

An efficient computational algorithm for solving the nonlinear Lane-Emden type equations

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Abstract—In this paper we propose a class of second derivative multistep methods for solving some well-known classes of Lane-Emden type equations which are nonlinear ordinary differential equations on the semi-infinite domain. These methods, which have good stability and accuracy properties, are useful in deal with stiff ODEs. We show superiority of these methods by applying them on the some famous Lane-Emden type equations.

Keywords—Lane-Emden type equations, Nonlinear ODE, Stiff problems, Multistep methods, Astrophysics.

I. INTRODUCTION

MANY problems in science and engineering arise in unbounded domains. Different spectral methods have been proposed for solving problems on unbounded domains. The most common method is through the use of polynomials that are orthogonal over unbounded domains, such as the Hermite spectral and the Laguerre spectral methods [2], [12], [24], [25], [27], [38], [55], [57].

Guo [26], [28], [30] proposed a method that proceeds by mapping the original problem in an unbounded domain to a problem in a bounded domain, and then using suitable Jacobi polynomials to approximate the resulting problems.

Another approach is replacing the infinite domain with $[-L, L]$ and the semi-infinite interval with $[0, L]$ by choosing L , sufficiently large. This method is named as the domain truncation [5].

Christov [11] and Boyd [6], [7] used another effective direct approach for solving such problems that are based on rational approximations. They developed some spectral methods on unbounded intervals by using mutually orthogonal systems of rational functions. Boyd [7] defined a new spectral basis, named rational Chebyshev functions on the semi-infinite interval, by mapping it to the Chebyshev polynomials. Guo et al. [29] introduced a new set of rational Legendre functions which are mutually orthogonal in $L^2(0, +\infty)$. They applied a spectral scheme using the rational Legendre functions for solving the Korteweg-de Vries equation on the half line. Boyd et al. [4] applied pseudospectral methods on a semi-infinite interval and compared the rational Chebyshev, Laguerre and the mapped Fourier sine methods.

Authors of [41]–[46] applied the spectral method to solve the nonlinear ordinary differential equations on semi-infinite intervals. Their approach is based on the rational Tau and collocation methods.

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Lane-Emden type equations are nonlinear ordinary differential equations on semi-infinite domain. They are categorized as singular initial value problems. These equations describe the temperature variation of a spherical gas cloud under the mutual attraction of its molecules and subject to the laws of classical thermodynamics. The polytropic theory of stars essentially follows out of thermodynamic considerations, that deals with the issue of energy transport, through the transfer of material between different levels of the star. These equations are one of the basic equations in the theory of stellar structure and has been the focus of many studies [1]–[3], [10], [19], [31], [37], [39], [47]–[50], [54], [58], [59], [61].

We simply begin with the Poisson equation and the condition for hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{GM(r)}{r^2},$$

$$\frac{dM(r)}{dr} = 4\pi\rho r^2,$$

where G is the gravitational constant, P is the pressure, $M(r)$ is the mass of a star at a certain radius r , and ρ is the density, at a distance r from the center of a spherical star. The combination of these equations yields the following equation, which as should be noted, is an equivalent form of the Poisson equation

$$\frac{1}{r^2} \frac{d}{dr} \left(\frac{r^2}{\rho} \frac{dP}{dr} \right) = -4\pi G\rho.$$

From these equations one can obtain the Lane-Emden equation through the simple supposition that the density is simply related to the pressure, while remaining independent of the temperature. We already know that in the case of a degenerate electron gas, the pressure and density are $\rho \sim P^{\frac{3}{5}}$, assuming that such a relation exists for other states of the star, we are led to consider a relation of the following form:

$$P = K\rho^{1+\frac{1}{m}},$$

where K and m are constants, at this point it is important to note that m is the polytropic index which is related to the ratio of specific heats of the gas comprising the star. Based upon these assumptions we can insert this relation into our first equation for the hydrostatic equilibrium condition and from this equation we have

$$\left[\frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m}-1} \right] \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dy}{dr} \right) = -y^m,$$

where the additional alteration to the expression for density has been inserted with λ representing the central density of

the star and y that of a related dimensionless quantity that are both related to ρ through the following relation

$$\rho = \lambda y^m.$$

Additionally, if place this result into the Poisson equation, we obtain a differential equation for the mass, with a dependence upon the polytropic index m . Though the differential equation is seemingly difficult to solve, this problem can be partially alleviated by the introduction of an additional dimensionless variable x , given by the following:

$$r = ax,$$

$$a = \left[\frac{K(m+1)}{4\pi G} \lambda^{\frac{1}{m}-1} \right]^{\frac{1}{2}}.$$

Inserting these relations into our previous equations we obtain the famous form of the Lane-Emden equations, given in the following:

$$\frac{1}{x^2} \frac{d}{dx} \left(x^2 \frac{dy}{dx} \right) = -y^m.$$

Taking these simple relations we will have the standard Lane-Emden equation with $g(y) = y^m$,

$$y'' + \frac{2}{x}y' + y^m = 0, \quad x > 0. \quad (1)$$

At this point it is also important to introduce the boundary conditions which are based upon the following boundary conditions for hydrostatic equilibrium and normalization consideration of the newly introduced quantities x and y . What follows for $r = 0$ is

$$r = 0 \rightarrow x = 0, \quad \rho = \lambda \rightarrow y(0) = 1. \quad (2)$$

As a result an additional condition must be introduced in order to maintain the condition of Eq. (2) simultaneously:

$$y'(0) = 0.$$

In other words, the boundary conditions are as follows

$$y(0) = 1, \quad y'(0) = 0.$$

The values of m which are physically interesting, lie in the interval $[0,5]$. The main difficulty in the analysis of this type of equation is the singularity behavior occurring at $x = 0$. Exact solutions for Eq. (1) are known only for $m = 0, 1$ and 5 . For other values of m the standard Lane-Emden equation is to be integrated numerically. Thus we decided to present a new and efficient technique to solve it numerically.

On the other hand, in recent years, the study of numerical methods for solving stiff initial-value problems for ordinary differential equations has reached a certain maturity. There now exist some excellent codes which are both efficient and reliable for solving these particular classes of problems. For example, as Enright [23] used second derivative of solution in his algorithm, Cash [8] and Ismail [34] introduced second derivative multistep methods that have good stability properties. These methods are A-stable of high orders. One class of the these efficient methods, that have good stability and accuracy properties, is a new class of second derivative

multistep methods (SDMMs) that have introduced by Hojjati et al. [32]. The main superiority of this new class of methods leads us to apply them to solve Lane-Emden type equations.

This paper is arranged as follows: in section 2, we review the proposed methods for solving Lane-Emden equations. In section 3, we briefly describe the new second derivative multistep methods that we use to solve Lane-Emden equation. Finally in section 4, we apply the new SDMMs for numerical solution of some famous ordinary differential equations of Lane-Emden type.

II. OUT OF PAPER

Recently, many analytical methods have been used to solve Lane-Emden equations. The main difficulty arises in the singularity of the equations at $x = 0$. Currently, most techniques which were used in handling the Lane-Emden-type problems are based on either series solutions or perturbation techniques.

Bender et al. [3] proposed a new perturbation technique based on an artificial parameter δ , the method is often called δ -method.

Mandelzweig et al. [39] used the quasilinearization approach to solve the standard Lane-Emden equation. This method approximates the solution of a nonlinear differential equation by treating the nonlinear terms as a perturbation about the linear ones, and unlike perturbation theories is not based on the existence of some small parameters.

Shawagfeh [54] applied a nonperturbative approximate analytical solution for the Lane-Emden equation using the Adomian decomposition method. His solution was in the form of a power series. He used Padé approximants method [15], [17] to accelerate the convergence of the power series.

In [58], Wazwaz employed the Adomian decomposition method [18], [22] with an alternate framework designed to overcome the difficulty of the singular point. It was applied to the differential equations of Lane-Emden type. Further author of [59] used the modified decomposition method for solving the analytical treatment of nonlinear differential equations such as the Lane-Emden equation.

Liao [37] provided an analytical algorithm for Lane-Emden type equations. This algorithm logically contains the well-known Adomian decomposition method. Different from all other analytical techniques, this algorithm itself provides us with a convenient way to adjust convergence regions even without Padé technique.

J.-H He [31] employed Ritz's method to obtain an analytical solution of the problem. By the semi-inverse method, a variational principle is obtained for the Lane-Emden equation which can gave much numerical convenience when applied to finite element methods or Ritz methods.

Parand et al. [42], [43], [46] presented two numerical techniques to solve higher ordinary differential equations such as Lane-Emden. Their approach was based on the rational Chebyshev and rational Legendre tau methods.

Ramos [47]–[50] solved Lane-Emden equations through different methods. Author of [48] presented the linearization method for singular initial-value problems in second-order

ordinary differential equations such as Lane-Emden. These methods result in linear constant-coefficients ordinary differential equations which can be integrated analytically, thus yielding piecewise analytical solutions and globally smooth solutions. Later this author [50] developed piecewise-adaptive decomposition methods for the solution of nonlinear ordinary differential equations. In [49], series solutions of the Lane-Emden type equation have been obtained by writing this equation as a Volterra integral equation and assuming that the nonlinearities are sufficiently differentiable. These series solutions have been obtained by either working with the original differential equation or transforming it into an ordinary differential equation that does not contain the first-order derivatives. Series solutions to the Lane-Emden type equation have also been obtained by working directly on the original differential equation or transforming it into a simpler one.

Yousefi [61] presented a numerical method for solving the Lane-Emden equations. He converted Lane-Emden equations to integral equations, using integral operator, and then he applied Legendre wavelet approximations.

Bataineh et al. [2] presented an algorithm based on homotopy analysis method (HAM) [14] to obtain the exact and/or approximate analytical solutions of the singular IVPs of the Emden-Fowler type equation.

In [10], Chowdhury et al. presented an algorithm based on the homotopy-perturbation method (HPM) [16], [52], [53] to solve singular IVPs of time-independent equations.

Aslanov [1] introduced a further development in the Adomian decomposition method to overcome the difficulty at the singular point of non-homogeneous, linear and non-linear Lane-Emden-like equations.

Dehghan and Shakeri [19] applied an exponential transformation to the Lane-Emden type equations to overcome the difficulty of a singular point at $x = 0$ and solved the resulting nonsingular problem by the variational iteration method [20], [21].

Yildirim et al. [60] presented approximate-exact solutions of a class of Lane-Emden type singular IVPs problems, by the variational iteration method.

Marzban et al. [40] used a method based upon hybrid function approximations. They used the properties of hybrid of block-pulse functions and Lagrange interpolating polynomials together for solving the nonlinear second-order initial value problems and the Lane-Emden equation.

Recently, Singh et al. [56] provided an efficient analytic algorithm for Lane-Emden type equations using modified homotopy analysis method, also they used some well-known Lane-Emden type equations as test examples.

We refer the interested reader to [35], [36] for analysis of the Lane-Emden equation based on the Lie symmetry approach.

III. SECOND DERIVATIVE MULTISTEP METHOD

In this section, we detail the properties of a second derivative multistep method (SDMM), that has introduced by Hojjati et. al [32]. Consider the stiff initial value problem

$$y'(x) = f(x, y(x)), \quad y(a) = y_0, \quad (3)$$

on the finite interval $I = [a, b]$. Let $\Delta = \{a = x_0 < x_1 < \dots < x_n = b\}$ be a partition of $[a, b]$ with step size h , that is

$$x_{i+1} - x_i = h, i = 0, \dots, n - 1.$$

In (3), $y : [x_0, x_N] \rightarrow \mathbb{R}^m$ and $f : [x_0, x_N] \times \mathbb{R}^m \rightarrow \mathbb{R}^m$ are continuous.

A potentially good numerical method for the solution of stiff systems of ODEs must have good accuracy and some reasonably wide region of absolute stability. A-stability requirement puts a severe limitation on the choice of a suitable methods for stiff problems. One of the main directions of search for higher order A-stable multistep methods is using high order derivatives of the solutions. By applying second derivative of solution in algorithm of linear multistep methods, a new class of methods has been introduced. These methods are known as second derivative multistep methods (SDMM).

The general SDMM can be written in the form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \sum_{j=0}^k \beta_j f_{n+j} + h^2 \sum_{j=0}^k \gamma_j g_{n+j}$$

where $\alpha_j, \beta_j, \gamma_j$ are parameters to be determined and $g_{n+j} = f_{n+j}^{(1)}$.

The new SDMM, takes the following general form

$$\sum_{j=0}^k \hat{\alpha}_j y_{n+j} = h \hat{\beta}_k f_{n+k} + h^2 (\hat{\gamma}_k g_{n+k} - \hat{\gamma}_{k+1} g_{n+k+1}), \quad (4)$$

where $\hat{\alpha}_k = 1$ and the other coefficients are chosen so that equation (4) has order $k+2$. The coefficients of k -step methods of class (4) are given in [32]. This method has equipped by super-future point technique (using y_{n+k+1}) and uses second derivative of solutions. Assuming that the solution values $y_n, y_{n+1}, \dots, y_{n+k-1}$ are available, the way in which (4) is used in practice is applying the predictor that has following form

$$\sum_{j=0}^k \alpha_j y_{n+j} = h \beta_k f_{n+k} + h^2 \gamma_k g_{n+k}, \quad (5)$$

where $\alpha_k = 1$ and the other coefficients are chosen so that equation (5) has order $k+1$. The coefficients of k -step methods of class (5) are given in [32].

The SDMM approach [32] goes as follows:

Step 1. Use the predictor (5) to compute the first predictor \bar{y}_{n+k} , assuming that approximate solutions y_{n+j} have been computed at x_{n+j} , for $0 \leq j \leq k - 1$,

$$\bar{y}_{n+k} + \sum_{j=0}^{k-1} \alpha_j y_{n+j} = h \beta_k f(x_{n+k}, \bar{y}_{n+k}) + h^2 \gamma_k g(x_{n+k}, \bar{y}_{n+k}).$$

Step 2. Use the predictor (5) to compute the second predictor \bar{y}_{n+k+1} ,

$$\bar{y}_{n+k+1} + \bar{\alpha}_{k-1} y_{n+k} + \sum_{j=0}^{k-2} \alpha_j y_{n+j} = h \beta_k f(x_{n+k+1}, \bar{y}_{n+k+1}) + h^2 \gamma_k g(x_{n+k+1}, \bar{y}_{n+k+1}).$$

Step 3. Evaluate $\bar{g}_{n+k+1} = g(x_{n+k+1}, \bar{y}_{n+k+1})$.

Step 4. Compute y_{n+k} as the solution of

$$\sum_{j=0}^k \hat{\alpha}_j y_{n+j} = h \hat{\beta}_k f_{n+k} + h^2 (\hat{\gamma}_k g_{n+k} - \hat{\gamma}_{k+1} \bar{g}_{n+k+1}).$$

TABLE I
 A(α)-STABILITY OF NEW SDMM

k	p	α	k	p	α
1	3	90°	7	9	85.3°
2	4	90°	8	10	80.5°
3	5	90°	9	11	73.5°
4	6	90°	10	12	61.9°
5	7	89.8°	11	13	50.3°
6	8	88.3°	12	14	29.9°

The method is a k -step method of order $k + 2$. It is A-stable up to order 6. The corresponding (approximate) regions of A(α)-stability are given in Table I. For more details see [32].

IV. NUMERICAL RESULTS

In this section, we apply the new SDMM for numerical solution of some ordinary differential equations of Lane-Emden type. In general, the Lane-Emden type equations are formulated as

$$y''(x) + \frac{\alpha}{x}y(x) + f(x)g(y) = h(x), \quad \alpha x \geq 0, \quad (6)$$

with initial conditions

$$\begin{aligned} a. \quad & y(0) = A, \\ b. \quad & y'(0) = B, \end{aligned} \quad (7)$$

where α , A and B are real constants and $f(x)$, $g(y)$ and $h(x)$ are some given functions. For other special forms of $g(y)$, the well-known Lane-Emden equations were used to model several phenomena in mathematical physics and astrophysics such as the theory of stellar structure, the thermal behavior of a spherical cloud of gas, isothermal gas spheres and the theory of thermionic currents [9], [51]. Here we consider various $f(x)$, $g(y)$, A and B , in two cases homogeneous ($h(x) = 0$) and non-homogeneous ($h(x) \neq 0$).

A possible way of solving problem near $x = 0$ is to find a series solution $y = 1 + a_2x^2 + a_4x^4 + \dots$. Substitution this into the equation the coefficients can be found in turn. This series will work if x is small enough. When we have enough values for $x > 0$, we start off an ordinary initial value solver.

Example 1. (The standard Lane-Emden equation)

For $f(x) = 1$, $g(y) = y^m$, $A = 1$ and $B = 0$, Eq. (6) is the standard Lane-Emden equation that was used to model the thermal behavior of a spherical cloud of gas acting under the mutual attraction of its molecules and subject to the classical laws of thermodynamics [13], [54].

$$y''(x) + \frac{2}{x}y'(x) + y^m(x) = 0, \quad x \geq 0, \quad (8)$$

subject to the initial conditions

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned}$$

TABLE II

COMPARISON THE FIRST ZERO OF y , BETWEEN BENDER AND PRESENT METHOD FOR $m = 2, 3, 4$

m	Present method	Bender	Exact value
2	4.3528	4.3603	4.35287460
3	6.8960	7.0521	6.89684862
4	14.9710	17.967	14.9715463

TABLE III

APPROXIMATION OF $y(x)$ FOR $m = 3$ USING PRESENT METHOD

x	Present method	Horedt
0.5	0.959839069882982	0.959839
1.0	0.855057568546280	0.855058
5.0	0.110819835159975	0.110820
6.0	0.043737983910256	0.043738
6.896	0.000036011165997	0.000036

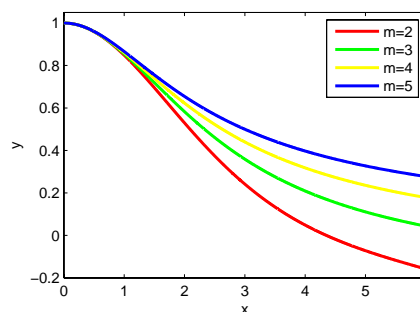


Fig. 1. The results of Ex. 1

where $m \geq 0$ is constant. Substituting $m = 0, 1$ and 5 into Eq. (8) leads to the exact solution

$$\begin{aligned} y(x) &= 1 - \frac{1}{3!}x^2, \\ y(x) &= \frac{\sin(x)}{x}, \\ y(x) &= \left(1 + \frac{x^2}{3}\right)^{-1/2}, \end{aligned}$$

respectively. In other cases there aren't any analytic exact solutions. We apply the SDMM to solve the standard Lane-Emden Eq. (8) for $m = 2, 3, 4$ and 5 . In Table II, we calculate the first zero of y of the Lane-Emden equation in the cases $m = 2, 3, 4$ and compare it with the results of Bender [3].

In Table III, the approximation values of y obtained from SDMM are compared with the results of Horedt [33].

In Fig. 1, we plot the results of applying SDMM on Lane-Emden equation in the cases $m = 2, 3, 4, 5$.

Example 2. The white-dwarf equation

In this model we consider the "white-dwarf" equation

$$y'' + \frac{2}{x}y' + (y^2 - C)^{3/2} = 0, \quad (9)$$

introduced by Davis [13] and Chandrasekhar [9] in his study of the gravitational potential of the degenerate white-dwarf stars. The initial conditions of (9) are

$$y(0) = 1, y'(0) = 0.$$

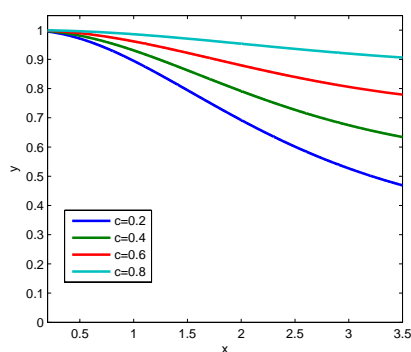


Fig. 2. The results of Ex. 2

It is clear that (9) is of Lane-Emden type where $f(y) = (y^2 - C)^{3/2}$. If $C = 0$, (9) reduces to Lane-Emden equation of index $m = 3$. Fig. 2 shows the approximation plot of $y(x)$ for few values of C .

Example 3. For $f(x) = 1, g(y) = 4(2e^y + e^{y/2}), A = 0$ and $B = 0$, Eq. (6) will be one of the Lane-Emden type equations that is to solve.

$$y''(x) + \frac{2}{x}y'(x) + 4(2e^y + e^{y/2}) = 0, \quad x \geq 0, \quad (10)$$

subject to the initial conditions

$$\begin{aligned} y(0) &= 0, \\ y'(0) &= 0, \end{aligned}$$

which has the following analytical solution:

$$y(x) = -2 \ln(1 + x^2). \quad (11)$$

We solved this problem by using SDMM and reported the results in Table IV.

TABLE IV
 THE RESULTS OF EX. 3

x	Exact solution	Present method	Absolute Error
0.25	-1.21249243632870E-1	-1.21249243632901E-1	3.09E-14
0.50	-4.46287102628420E-1	-4.46287102628442E-1	2.29E-14
0.75	-8.92574205256839E-1	-8.92574205256853E-1	1.42E-14
1.00	-1.38629436111989E+0	-1.38629436111990E+0	6.88E-15

Example 4. For $f(x) = 1, g(y) = -6y - 4y \ln(y), A = 1$ and $B = 0$, Eq. (6) will be one of the Lane-Emden type equations that is:

$$y''(x) + \frac{2}{x}y'(x) - 6y(x) = 4y(x) \ln(y(x)), \quad x \geq 0, \quad (12)$$

subject to the initial conditions

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned}$$

which has the following analytical solution:

$$y(x) = e^{-x^2}. \quad (13)$$

The obtained results of applying SDMM to this problem are reported in Table V.

TABLE V
 THE RESULTS OF EX. 4

x	Exact solution	Present method	Absolute Error
0.25	1.06449445891786E+0	1.06449445891767E+0	1.84E-13
0.50	1.28402541668774E+0	1.28402541668749E+0	2.48E-13
0.75	1.75505465696030E+0	1.75505465695991E+0	3.92E-13
1.00	2.71828182845904E+0	2.71828182845834E+0	7.09E-13

Example 5. For $f(x) = -2(2x^2 + 3), g(y) = y, A = 1$ and $B = 0$, Eq. (6) will be one of the Lane-Emden type equations that is

$$y''(x) + \frac{2}{x}y'(x) - 2(2x^2 + 3)y = 0, \quad x \geq 0, \quad (14)$$

subject to the boundary conditions

$$\begin{aligned} y(0) &= 1, \\ y'(0) &= 0, \end{aligned}$$

which has the following analytical solution:

$$y(x) = e^{-x^2}. \quad (15)$$

We apply SDMM to solve the equation (14) and report the results in Table VI.

TABLE VI
 THE RESULTS OF EX. 5

x	Exact solution	Present method	Absolute Error
0.25	1.06449445891786E+0	1.06449445891768E+0	1.77E-13
0.50	1.28402541668774E+0	1.28402541668753E+0	2.14E-13
0.75	1.75505465696030E+0	1.75505465696000E+0	2.93E-13
1.00	2.71828182845904E+0	2.71828182845859E+0	4.54E-13

V. CONCLUSION

We applied an efficient class of methods for solving Lane-Emden type equations. The singularity of the equations at $x = 0$ causes that in beginning of the integration interval we deal with a stiff problem. Stability properties of the mentioned methods let us to overcome this difficulty and get reasonable results.

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