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Iterative methods for computing the weighted Minkowski inverses of matrices in Minkowski space

Xiaoji Liu, Yonghui Qin,

Abstract—In this note, we consider a family of iterative formula for computing the weighted Minskowski inverses $A^{\bigoplus}_{M,N}$ in Minskowski space, and give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Keywords—iterative method, the Minskowski inverse, $A^{\bigoplus}_{M,N}$ inverse.

I. Introduction

In this paper, let $M_{m,n}$ denotes the set of all $m-by-n$ complex matrices in Minskowski space. When $m=n$, M_n is instead of $M_{m,n}$. Let A^*,$\|A\|$,$R(A)$,$N(A)$,$A^\dagger$ and $\sigma(A)$ stand for conjugate transpose, spectrum norm, range, null space, Moore-Penrose inverse and spectrum of matrix A.

In the following , we give some notations and lemmas for the Minskowski inverse in Minskowski space.

Let G be the Minskowski metric tensor defined by

$$Gu = (u_0, -u_1, -u_2, \cdots, -u_n). \tag{1}$$

where $u \in C^n$ is an element of the space of complex $n-$tuples.

For $G \in M_n$, it defined by

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}; \quad G^* = G; \quad \text{and} \quad G^2 = I_n. \tag{2}$$

For $A \in M_{m,n}$ x and $y \in C^n$ in Minskowski space μ, using (1) we define the Minskowski conjugate matrix $A^\sim$ of A as follow

$$\begin{aligned} (Ax, y) &= [Ax, Gy] = [x, A^*Gy] \\ &= [x, G(GA^*G)y] \\ &= [x, GA^\sim y] = (x, A^\sim y) \end{aligned} \tag{3}$$

where $A^\sim = GA^*G$ (see [4]).

Definition 1[4, Definition 2] For $A \in M_{m,n}$ in Minskowski space μ, the Minskowski conjugate matrix $A^\approx$ of A is defined as

$$A^\approx = G_1 A^* G_2 \tag{4}$$

where G_1, G_2 are Minskowski metric matrices of $n \times n$ and $m \times m$, respectively. Obviously, (see [4])if $A, B \in M_{m,n}$ and $C \in M_{n,l}$, then

$$\begin{aligned} &(A+B)^\approx = A^\approx + B^\approx, \\ &(AC)^\approx = C^\approx A^\approx, \\ &(A^\approx)^\approx = A^\approx, \\ &(A^\approx)^* = (A^*)^\approx. \end{aligned}$$

Analogous to Moore-Penrose inverse of A, we give the following definition of the Minskowski space $A^{\bigoplus}_{M,N}$ of A.

Definition 2 [4, Definition 1] Let $A \in M_{m,n}$, $M \in M_m$ and $N \in M_n$ be positive definite matrices. if there exists B such that

$$\begin{aligned} &ABA = A, BAB = B, \\ &MAB \text{ and } NBA \ \text{ are } \ M-\text{symmetric}. \end{aligned}$$

then B is the weighted Minskowski inverse of A (denoted by $A^{\bigoplus}_{M,N}$). When $M = I_m$ and $N = I_n$, $A^{\bigoplus}_{M,N}$ reduces to the Minskowski inverse and denoted by $A^{\bigoplus}$.

Lemma 1[4, Lemma 5] Let $A \in M_{m,n}$ be a matrix in μ, and let $M \in M_m$ and $N \in M_n$ be positive definite matrices. Then

$$A^{\bigoplus}_{M,N} = (A^\propto)^{-1} A^\approx \tag{5}$$

where$A^\approx = N^{-1}G_1A^*G_2M$ and $A^\propto = A^\approx A|_{R(A^\approx)}$ is the restriction of $A^\approx A$ on $R(A^\approx)$.

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II. Conclusion

In this section, we will use a family of iterative formula which be defined in [3] for computing the Minkowski inverse $A^{\bigoplus}_{M,N}$ in Minskowski space. And also give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Theorem 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n\times m}$ as follow

$$X_{k+1} = X_k(3I - 3AX_k + (AX_k)^2) \tag{6}$$

and if we take $X_0 = Y \in C^{n\times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{7}$$

then the sequence (6) converges to $A^{\bigoplus}_{M,N}$ if and only if

$$\rho(I - YA) < 1 \ (\text{or} \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{3^k}}{\|A\|} \tag{8}$$

Xiaoji Liu is with College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, P.R. China. xiaojiliu72@yahoo.com.cn. Tel. +86-0771-3264782.

Yonghui Qin is with College of Mathematics and Computer Science, Guangxi University for Nationalities, Nanning 530006, P.R. China. yonghui1676@163.com.

where

$$q = \min\{\rho(I - YA) < 1, \rho(I - AY) < 1\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: Let $e_k = Y(I - AX_k)$, by the iteration (6), we have

$$\begin{aligned} e_{k+1} &= Y(I - AX_{k+1}) \\ &= Y(I - AX_k(3I - 3AX_k + (AX_k)^2)) \\ &= \cdots \\ &= Y(I - AX_0)^{3^k} = Ye_0^{3^k} \end{aligned} \tag{9}$$

i.e.$Y = YAX_\infty$ when $k \to \infty$. In the following, we present

$$\lim_{k\to\infty} X_k = X_\infty.$$

Sufficient: From

$$\rho(I - YA) < 1,$$

we easily have

$$\rho(I - AY) < 1,$$

since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

We also easily prove that YA is invertible on $R(YA)$ and AY is invertible on $R(AY)$.

From above, we can show that

$$X_\infty = Y(AY)\mid_{R(AY)}^{-1} = (YA)\mid_{R(YA)}^{-1} Y. \tag{10}$$

we also get

$$\begin{aligned} &AX_\infty A = AY(AY)|_{R(AY)}^{-1}A = A, \\ &X_\infty AX_\infty = Y(AY)|_{R(AY)}^{-1}, \\ &AY(AY)|_{R(AY)}^{-1} = Y(AY)|_{R(AY)}^{-1}, \\ &MAY(AY)|_{R(AY)}^{-1} = I|_{R(AY)}, \\ &N(YA)|_{R(AY)}^{-1}YA = I_{R(YA)} \end{aligned}$$

i.e. we can prove $X_\infty = A^{\bigoplus}_{M,N}$. It show that (6) converges to $A^{\bigoplus}_{M,N}$.

Necessary: If (6) converges to $A^{\bigoplus}_{M,N}$. By (9), we have

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Finally we will consider the error of two adjacent iterations between X_{k+1} and X_k in the following.

$$\begin{aligned} AA^{\bigoplus}_{M,N} - AX_{k+1} &= AA^{\bigoplus}_{M,N} - AA^{\bigoplus}_{M,N}AX_{k+1} \\ &= AA^{\bigoplus}_{M,N}(I - AX_k)^3 \\ &= (AA^{\bigoplus}_{M,N} - AX_k)^3 \\ &= A^3(A^{\bigoplus}_{M,N} - X_k)^3 \end{aligned} \tag{11}$$

So we have

$$\|A^{\bigoplus}_{M,N} - X_k\| \le \frac{q^{3^k}}{\|A\|} \tag{12}$$

and

$$\|A^{\bigoplus}_{M,N} - X_{k+1}\| \le \|A\|^2 \|A^{\bigoplus}_{M,N} - X_k\|^3 \tag{13}$$

From (11)-(13) we get

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|X_{k+1} - A^{\bigoplus}_{M,N} + A^{\bigoplus}_{M,N} - X_k\| \\ &\le \|A^{\bigoplus}_{M,N} - X_{k+1}\| + \|A^{\bigoplus}_{M,N} - X_k\| \\ &\le (1 + \|A\|^2)\|A^{\bigoplus}_{M,N} - X_k\| \\ &\le (1 + \|A\|^2)\frac{q^{3^k}}{\|A\|} \end{aligned} \tag{14}$$

By (14), it prove that (16) holds.

Corollary 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n\times m}$ as (6) and if we take $X_0 = Y \in C^{n\times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{15}$$

then the sequence (6) converges to $A^{\bigoplus}$ if and only if

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \le (1 + \|A\|^2)\frac{q^{3^k}}{\|A\|} \tag{16}$$

where $q = \min\{\rho(I - YA) < 1, \rho(I - AY) < 1\}$.

Theorem 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n\times m}$ as follow,

$$\begin{aligned} X_{k+1} = X_k[mI - \frac{m(m-1)}{2}AX_k + \cdots + (-AX_m)^{m-1}], \\ k = 0, 1, \cdots, m = 2, 3, \cdots. \end{aligned} \tag{17}$$

and if we take $X_0 = G_1A^*G_2 = Y \in C^{n\times m}$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{18}$$

then (17) converge to $A^{\bigoplus}_{M,N}$ if and only

$$\rho(I - YA) < 1(or\rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \le (1 + \|A\|^2)\frac{q^{m^k}}{\|A\|} \tag{19}$$

where

$$q = \min\{\rho(I - YA) < 1, \rho(I - AY) < 1\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

In the following , we will consider another the iterative formula for computing the weighted Minskowski inverse $A^{\bigoplus}_{M,N}$ in Minskowski space.

Theorem 3 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n\times m}$ as

$$\begin{aligned} X_k = X_{k-1} + \beta Y(I_y - AX_{k-1}), \\ \beta \in C\backslash\{0\}, k = 1, 2, \cdots. \end{aligned} \tag{20}$$

and if we take $X_0 = Y \in C^{n\times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{21}$$

then (20) converges to $A^{\bigoplus}_{M,N}$ if and only if

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore,

$$\|X_{k+1} - X_k\| \leq q^k|\beta|\ \|Y\|\|(I_y - AX_0)\| \tag{22}$$

where

$$q = \min\{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: By iteration (27), we have

$$X_{k+1} = (I_n - \beta YA)X_k + \beta Y \tag{23}$$

Hence

$$\begin{aligned} X_{k+1} - X_k &= (I_n - \beta YA)(X_k - X_{k-1}) \\ &= \cdots \\ &= (I_n - \beta YA)^k(X - X_0) \\ &= \beta(I_n - \beta YA)^k Y(I_m - AX_0) \\ &= \beta Y(I_n - \beta AY)^k(I_m - AX_0) \quad (24) \end{aligned}$$

From (24), we obtain

$$\begin{aligned} YA(X_k - X_0) &= \beta YA[(I_n - \beta YA)^{k-1} \\ &\quad + \cdots + (I_n - \beta YA) + I_n]Y(I_y - AX_0) \\ &= [I_n - (I_n - \beta AY)^k]Y(I_m - AX_0) \quad (25) \end{aligned}$$

Similarly, we get

$$YA(X_k - X_0) = Y[I_n - (I_n - \beta AY)^k](I_m - AX_0) \tag{26}$$

By (25)(26), we prove that (20) converges to $A^{\bigoplus}_{M,N}$ if and only if $\rho(I - \beta YA) < 1$ (or $\rho(I - \beta AY) < 1$), respectively. From

$$\rho(I - YA) < 1,$$

we have

$$\rho(I - AY) < 1,$$

Since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

As the proof in Theorem 1, we obtain

$$X_\infty = (YA)|^{-1}_{R(YA)}Y = Y(AY)|^{-1}_{R(AY)}.$$

Let $\lim_{k\to\infty} X_k = X_\infty$ and by (25)(26), we show that

$$\begin{aligned} \lim_{k\to\infty} X_k &= (YA)|^{-1}_{R(YA)}Y(I_y - AX_0) + X_0 \\ &= (YA)|^{-1}_{R(YA)}Y \end{aligned}$$

Using the definition of the weighted Minskowski inverse, we obtain

$$X_\infty = (YA)|^{-1}_{R(YA)}Y = Y(AY)|^{-1}_{R(AY)} = A^{\bigoplus}_{M,N}.$$

From (24), we can get

$$\begin{aligned} \|X_{k+1} - X_k\| &= |\beta|\|Y(I_x - \beta YA)^k(I_y - AX_0)\| \\ &\leq q^k|\beta|\ \|Y\|\|(I_y - AX_0)\|. \end{aligned}$$

Corollary 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n\times m}$ as(20) and if we take $X_0 = Y \in C^{n\times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{27}$$

then (20) converges to $A^{\bigoplus}$ if and only if

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore,

$$\|X_{k+1} - X_k\| \leq q^k|\beta|\ \|Y\|\|(I_y - AX_0)\| \tag{28}$$

where $q = \min\{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}$.

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