = \varepsilon^2 t$, where ε is a smallness parameter and c_g is the normalized group velocity.

Now substituting the expansion (7) in Eqs. (6) and then equating from both sides the coefficients of exp (iψ), exp (i2ψ) & terms independent of ψ we obtain three sets of equations which we call I, II and III. To solve these three sets of equations we make the following perturbation expansion for

the field quantities n_{eo}, u_{eo}, ϕ_0 n_{es}, u_{es}, ϕ_s, n_{io} and u_{io} which we denote by A :

$$A = A^{(1)} + \varepsilon A^{(2)} + \varepsilon^2 A^{(3)} + \ldots \tag{8}$$

Solving the lowest order equations obtained from the set of equations I after substituting the expansion (81) we get

$$n_{e1}^{(1)} = -k^2\phi_1^{(1)}, u_{e1}^{(1)} = -(\omega - ku_0)k\phi_1^{(1)} \tag{9}$$

And the linear dispersion relation,

$$\omega^2 = 1 + k^2\left(3G\alpha^2 + \frac{k^2 H^2}{4}\right) \tag{10}$$

In the dimensional form the dispersion relation becomes:

$$\omega^2 = \omega_{pe}^2 + 3G\alpha^2 k^2 V_{Fe}^2 + \frac{k^4 V_{Fe}^4 H^2}{4\omega_{pe}^2}$$
$$= \omega_{pe}^2 + 3Gk^2 V_{Te}^2 + \frac{k^4 V_{Fe}^4 H^2}{4\omega_{pe}^2} \tag{11}$$

The degeneracy parameter G determines the transition between the ultra cold and thermal cases. In the low twmperature limit $\beta\mu\to\infty$, $\mu \approx E_F \equiv (mV_{Fe}^2)/2$ and $G \approx 2\beta E_F/5$, then the dispersion relation (11) takes the form

$$\omega^2 = \omega_{pe}^2 + \frac{3}{5}k^2 V_{Fe}^2 + \frac{k^4 V_{Fe}^4 H^2}{4\omega_{pe}^2} \tag{12}$$

which is similar to the dispersion relation for electron plasma waves in a quantum plasma obtained by using one dimensional QHD Model. In the high temperature limit $\beta\mu\to-\infty$ so that G→1 and then the dispersion relation (11) reduces to the Bohm-Gross dispersion relation for electron plasma waves in a hot plasma

$$\omega^2 = \omega_{pe}^2 + 3k^2 V_{Te}^2 + \frac{k^4 V_{Fe}^4 H^2}{4\omega_{pe}^2} \tag{13}$$

In the high temperature limit the last term on the RHS may be neglected and then one gets the well known Bohm-Gross dispersion relation of electron plasma waves in a hot plasma. Fig. 1 shows the linear dispersion characteristics for different values of G. The electron degeneracy parameter is found to increase the slope of the dispersion curve. i.e.as the value of G increases the wave frequency increases for a given k.

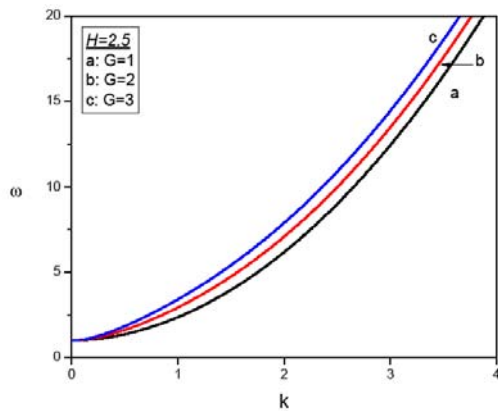World Academy of Science, Engineering and Technology
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Fig. 1 Linear dispersion curve for different value of G

The group velocity $c_g = d\omega / dk$ is obtained from the dispersion relation (13) as:

$$c_g = \left(3G\alpha^2 k + \frac{H^2 k^3}{2} \right) \left(1 + 3G\alpha^2 k^2 + \frac{H^2 k^4}{4} \right)^{-1/2} \tag{14}$$

Fig. 2 shows the plot of group velocity versus wavenumber for different values of electron degeneracy parameter G. It is found that in the high k-region quantum diffraction term dominated and the group velocity is almost independent of G. In the small wavenumber region the G term dominates thus contributing to the nonlinear effect due to degeneracy pressureIn this region group velocity increases with increase in G.
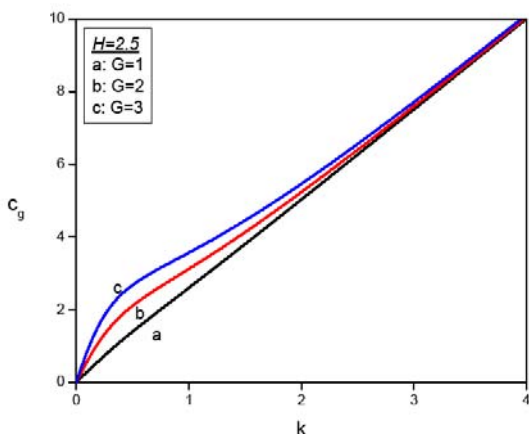

Fig. 2 Group velocity versus wavenuber for different G

We now obtain the second harmonic terms in the lowest order equation which are obtained from the set of equations II by substituting the perturbation expansion.

$$\phi_2^{(1)} = -b_2 \phi_1^{(1)2}$$

$$n_{e2}^{(1)} = 4k^2 b_2 \phi_1^{(1)2}$$

$$u_{e2}^{(1)} = \omega k \left(4b_2 - k^2 \right) \phi_1^{(1)2} \tag{15}$$

$$b_2 = \frac{k^2 \left(3\omega^2 + 3G\alpha^2 k^2 - \frac{H^2 k^4}{4} \right)}{8 \left(\omega^2 - 3G\alpha^2 k^2 - H^2 k^4 - \frac{1}{4} \right)} \tag{16}$$

The zeroth harmonic quantities are obtained from the solutions of the lowest order equations obtained from the set of equations III after substituting the perturbation expansion (8)

$$\phi_0^{(1)} = b_0 \phi_1^{(1)2}$$

$$n_{e0}^{(1)} = n_{i0}^{(1)} = b_1 \phi_1^{(1)2}$$

$$u_{e0}^{(1)} = \left(b_1 c_g - 2\omega k^3 \right) \phi_1^{(1)2} \tag{17}$$

$$u_{i0}^{(1)} = b_1 c_g \phi_1^{(1)2}$$

Where

$$b_0 = \frac{k^2 \left(1 + 12 \cdot G\alpha^2 k^2 + 3H^2 k^3 \right) \left(c_g^2 - \sigma \right)}{\left(\left(c_g^2 - \sigma \right) + \mu \left(c_g^2 - 3G\alpha^2 \right) \right)} \tag{18}$$

$$b_1 = \frac{\mu b_0}{\left(c_g^2 - \sigma \right)}$$

The first harmonic quantities in the second order are obtained from the solutions (9) by replacing $-i\omega$ by $(-i\omega - \varepsilon c_g \frac{\partial}{\partial \xi} + \varepsilon^2 \frac{\partial}{\partial \tau})$ and ik by $(ik + \varepsilon \frac{\partial}{\partial \xi})$ and then picking out order ε terms. Thus we obtain

$$\phi_1^{(2)} = 0 \tag{19}$$

$$n_{e1}^{(2)} = 2ik \frac{\partial \phi_1^{(1)}}{\partial \xi}$$

$$u_{e1}^{(2)} = i(\omega + kc_g) \frac{\partial \phi_1^{(1)}}{\partial \xi}$$

Now collecting coefficients of ε^3 from both sides of the sets of equations I after substituting the perturbation expansion (8) we get a set of equations for the first harmonic quantities in the third order. Using the above solutions and after proper elimination we obtain the following desired Nonlinear Schrödinger equation (NLSE) describing the nonlinear evolution of the wave amplitude

$$i\frac{\partial \alpha}{\partial \tau} + P\frac{\partial^2 \alpha}{\partial \xi^2} = Q\alpha^2 \alpha^* \tag{20}$$

where $\alpha = \phi_1^{(1)}$

The group dispersion coefficient

$$P = \frac{1}{2}\frac{dc_g}{dk} = \frac{1}{2\omega}\left[3G\alpha^2 - c_g^2 + \frac{3}{2}H^2 k^2 \right] \tag{21}$$

and the nonlinear coefficient

$$Q = \frac{\left[\begin{array}{c} \omega^2 \left[b_1 k^2 - 6k^6 + 12b_2 k^4 \right] + \omega 2 b_1 k^3 c_g \\ + 3G\alpha^2 k^4 \left(b_1 + 4b_2 k^2 \right) - \left(\frac{H^2 k^6}{8} \right) \left(b_1 + 20b_2 \right) \end{array} \right]}{2\omega k^2} \tag{22}$$

Note that both the coefficients P and Q depend on the quantum parameter H.

IV. MODULATIONAL INSTABILITY

The amplitude modulation of electron plasma waves can be investigated by using the NLS equation (20) which includes corrections due to finite temperature and quantum diffraction. The NLS equation has been studied extensively by many authors in connection to the nonlinear propagation of varies types of waves in plasma. It has been found that under certain conditions, an initially uniform wavetrain gets converted into a spatially modulated wave; such modulated waves are found to be energitically favourable. This effect is known as modulational instability. It is well known from the available literature that a uniformly propagating plasma wavetrain is modulatioanally stable when PQ>0 and unstable when PQ<0. The growth rate of such modulational instability attains a maximum value, $g_m = |Q| . \alpha_0^2$ corresponding to the wavenumber $l_m = |Q \ / \ P|^{1/2} . |\alpha_0|$ of the modulation. Thus we find that the instability condition depends on the sign of the product PQ. Numerical computation of P and Q by using the expressions (21) and (22) for different values of k in terms of the system parameters shows that in low and high k-regions seperated by a stable region in k-space. The stability region in k-space increases with increase in the degeneracy parameter G but it decreases with increase in the quantum diffraction parameter H. The dependence of growth rate of instability on G depends on the region of instability in k-space. In the low k-region the instability growth rate is higher for higher G (Fig.3) but in the high k-region the instability growth rate decreases with increase in G (Fig.4).
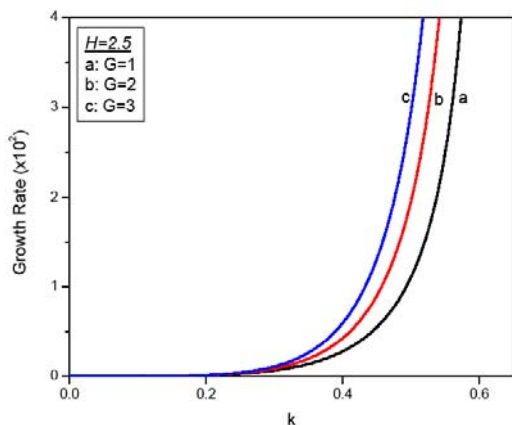

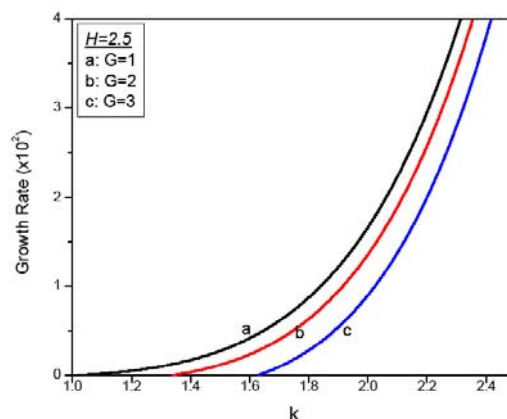Fig. 3 Growth rate of instability vs. k for different G for low k

Fig. 4 Growth rate of instability vs. k for different G for high k

V. DISCUSSION AND CONCLUSION

As the degeneracy parameter G determines the transition from ultracold to thermal cases it is important to know its value. Table I shows the values of G for certain practical plasmas. Finally we would like to point out that the investigation presented here may be helpful in the understanding of the basic features of long wavelength electron plasma waves in dense and hot plasmas such as can be found in white dwarfs, neutron stars and intense laser-solid plasma experiments.

TABLE I
DEGENERACY PARAMETER FOR DIFFERENT KIND OF PLASMA

Type of Plasma	Density (m^{-3})	Temperature (K)	G
Tokamak	10^{20}	10^{18}	1
Inertial Confinement Fusion	10^{32}	10^{8}	1
Metal and Metal clusters	10^{28}	10^{4}	1.4
Jupiter	10^{32}	10^{4}	1.4
White Dwarf	10^{35}	10^{8}	4

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