Iterative methods for computing the weighted Minkowski inverses of matrices in Minkowski space

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Abstract—In this note, we consider a family of iterative formula for computing the weighted Minkowski inverses $A_{M,N}^\oplus$ in Minkowski space, and give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Keywords—iterative method, the Minkowski inverse, $A_{M,N}^\oplus$ inverse.

I. INTRODUCTION

In this paper, let $M_{m,n}$ denote the set of all $m - by - n$ complex matrices in Minkowski space. When $m = n$, $M_n$ is instead of $M_{m,n}$. Let $R(A)$, $N(A)$, $A^\dagger$ and $\sigma(A)$ stand for conjugate transpose, spectrum norm, range, null space, Moore-Penrose inverse and spectrum of matrix $A$.

In the following, we give some notations and lemmas for the Minkowski inverse in Minkowski space.

Let $G$ be the Minkowski metric tensor defined by

$$G_u = (u_0, -u_1, -u_2, \ldots, -u_n).$$

where $u \in C^n$ is an element of the space of complex $n$-tuples.

For $G \in M_n$, it defined by

$$G = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & I_{n-1} \end{pmatrix} : G^* = G; \text{ and } G^2 = I_n.$$ (1)

For $A \in M_{m,n}$ and $y \in C^n$ in Minkowski space $\mu$, using (1) we define the Minkowski conjugate matrix $A^\sim$ of $A$ as follow

$$(Ax, y) = [Ax, Gy] = [x, A^\dagger Gy] = [x, G(AA^\dagger G)y] = [x, GA^\sim y] = (x, A^\sim y)$$ (2)

where $A^\sim = GA^\dagger G$ (see [4]).

Definition 1[4, Definition 2] For $A \in M_{m,n}$ in Minkowski space $\mu$, the Minkowski conjugate matrix $A^\sim$ of $A$ is defined as

$$A^\sim = G_1A^\dagger G_2$$ (3)

where $G_1, G_2$ are Minkowski metric matrices of $n \times n$ and $m \times m$, respectively. Obviously, (see [4]) if $A, B \in M_{m,n}$ and

$$C \in M_{n,m},$$

then

$$(A + B)^\sim = A^\sim + B^\sim,$$

$$(AC)^\sim = C^\sim A^\sim,$$

$$(A^\sim)^\sim = A^\sim,$$

$$(A^\sim)^\dagger = (A^\dagger)^\sim.$$ (4)

Analogous to Moore-Penrose inverse of $A$, we give the following definition of the Minkowski space $A_{M,N}^\oplus$ of $A$.

Definition 2 [4, Definition 1] Let $A \in M_{m,n}$, $M \in M_m$ and $N \in M_n$ be positive definite matrices. If there exists $B$ such that

$$ABA = A, BAB = B,$$

$MAB$ and $NBA$ are $M$-symmetric.

then $B$ is the weighted Minkowski inverse of $A$ (denoted by $A_{M,N}^\oplus$). When $M = I_m$ and $N = I_n$, $A_{M,N}^\oplus$ reduces to the Minkowski inverse and denoted by $A_{M}^\oplus$.

Lemma 1 [4, Lemma 5] Let $A \in M_{m,n}$ be a matrix in $\mu$, and let $M \in M_m$ and $N \in M_n$ be positive definite matrices. Then

$$A_{M,N}^\oplus = (A^\dag)^{-1}A^\sim$$ (5)

where $A^\sim = N^{-1}G_1A^\dagger G_2M$ and $A^\sim = A^\sim A|_{R(A^\sim)}$ is the restriction of $A^\sim A$ on $R(A^\sim)$.

II. CONCLUSION

In this section, we will use a family of iterative formula which be defined in [3] for computing the Minkowski inverse $A_{M,N}^\oplus$ in Minkowski space. And also give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Theorem 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \subset C^{n \times m}$ as follow

$$X_{k+1} = X_k(3I - 3AX_k + (AX_k)^2)$$ (6)

and if we take $X_0 = Y \subset C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1$$ (7)

then the sequence (6) converges to $A_{M,N}^\oplus$ if and only if

$$\rho(I - YA) < 1 \text{ (or} \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2)\frac{\|X_k\|^4}{\|A\|^4}.$$ (8)
where
\[ q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \} \]
and \( M \in M_n, N \in M_n \) be positive definite matrices, respectively.

**Proof:** Let \( e_k = Y(I - AX_k) \), by the iteration (6), we have
\[
e_{k+1} = Y(I - AX_{k+1}) = Y(I - AX_k(I - 3AX_k + (AX_k)^2)) = \cdots = Y(I - AX_0)^{q_k} = Ye_0^{q_k}
\]
i.e. \( Y = YAX_\infty \) when \( k \to \infty \). In the following, we present
\[ \lim_{k \to \infty} X_k = X_\infty. \]
**Sufficient:** From \( \rho(I - YA) < 1 \), we easily have \( \rho(I - AY) < 1 \), since
\[ \sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}. \]
We also easily prove that \( YA \) is invertible on \( R(YA) \) and \( AY \) is invertible on \( R(YA) \).

From above, we can show that
\[ X_\infty = Y(AY)^{-1}R(AY) = (YA) Y^{-1}R(AY). \]
we also get
\[
AX_\infty A = AY(AY)^{-1}R(AY) A = A, \\
X_\infty AX_\infty = Y(AY)^{-1}R(AY), \\
AY(AY)^{-1}R(AY) = Y(AY)^{-1}R(AY), \\
MAY(AY)^{-1}R(AY) = I, \\
N(YA)^{-1}R(AY) Y A = I, \\
\]
i.e. we can prove \( X_\infty = A^{\oplus}_{M,N} \). It show that (6) converges to \( A^{\oplus}_{M,N} \).
**Necessary:** If (6) converges to \( A^{\oplus}_{M,N} \). By (9), we have
\[ \rho(I - YA) < 1(\text{or } \rho(I - AY) < 1). \]
Finally we will consider the error of two adjacent iterations between \( X_{k+1} \) and \( X_k \) in the following.
\[
AA^{\oplus}_{M,N} - AX_{k+1} = AA^{\oplus}_{M,N} - AA^{\oplus}_{M,N} AX_k = A^3(A^{\oplus}_{M,N} - AX_k)^3
\]
and
\[
\| A^{\oplus}_{M,N} - X_{k+1} \| \leq \| A \|^2 \| A^{\oplus}_{M,N} - X_k \|^3 \]
From (11)-(13) we get
\[
\| X_{k+1} - X_k \| \leq \| X_{k+1} - A^{\oplus}_{M,N} + A^{\oplus}_{M,N} - X_k \|
\leq \| A^{\oplus}_{M,N} - X_{k+1} \| + \| A^{\oplus}_{M,N} - X_k \|
\leq (1 + \| A \|^2) \| A^{\oplus}_{M,N} - X_k \|
\leq (1 + \| A \|^2) q_k \| A \|
\]
By (14), it prove that (16) holds.

**Corollary 1** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\}_{k=0}^\infty \in C^{n \times m} \) as (6) and if we take \( X_0 = Y \in C^{n \times m} \) and \( Y \neq YAX_0 \) such that
\[ \| e_0 \| = \| I - AX_0 \| < 1 \]
then the sequence (6) converges to \( A^{\oplus} \) if and only if
\[ \rho(I - YA) < 1(\text{or } \rho(I - AY) < 1). \]
Furthermore, we have
\[ \| X_{k+1} - X_k \| \leq (1 + \| A \|^2) q_k \| A \| \]
where \( q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \} \).

**Theorem 2** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\}_{k=0}^\infty \in C^{n \times m} \) as follow,
\[
X_{k+1} = X_k[mI - \frac{m(m-1)}{2}AX_k + \cdots + (-AX_k)^m-1],
\]
and if we take \( X_0 = G_1A^*G_2 \in C^{n \times m} \) such that
\[ \| e_0 \| = \| I - AX_0 \| < 1 \]
then (17) converge to \( A^{\oplus}_{M,N} \) if and only
\[ \rho(I - YA) < 1(\text{or } \rho(I - AY) < 1). \]
Furthermore, we have
\[ \| X_{k+1} - X_k \| \leq (1 + \| A \|^2) q_k \| A \|^m \]
where
\[ q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \} \]
and \( M \in M_n, N \in M_n \) be positive definite matrices, respectively.

In the following, we will consider another the iterative formula for computing the weighted Minkowski inverse \( A^{\oplus}_{M,N} \) in Minkowski space.

**Theorem 3** Let \( A \in M_{m,n} \), define the sequence \( \{X_k\} \in C^{n \times m} \) as
\[
X_k = X_{k-1} + \beta Y(Iy - AX_{k-1}),
\]
\[ \beta \in C \setminus \{0\}, k = 1, 2, \ldots. \]
and if we take $X_0 = Y \in \mathbb{C}^{n \times m}$ and $Y \neq YA$ such that

$$\|e_0\| = \|I - AX_0\| < 1$$

then (20) converges to $A^{\bigoplus}_{M,N}$ if and only if

$$\rho(I - YA) < 1 \text{ (or } \rho(I - AY) < 1).$$

Furthermore,

$$\|X_{k+1} - X_k\| \leq q^k \|Y\| \|I - AX_0\||(I - A)(I - AX_0)\|$$

where

$$q = \min \{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: By iteration (27), we have

$$X_{k+1} = (I_n - \beta YA)X_k + \beta Y$$

Hence

$$X_{k+1} - X_k = (I_n - \beta YA)(X_k - X_k) = \ldots = (I_n - \beta YA)^k (X - X_0)$$

$$= \beta(I_n - \beta YA)^k Y (I_m - AX_0)$$

$$= \beta Y (I_n - \beta YA)^k (I_m - AX_0)$$

From (24), we obtain

$$YA(X_k - X_0) = \beta Y A [(I_n - \beta YA)^{k-1} + \ldots + (I_n - \beta YA) + I_n] Y (I_m - AX_0)$$

Similarly, we get

$$YA(X_k - X_0) = Y [I_n - (I_n - \beta YA)^k] Y (I_m - AX_0)$$

By (25)(26), we prove that (20) converges to $A^{\bigoplus}_{M,N}$ if and only if $\rho(I - \beta YA) < 1$ (or $\rho(I - \beta AY) < 1$), respectively.

From

$$\rho(I - YA) < 1,$$

we have

$$\rho(I - AY) < 1,$$

Since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$ 

As the proof in Theorem 1, we obtain

$$X_\infty = (YA)^{-1}_{R(YA)} Y = Y(AY)^{-1}_{R(AY)} Y.$$

Let $\lim_{k \to \infty} X_k = X_\infty$ and by (25)(26), we show that

$$\lim_{k \to \infty} X_k = (YA)^{-1}_{R(YA)} Y (I_m - AX_0) + X_0$$

$$= (YA)^{-1}_{R(YA)} Y$$

Using the definition of the weighted Minkowski inverse, we obtain

$$X_\infty = (YA)^{-1}_{R(YA)} Y = Y(AY)^{-1}_{R(AY)} Y = A^{\bigoplus}_{M,N}.$$