World Academy of Science, Engineering and Technology
International Journal of Aerospace and Mechanical Engineering
Vol:4, No:9, 2010

4D Flight Trajectory Optimization Based on Pseudospectral Methods

Kouamana Bousson, Paulo Machado,

Abstract—The optimization and control problem for 4D trajectories is a subject rarely addressed in literature. In the 4D navigation problem we define waypoints, for each mission, where the arrival time is specified in each of them. One way to design trajectories for achieving this kind of mission is to use the trajectory optimization concepts. To solve a trajectory optimization problem we can use the indirect or direct methods. The indirect methods are based on maximum principle of Pontryagin, on the other hand, in the direct methods it is necessary to transform into a nonlinear programming problem. We propose an approach based on direct methods with a pseudospectral integration scheme built on Chebyshev polynomials.

Keywords—Pseudospectral Methods, Trajectory Optimization, 4D Trajectories

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

I. Introduction

THE trajectory optimization was always a important area in aeronautic and aerospace industries. This methodology allows generating, for a vehicle, a trajectory that can take into account fuel consumption, specified final time and many others requirements. The common applications of trajectory optimization in robotics, space applications is in minimum time problem and may be with position constraints. Few works addressed trajectories defined by waypoints. One of these [1] intends to design trajectories in minimum time and with waypoints constraints. The current work proposes a new approach to design 4D trajectories defined by waypoints. The difference between 3D trajectories and 4D trajectories is that in each waypoint the time is also specified for 4D trajectories. Although the 4D trajectories have been used in the flight planning of civil aircraft and in avoidance collision of Air Traffic Control facilities, there is no work that specifies a systematic method for to design this kind of trajectories.

The trajectory optimization can be solved by optimal control techniques. There are two ways to resolve optimal control problems, by direct and indirect methods [2]. The indirect methods are based on Pontryagin maximum that transforms the optimal control problem into Euler-Lagrange equations. On the other hand, the direct methods transform the optimal control problem into a nonlinear programming problems. In the current work we address only the direct methods to solve 4D trajectory optimization problem.

To transform an optimal control problem into nonlinear programming problem it is needed to parameterize the state and the control [3]. These parameterization techniques have an important role in a convergence and accuracy of the solution. The most known techniques are based on local integration schemes (Collocation Methods) [4]–[6]. Recently some works have been presented another way of state parameterization, these methods are known as Pseudospectral Methods [7]–[9].

K. Bousson is with the Department of Aerospace Sciences, University of Beira Interior, Covilhã, Portugal, e-mail: k1bousson@yahoo.com
P. Machado is PhD Student in Department of Aerospace Sciences, University of Beira Interior, Covilhã, Portugal, e-mail: pffmachado@gmail.com

II. Problem Statement

The main goal of guidance applied to navigation is to provide a reference Velocity V_{ref}, Path Angle γ_{ref} and Heading ψ_{ref} to enable the aircraft go through predefined sequence of waypoints $P_0, P_1, \ldots, P_N$.

Most of the approaches consider the waypoints defined by tri-dimensional positions $P_k = (\lambda_k, \varphi_k, h_k),\ k = 0, 1, ..., N$ and do not take into account the time. We redefined this approach and added the time restriction to the waypoint $P_k = (\lambda_k, \varphi_k, h_k, \tau_k),\ k = 0, 1..., N$.

The problem to be solved is to guide the aircraft to navigate along a specified sequence of waypoints $P_k = (\lambda_k, \varphi_k, h_k, \tau_k),\ k = 0, 1..., N$, minimizing the arrival delay at each waypoint.

III. Proposed Method

Optimal control consists in finding a control vector $u(\tau)$ that minimize a cost functional defined as:

$$J(u(\tau)) = \Phi(\tau_f, x(\tau_f), u(\tau_f), p) + \int_{\tau_0}^{\tau_f} L(\tau, x(\tau), u(\tau), p) d\tau \tag{1}$$

where $x \in \Re^n$ is the state vector, p the associate parameters, τ the time and $u \in \Re^m$ the control vectrol, subject to the following constraints:

$$\begin{gathered} \dot{x} = f(\tau, x(\tau), u(\tau), p) \\ \tau_0 \leq \tau \leq \tau_f \\ p_{min} \leq p \leq p_{max} \end{gathered} \tag{2}$$

The cost functional (1), can also be subject to equality and inequality constraints:

$$h(\tau, x(\tau), u(\tau), p) = 0 \tag{3}$$

$$g(\tau, x(\tau), u(\tau), p) \leq 0 \tag{4}$$

as well as to boundary conditions:

$$\Psi(x(\tau_0), x(\tau_f)) = 0 \tag{5}$$

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

A. Pseudospectral

The pseudospectral methods have been developed for direct methods [7], [8]. The main goal is to find the optimal trajectories of the nonlinear systems of high order. Lagrange and Chebychev polynomials are used in these methods to approximate the state variables. The procedure for approximating the state and control variables is based on Legendre polynomials built on Chebychev nodes. Consider $N+1$ the number of nodes that define some trajectory, the Chebychev polynomials are defined in trigonometric form as

$$T_N(t) = \cos(N\cos^{-1}(t)) \tag{6}$$

The nodes $t_k = cosk\pi/N$ are defined on the interval $t \in [1,1]$ with $k = 0,1,,N$. Now, we consider the following Lagrange polynomials built on Chebychev nodes as discussed above,

$$\begin{aligned}\varphi_k(t) &= \frac{(-1)^{k+1}(1-t^2)\dot{T}_N(t)}{c_k N^2(t-t_k)} \\ &= \frac{2}{Nc_k}\sum_{l=0}^{N}\frac{T_l(t_k)T_l(t)}{c_l}\end{aligned} \tag{7}$$

with,

$$c_k = \begin{cases} 2 & if \quad k = 0 \vee k = N \\ 1 & if \quad k = 1,\ldots,N-1 \end{cases} \tag{8}$$

It can be noticed that each Lagrange polynomial is such that,

$$\varphi_k(t_l) = \delta_{kl} = \begin{cases} 1 & if \quad k = l \\ 0 & if \quad k \neq l \end{cases} \tag{9}$$

In this method, the state equation is approximated by polynomials only computed in nodes, then the state equation can be rewritten in the following form:

$$\dot{x}_i^N(t_k) = \sum_{l=0}^{N} C_{kl}x_i(t_l) \tag{10}$$

where,

$$\frac{dC_l}{dt}(t_k) = C_{kl} = \begin{cases} \frac{(2N^2+1)}{6} & k = l = 0 \\ -\frac{(2N^2+1)}{6} & k = l = N \\ -\frac{t_l}{2(1-t_l^2)} & 1 \leq k = l \leq N-1 \\ \frac{c_k(-1)^{k+l}}{c_l(t_k-t_l)} & k \neq l \end{cases} \tag{11}$$

and the parameterization of the state and control variables can be defined in the following way,

$$x_i^N(t) = \sum_{k=0}^{N} x_i(t_k)\varphi_k(t) \tag{12}$$

$$u_j^N(t) = \sum_{k=0}^{N} u_j(t_k)\varphi_k(t) \tag{13}$$

with n and m as been the number of state and control variables respectively. Consider y as all set variables of nonlinear programming,

$$y = [x_{il}, u_{jl}, \tau_0, \tau_f] \tag{14}$$

with $i = 1,\ldots,n$, $j = 1,\ldots,m$ and $l = 0,\ldots,N$.

Because the problem of the optimal control is formulated over the time interval $[\tau_0, \tau_f]$ and the Chebychev nodes are defined in $[1,1]$ interval, we need the following transformation,

$$\tau = \frac{(\tau_f - \tau_0)t + (\tau_f + \tau_0)}{2} \tag{15}$$

then we can formulate the optimal control problem using the Chebychev pseudospectral method,

$$\begin{gathered} \min_{u\in\Re} J(u(\tau)) \\ subject\,to \\ \left(\frac{\tau_f-\tau_0}{2}\right) f(x_{il},u_{jl}) - \sum_{l=0}^{N} C_{kl}x_i(t_l) = 0 \\ \Psi_0(x_{0l},\tau_0) = 0 \\ \Psi_f(x_{Nl},\tau_f) = 0 \\ g(y) \leq 0 \end{gathered} \tag{16}$$

B. Modeling 4D Navigation Problem

1) Problem Formulation: Let us consider the point P defined in the geodetic reference λ, φ, h and a set of points $P_i = P_0, P_1, \ldots, P_n$ in that n is the number of the points. If we consider the arrival time in each point τ_i then the $4D$ waypoint can be defined in the following way,

$$(P_i, \tau_i) \rightarrow (P_i, [\tau_i^1, \tau_i^2]) \tag{17}$$

where $[\tau_i^1, \tau_i^2]$ is the time interval tolerated for arrive at a determined waypoint. If $s(\tau)$ is the current position, the $4D$ navigation problem is solved by the next equation,

$$||P_i - s(\tau_i)|| \leq \epsilon \quad \forall i \in \Re^n \tag{18}$$

where ϵ is the tolerance distance that an aircraft can be over fly a waypoint.

2) Dynamic Model: The following differential equations are the dynamic model used to modeling the problem,

$$\dot{\lambda} = \frac{V\cos\gamma\cos\psi}{(h+Re)\cos\varphi} \tag{19}$$

$$\dot{\varphi} = \frac{V\cos\gamma\sin\psi}{(h+Re)} \tag{20}$$

$$\dot{h} = V\sin\gamma \tag{21}$$

$$\dot{V} = u_1 \tag{22}$$

$$\dot{\gamma} = u_2 \tag{23}$$

$$\dot{\psi} = u_3 \tag{24}$$

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

where λ, φ and h are the longitude, latitude and altitude respectively, the V, γ and ψ are the velocity, path angle and heading respectively and Earth radius Re . The state and control vector are composed by $x = [\lambda, \varphi, h, V, \gamma, \psi]$ and $u = [u_1, u_2, u_3]$ respectively. We will only use the navigation equations above. Consider the following constraints:

$$u_i \in [u_i^{min}, u_i^{max}] \; ; \; i = 0, \ldots, 3 \quad (25)$$
$$\dot{V} \in [V^{min}, V^{max}] \quad (26)$$
$$\dot{\gamma} \in [\gamma^{min}, \gamma^{max}] \quad (27)$$

Therefore, the trajectories generated are always within the aerodynamic and structural limits.

C. Resolution

The 4D trajectory optimization problem above can be re-stated as:

$$\min_u J(u) = ||P_f - s(\tau_f)||_Q^2 = (P_f - s(\tau_f))^T Q (P_f - s(\tau_f))$$

$$subject\,to$$

$$\begin{gathered} \dot{x} = f(x,u) = 0 \\ ||P_i - s(\tau_i)|| \leq \epsilon \\ u_i^{min} \leq u_i \leq u_i^{max} \\ V^{min} \leq V \leq V^{max} \\ \gamma^{min} \leq \gamma \leq \gamma^{max} \end{gathered} \quad (28)$$

where J is the cost functional, τ_f is the desired flight duration of the trajectory, Q is real symmetric positive definite matrix and x is the state vector. In current paper the pseudo-spectral methods are chosen to solve the optimal control problem. The $4D$ navigation problem re-stated using the pseudospectral technique explained above takes the next shape:

$$\min_y J(y) = ||(\lambda_M, \varphi_M, h_M) - (\lambda(t_N), \varphi(t_N), h(t_N)||_Q^2$$

$$subject\,to$$

$$\begin{gathered} \left(\frac{\tau_f - \tau_0}{2}\right) f(x_k, u_k) - \sum_{l=0}^{N} C_{kl} x(t_l) = 0 \\ ||P_i - s(t_i)|| \leq \epsilon \\ u_j^{min} \leq u_j \leq u_j^{max} \\ V^{min} \leq V \leq V^{max} \\ \gamma^{min} \leq \gamma \leq \gamma^{max} \end{gathered} \quad (29)$$

with $k = 0, 1, \ldots, N$, t_i are the Chebychev nodes corresponded to the waypoints and $i = 0, 1, \ldots, M$ where M is number of waypoints and C_{kl} as defined in equation (11).

IV. Application

In this section are presented two applications, the first is a typical commercial flight and the second mission is a flight in circuit. In both applications the air vehicle used is the UAV Skygu@rdian constructed in University of Beira Interior. In both situations were applied the Pseudospectral method and a Collocation method with trapezoidal integration scheme. For the solution search of the problem, it was used the *fmincon* function of optimization toolbox of MatLab for solve the nonlinear programming problem. The computer used for test and simulation was an *Acer Aspire 1690* with 2.0 GHz CPU and 1GB of RAM.

TABLE I
List of Waypoints for the Straight Mission

	λ	φ	h	τ
N°	[deg min sec]	[deg min sec]	[m]	[hour]
1	7° 29' 35.00" W	39° 49' 25.71" N	400	0.000
2	7° 29' 37.00" W	39° 50' 34.82" N	500	0.035
3	7° 29' 39.00" W	39° 51' 33.38" N	600	0.070
4	7° 29' 41.00" W	39° 52' 39.86" N	600	0.080
5	7° 29' 41.50" W	39° 54' 50.26" N	700	0.120
6	7° 29' 42.00" W	39° 56' 55.38" N	800	0.165
7	7° 29' 45.00" W	39° 59' 15.04" N	800	0.210
8	7° 29' 47.00" W	40° 01' 17.12" N	800	0.245
9	7° 29' 49.00" W	40° 03' 45.92" N	800	0.280
10	7° 29' 51.00" W	40° 05' 31.38" N	800	0.325
11	7° 29' 53.00" W	40° 08' 12.56" N	800	0.370
12	7° 29' 55.00" W	40° 11' 06.43" N	750	0.415
13	7° 30' 00.00" W	40° 14' 07.43" N	650	0.450
14	7° 30' 02.00" W	40° 17' 02.02" N	600	0.480

TABLE II
Results of Final Values In Straight Mission

	$\lambda(\tau_f)$	$\varphi(\tau_f)$	h(τ_f)	J
	[rad]	[rad]	[km]	–
Collocation	-0.1309	0.7031	0.5998	2.8×10^{-8}
Pseudospectral	-0.1309	0.6999	0.5058	3.7×10^{-4}

A. Example I

The first example is a straight flight typically of civil flights. The table I show the waypoints list. Each waypoint is defined in geodetic coordinates (λ, φ, h) and must be specified the desired time (τ) to reach it. In this specific mission, both methods of parameterization achieve a solution for the problem, the final values of position as well as the cost function are represented in table II.

When applied the Collocation technique, we consider the nodes coincident with the waypoints, this is valid because the time was expressed in hours and the difference between waypoints is small that allows an acceptable step of integration. We tried some distributions of collocation nodes but finally we found that satisfactory results could only be achieved with sufficiently high number of nodes.

As in Pseudospectral method the nodes are specified on Chebychev nodes in $[-1, 1]$ interval, the problem with node placement does not arise. We considered 20 nodes for this example because the method converged accurately using this number of nodes, given practically the same result as when higher numbers of nodes are used.

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

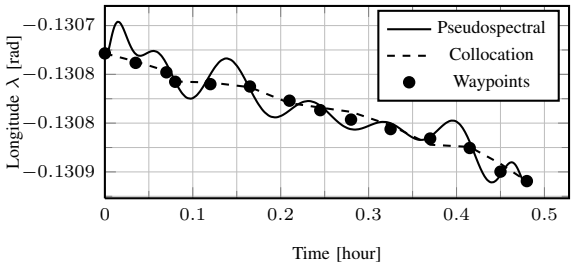


Fig. 1. Longitude vs Time for Example I

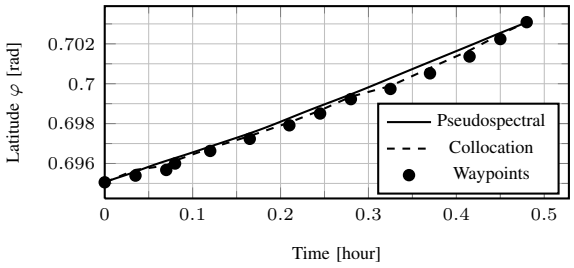


Fig. 2. Latitude vs Time for Example I

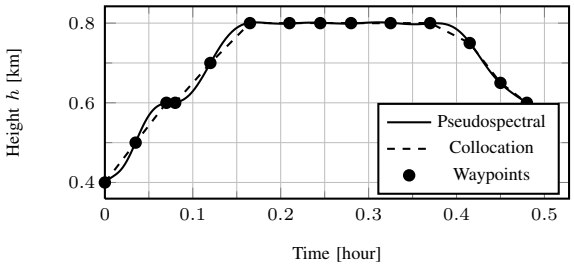


Fig. 3. Height vs Time for Example I

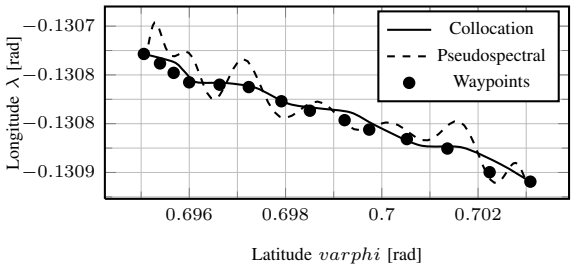


Fig. 4. Height vs Time for Example I

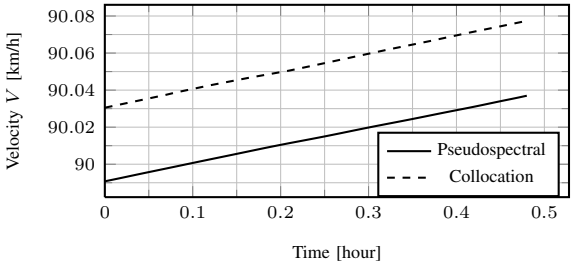


Fig. 5. Velocity vs Time for Example I

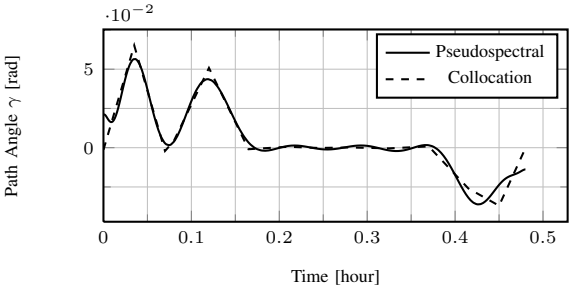


Fig. 6. Path Angle vs Time for Example I

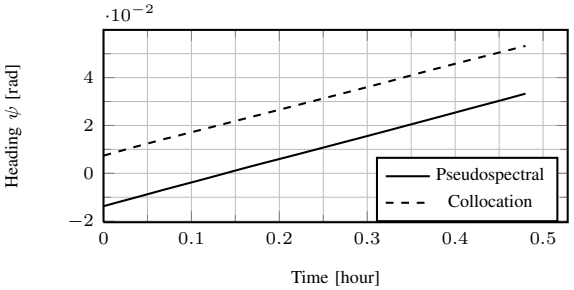


Fig. 7. Heading vs Time for Example I

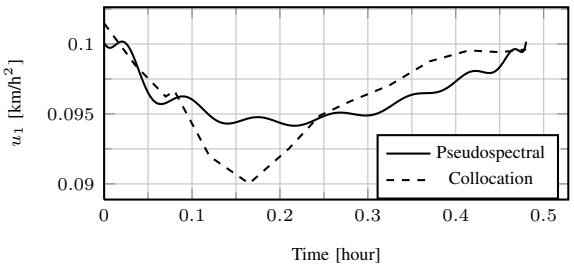


Fig. 8. Control$_1$ vs Time for Example I

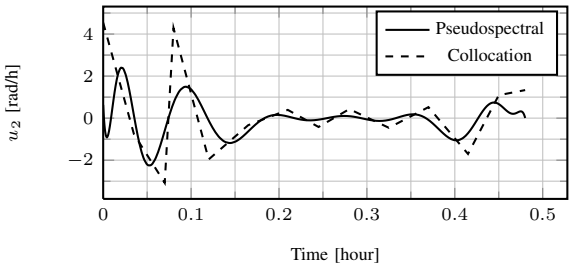


Fig. 9. Control$_2$ vs Time for Example I

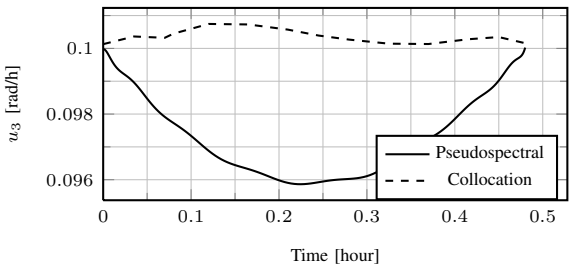


Fig. 10. Control$_3$ vs Time for Example I

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

TABLE III
LIST OF WAYPOINTS FOR THE CIRCUIT MISSION

	λ	φ	h	τ
N°	[deg min sec]	[deg min sec]	[m]	[hour]
1	7° 28' 45.66" W	40° 15' 54.29" N	700	0.000
2	7° 29' 37.84" W	40° 15' 55.50" N	750	0.014
3	7° 30' 28.55" W	40° 15' 57.79" N	800	0.023
4	7° 31' 36.32" W	40° 16' 38.79" N	1100	0.051
5	7° 32' 05.08" W	40° 17' 35.86" N	1500	0.072
6	7° 31' 27.13" W	40° 18' 19.47" N	1350	0.085
7	7° 30' 32.76" W	40° 18' 45.15" N	1250	0.097
8	7° 29' 45.49" W	40° 18' 58.96" N	1150	0.120
9	7° 28' 32.44" W	40° 19' 05.28" N	1000	0.136
10	7° 27' 24.73" W	40° 18' 53.08" N	850	0.157
11	7° 26' 56.44" W	40° 17' 52.36" N	810	0.175
12	7° 27' 09.48" W	40° 16' 45.91" N	760	0.193
13	7° 27' 37.41" W	40° 16' 14.20" N	730	0.213
14	7° 28' 14.22" W	40° 15' 59.65" N	710	0.221
15	7° 28' 45.66" W	40° 15' 54.29" N	700	0.232

It is possible to see in figures 1,2,3 the trajectories found by both methods representing the longitude, latitude and altitude respectively. The figures 5, 6, 7 represent the velocity, path angle and heading respectively, and figures 8, 9, 10 represent the controls. n figure 4 is represented the ground track.

The velocity in Fig. 5 shows a constant behavior because the arrival times at each waypoint were defined as such, the control u_1, that is, the variation of velocity, is the depicted in (Fig. 8), which is practically identical in both methods. The path angle (Fig. 6) generated by pseudospectral method is slightly smoother than the path angle generated by collocation method, but the differences can be seen better in Fig. 9 that represents the variation of the path angle (u_2), here the pseudospectral method shows a trajectory with more smoothness than the collocation method. The heading trajectory (Fig. 7) and control u_3 (Fig. 10) representing the variation of heading do not present significantly differences between the two methods.

This result shows that the pseudospectral method achieves a final solution almost equal to the collocation method but with the controls trajectories are smoother, which allows improving cost of the mission.

B. Example II

The second example proposed intends to show more features of the above methods. The mission is a round trip about Covilhã city, with waypoints described in Table III. The main difference between this mission and the last one is that the Collocation parameterization does not reach a feasible solution with the available computational resources. In Table IV are shown the final value of the position and that of the cost functional value.

For Collocation method we tried several sets of nodes, but none of these attempts achieved a feasible solution. On the other hand the pseudospectral method reached an acceptable solution. We used 25 nodes because the method converged for this number of nodes. For more than 25 nodes, the computation

TABLE IV
RESULTS OF FINAL VALUES IN CIRCUIT MISSION

	$\lambda(\tau_f)$	$\varphi(\tau_f)$	h(τ_f)	J
	[rad]	[rad]	[km]	–
Collocation	**	**	**	**
Pseudospectral	-0.1305	0.7026	0.5749	3.7×10^{-7}

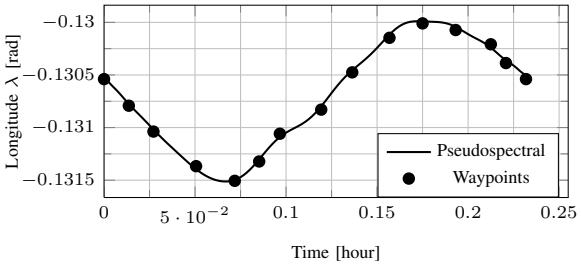


Fig. 11. Longitude vs Time for Example II

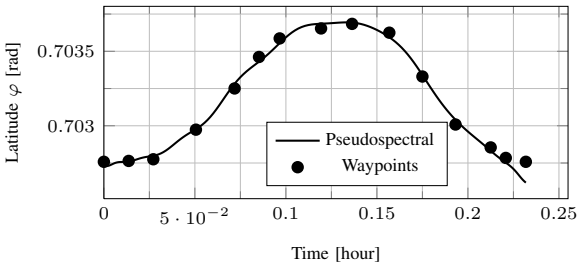


Fig. 12. Latitude vs Time for Example II

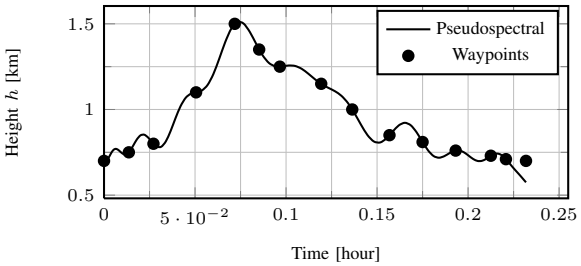


Fig. 13. Height vs Time for Example II

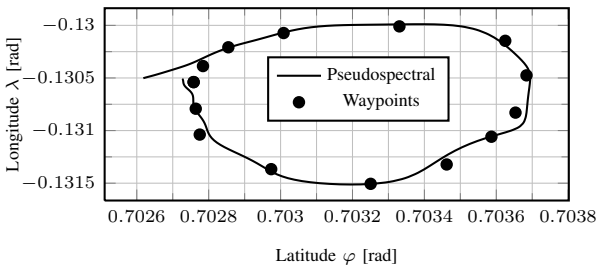


Fig. 14. Height vs Time for Example II

became very heavy in the framework of Matlab capabilities. Figures 11,12,13 representing longitude, latitude and altitude respectively. The figures 15, 16,17 represent velocity, path angle and heading respectively, and figures 18,19, 20 represent

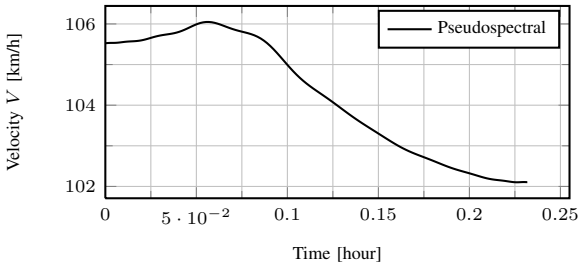


Fig. 15. Velocity vs Time for Example II

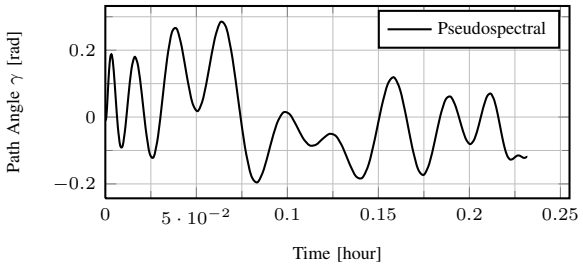


Fig. 16. Path Angle vs Time for Example II

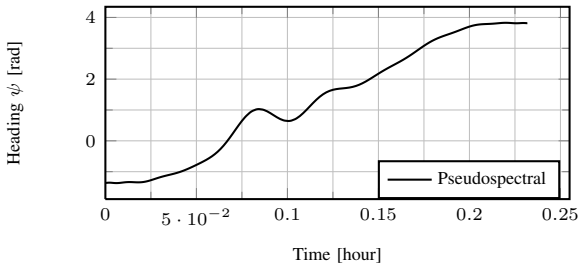


Fig. 17. Heading vs Time for Example II

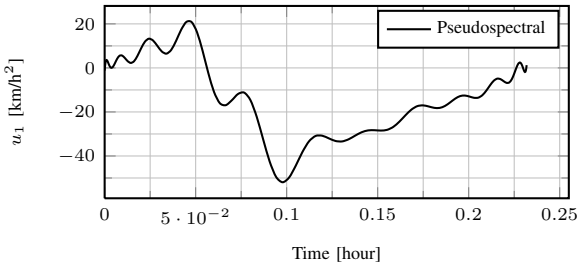


Fig. 18. Control$_1$ vs Time for Example II

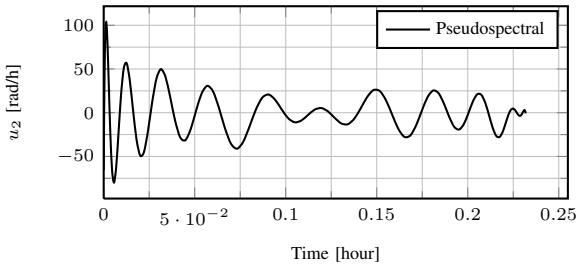


Fig. 19. Control$_2$ vs Time for Example II

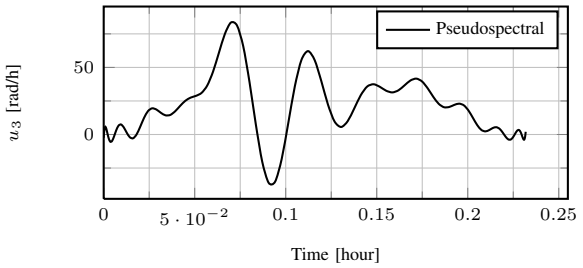


Fig. 20. Control$_3$ vs Time for Example II

the controls. The figure (14) represent the ground track. Although the cost functional, for this example, presents a low value it is not sufficient for trajectory overlap with the final waypoint, this is visible in Fig. 13 and happens because the optimization software can not refine the solution. The velocity (Fig.15) and velocity variation (Fig.18) have an almost constant behavior, this fact was expected because, similary to example I, the time arrival in waypoints was specified with this intention. Path angle (Fig.16) and path angle variation (Fig.19) show a behavior more oscillatory than the other variables. The increase of nodes solve this problem, nevertheless the path angle shall not exceed the aerodynamics limits of aircraft. Finally the heading (Fig.17) and its variation (Fig.20) show a behavior in accordance with the track. Also as in the first example, the pseudospectral methods give us a smoother trajectory, and with a better optimization tool than was used, the trajectory can be improved.

V. Conclusion

It was proposed a method for design to 4D optimal navigation trajectories define by waypoints. The 4D trajectories have the expected time of arrival at each waypoint in addition to the desired position. Also was proposed a Pseudospectral technique for parameterization of trajectory optimization problem built on Chebychev nodes. To examples were presented, in both cases also was applied the Collocation parameterization technique with trapezoidal integration scheme. The Pseudospectral methods achieved appropriate solutions for the two applications and the Collocation method only achieved a feasible solution for example I. Although the pseudospectral method found solutions for the two examples, what is relevant is that the controls trajectories were smoother than in the case of Collocation methods. In future work, the method presented in this paper will be improved and validated on full scale missions for unmanned aerial vehicles.

Acknowledgment

This research was conducted in the Aeronautics and Astronautics Research Center (AeroG) of the University of Beira Interior at Covilhã, Portugal, and supported by the Portuguese Foundation for Sciences and Technology (FCT).

References

[1] G. Moon and Y. Kim, "Flight path optimization passing through waypoints for autonomous flight control systems," *Engineering Optimization*, vol. 37, no. 7, pp. 755–774, October 2005.

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf

[2] J. T. Betts, "Survey of numerical methods for trajectory optimization," *Journal of Guidance, Control, and Dynamics*, vol. 21, no. 2, pp. 193–207, March-April 1998.
[3] D. G. Hull, "Conversion of optimal control problems into parameter optimization problems," *Journal of Guidance, Control, and Dynamics*, vol. 20, no. 1, pp. 57–60, January-February 1997.
[4] F. Benyah and L. S. Jennings, "Regularization for optimal control computation," Centre for Applied Dynamics and Optimization, Department of Mathematics, University of Westerm Australia, Report WA 6907, 2000.
[5] ——, "The l-curve in regulation of optimal control computation," *J. Austral. Math. Soc.*, vol. 40, pp. 138–172, September 1998.
[6] K. Bousson, "Single gridpoint dynamic programming for trajectory optimization," in *AIAA Atomospheric Flight Mechanics Conference and Exhibit*. AIAA, August 2003, pp. AIAA 2005–5902.
[7] F. Fahroo and I. M. Ross, "Costate estimation by a legendre pseudospectral method," *Journal of Guidance, Control, and Dynamics*, vol. 24, no. 2, pp. 270–277, March-April 2001.
[8] L. R. Lewis, I. M. Ross, and Q. Gong, "Pseudospectral Motion Planning Techniques for Autonomous Obstacle Avoidance," in *Proceedings of the 46th IEEE Conference on Decision and Control*, vol. 3, New Orleans, 2007, pp. 2210–2215.
[9] K. Bousson, "Chebychev pseudospectral trajectory optimization of differential inclusion models," in *SAE World Aviation Congress, 2003 Aerospace Congress and Exhibition, Montreal, Canada*. SAE, September 2003, pp. 2003–01–3044.

Kouamana Bousson is a Professor in the Department of Aerospace Sciences at the University of Beira Interior in Covilh, Portugal, since February 1995. Formerly, he was a Researcher at the LAAS Laboratory of French National Council for Scientific Research (CNRS) in Toulouse, France, from 1993 to 1995. His current research activities include trajectory optimization, optimal flight control systems, analysis and control of uncertain systems with applications to flight dynamics, electrical machines and chaotic systems. Dr. K. Bousson is Member of the Portuguese Association for Automatic Control (APCA) and Senior Member of the American Institute of Aeronautics and Astronautics (AIAA).

Paulo Machado received the M.Sc. degree in Aeronautical Engineering from University of Beira Interior, Portugal in 2008. Currently is a PhD student in the Department of Aerospace Sciences at the University of Beira Interior in Covilh, Portugal, since September 2009. His current research include optimal control, trajectory optimization applied to avoidance collision problems.

Open Science Index, Aerospace and Mechanical Engineering Vol:4, No:9, 2010 publications.waset.org/11036/pdf