

Iterative methods for computing the weighted Minkowski inverses of matrices in Minkowski space

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Abstract—In this note, we consider a family of iterative formula for computing the weighted Minkowski inverses $A_{M,N}^{\oplus}$ in Minkowski space, and give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Keywords—iterative method, the Minkowski inverse, $A_{M,N}^{\oplus}$ inverse.

I. INTRODUCTION

In this paper, let $M_{m,n}$ denotes the set of all $m - by - n$ complex matrices in Minkowski space. When $m = n$, M_n is instead of $M_{m,n}$. Let A^* , $\|A\|$, $R(A)$, $N(A)$, A^\dagger and $\sigma(A)$ stand for conjugate transpose, spectrum norm, range, null space, Moore-Penrose inverse and spectrum of matrix A .

In the following , we give some notations and lemmas for the Minkowski inverse in Minkowski space.

Let G be the Minkowski metric tensor defined by

$$Gu = (u_0, -u_1, -u_2, \dots, -u_n). \quad (1)$$

where $u \in C^n$ is an element of the space of complex n -tuples.

For $G \in M_n$, it defined by

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}; \quad G^* = G; \quad \text{and} \quad G^2 = I_n. \quad (2)$$

For $A \in M_{m,n}$ x and $y \in C^n$ in Minkowski space μ , using (1) we define the Minkowski conjugate matrix A^\sim of A as follow

$$\begin{aligned} (Ax, y) &= [Ax, Gy] = [x, A^*Gy] \\ &= [x, G(AA^*G)y] \\ &= [x, GA^\sim y] = (x, A^\sim y) \end{aligned} \quad (3)$$

where $A^\sim = GA^*G$ (see [4]).

Definition 1[4, Definition 2] For $A \in M_{m,n}$ in Minkowski space μ , the Minkowski conjugate matrix A^\sim of A is defined as

$$A^\sim = G_1A^*G_2 \quad (4)$$

where G_1, G_2 are Minkowski metric matrices of $n \times n$ and $m \times m$, respectively. Obviously, (see [4])if $A, B \in M_{m,n}$ and

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$C \in M_{n,l}$, then

$$\begin{aligned} (A + B)^\sim &= A^\sim + B^\sim, \\ (AC)^\sim &= C^\sim A^\sim, \\ (A^\sim)^\sim &= A^\sim, \\ (A^\sim)^* &= (A^*)^\sim. \end{aligned}$$

Analogous to Moore-Penrose inverse of A , we give the following definition of the Minkowski space $A_{M,N}^{\oplus}$ of A .

Definition 2 [4, Definition 1] Let $A \in M_{m,n}$, $M \in M_m$ and $N \in M_n$ be positive definite matrices. if there exists B such that

$$\begin{aligned} ABA &= A, BAB = B, \\ MAB \text{ and } NBA &\text{ are } M - \text{symmetric.} \end{aligned}$$

then B is the weighted Minkowski inverse of A (denoted by $A_{M,N}^{\oplus}$). When $M = I_m$ and $N = I_n$, $A_{M,N}^{\oplus}$ reduces to the Minkowski inverse and denoted by A^{\oplus} .

Lemma 1[4, Lemma 5] Let $A \in M_{m,n}$ be a matrix in μ , and let $M \in M_m$ and $N \in M_n$ be positive definite matrices. Then

$$A_{M,N}^{\oplus} = (A^\infty)^{-1}A^\sim \quad (5)$$

where $A^\sim = N^{-1}G_1A^*G_2M$ and $A^\infty = A^\sim A|_{R(A^\sim)}$ is the restriction of $A^\sim A$ on $R(A^\sim)$.

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II. CONCLUSION

In this section, we will use a family of iterative formula which be defined in [3] for computing the Minkowski inverse $A_{M,N}^{\oplus}$ in Minkowski space. And also give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Theorem 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n \times m}$ as follow

$$X_{k+1} = X_k(3I - 3AX_k + (AX_k)^2) \quad (6)$$

and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \quad (7)$$

then the sequence (6) converges to $A_{M,N}^{\oplus}$ if and only if

$$\rho(I - YA) < 1 \text{ (or } \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{3^k}}{\|A\|} \quad (8)$$

where

$$q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: Let $e_k = Y(I - AX_k)$, by the iteration (6), we have

$$\begin{aligned} e_{k+1} &= Y(I - AX_{k+1}) \\ &= Y(I - AX_k(3I - 3AX_k + (AX_k)^2)) \\ &= \dots \\ &= Y(I - AX_0)^{3^k} = Ye_0^{3^k} \end{aligned} \quad (9)$$

i.e. $Y = YAX_\infty$ when $k \rightarrow \infty$. In the following, we present

$$\lim_{k \rightarrow \infty} X_k = X_\infty.$$

Sufficient: From

$$\rho(I - YA) < 1,$$

we easily have

$$\rho(I - AY) < 1,$$

since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

We also easily prove that YA is invertible on $R(YA)$ and AY is invertible on $R(AY)$.

From above, we can show that

$$X_\infty = Y(AY)|_{R(AY)}^{-1} = (YA)|_{R(YA)}^{-1} Y. \quad (10)$$

we also get

$$\begin{aligned} AX_\infty A &= AY(AY)|_{R(AY)}^{-1} A = A, \\ X_\infty AX_\infty &= Y(AY)|_{R(AY)}^{-1}, \\ AY(AY)|_{R(AY)}^{-1} &= Y(AY)|_{R(AY)}^{-1}, \\ MAY(AY)|_{R(AY)}^{-1} &= I|_{R(AY)}, \\ N(YA)|_{R(AY)}^{-1} YA &= I_{R(YA)} \end{aligned}$$

i.e. we can prove $X_\infty = A_{M,N}^\oplus$. It show that (6) converges to $A_{M,N}^\oplus$.

Necessary: If (6) converges to $A_{M,N}^\oplus$. By (9), we have

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Finally we will consider the error of two adjacent iterations between X_{k+1} and X_k in the following.

$$\begin{aligned} AA_{M,N}^\oplus - AX_{k+1} &= AA_{M,N}^\oplus - AA_{M,N}^\oplus AX_{k+1} \\ &= AA_{M,N}^\oplus (I - AX_k)^3 \\ &= (AA_{M,N}^\oplus - AX_k)^3 \\ &= A^3 (A_{M,N}^\oplus - X_k)^3 \end{aligned} \quad (11)$$

So we have

$$\|A_{M,N}^\oplus - X_k\| \leq \frac{q^{3^k}}{\|A\|} \quad (12)$$

and

$$\|A_{M,N}^\oplus - X_{k+1}\| \leq \|A\|^2 \|A_{M,N}^\oplus - X_k\|^3 \quad (13)$$

From (11)-(13) we get

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|X_{k+1} - A_{M,N}^\oplus + A_{M,N}^\oplus - X_k\| \\ &\leq \|A_{M,N}^\oplus - X_{k+1}\| + \|A_{M,N}^\oplus - X_k\| \\ &\leq (1 + \|A\|^2) \|A_{M,N}^\oplus - X_k\| \\ &\leq (1 + \|A\|^2) \frac{q^{3^k}}{\|A\|} \end{aligned} \quad (14)$$

By (14), it prove that (16) holds.

Corollary 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n \times m}$ as (6) and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \quad (15)$$

then the sequence (6) converges to $A_{M,N}^\oplus$ if and only if

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{3^k}}{\|A\|} \quad (16)$$

where $q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \}$.

Theorem 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^\infty \in C^{n \times m}$ as follow,

$$\begin{aligned} X_{k+1} &= X_k [mI - \frac{m(m-1)}{2} AX_k + \dots + (-AX_m)^{m-1}], \\ &k = 0, 1, \dots, m = 2, 3, \dots \end{aligned} \quad (17)$$

and if we take $X_0 = G_1 A^* G_2 = Y \in C^{n \times m}$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \quad (18)$$

then (17) converge to $A_{M,N}^\oplus$ if and only

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore, we have

$$\|X_{k+1} - X_k\| \leq (1 + \|A\|^2) \frac{q^{m^k}}{\|A\|} \quad (19)$$

where

$$q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

In the following, we will consider another the iterative formula for computing the weighted Minkowski inverse $A_{M,N}^\oplus$ in Minkowski space.

Theorem 3 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n \times m}$ as

$$\begin{aligned} X_k &= X_{k-1} + \beta Y(I - AX_{k-1}), \\ &\beta \in C \setminus \{0\}, k = 1, 2, \dots \end{aligned} \quad (20)$$

and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \quad (21)$$

then (20) converges to $A_{M,N}^{\oplus}$ if and only if

$$\rho(I - YA) < 1 \text{ (or } \rho(I - AY) < 1 \text{)}.$$

Furthermore,

$$\|X_{k+1} - X_k\| \leq q^k |\beta| \|Y\| \|(I_y - AX_0)\| \quad (22)$$

where

$$q = \min \{ \rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1 \}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: By iteration (27), we have

$$X_{k+1} = (I_n - \beta YA)X_k + \beta Y \quad (23)$$

Hence

$$\begin{aligned} X_{k+1} - X_k &= (I_n - \beta YA)(X_k - X_{k-1}) \\ &= \dots \\ &= (I_n - \beta YA)^k (X - X_0) \\ &= \beta (I_n - \beta YA)^k Y (I_m - AX_0) \\ &= \beta Y (I_n - \beta AY)^k (I_m - AX_0) \end{aligned} \quad (24)$$

From (24), we obtain

$$\begin{aligned} YA(X_k - X_0) &= \beta YA[(I_n - \beta YA)^{k-1} \\ &\quad + \dots + (I_n - \beta YA) + I_n] Y (I_y - AX_0) \\ &= [I_n - (I_n - \beta AY)^k] Y (I_m - AX_0) \end{aligned} \quad (25)$$

Similarly, we get

$$YA(X_k - X_0) = Y[I_n - (I_n - \beta AY)^k](I_m - AX_0) \quad (26)$$

By (25)(26), we prove that (20) converges to $A_{M,N}^{\oplus}$ if and only if $\rho(I - \beta YA) < 1$ (or $\rho(I - \beta AY) < 1$), respectively. From

$$\rho(I - YA) < 1,$$

we have

$$\rho(I - AY) < 1,$$

Since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

As the proof in Theorem 1, we obtain

$$X_\infty = (YA)|_{R(YA)}^{-1} Y = Y(AY)|_{R(AY)}^{-1}.$$

Let $\lim_{k \rightarrow \infty} X_k = X_\infty$ and by (25)(26), we show that

$$\begin{aligned} \lim_{k \rightarrow \infty} X_k &= (YA)|_{R(YA)}^{-1} Y (I_y - AX_0) + X_0 \\ &= (YA)|_{R(YA)}^{-1} Y \end{aligned}$$

Using the definition of the weighted Minkowski inverse, we obtain

$$X_\infty = (YA)|_{R(YA)}^{-1} Y = Y(AY)|_{R(AY)}^{-1} = A_{M,N}^{\oplus}.$$

From (24), we can get

$$\begin{aligned} \|X_{k+1} - X_k\| &= |\beta| \|Y(I_x - \beta YA)^k (I_y - AX_0)\| \\ &\leq q^k |\beta| \|Y\| \|(I_y - AX_0)\|. \end{aligned}$$

Corollary 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n \times m}$ as(20) and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \quad (27)$$

then (20) converges to A^{\oplus} if and only if

$$\rho(I - YA) < 1 \text{ (or } \rho(I - AY) < 1 \text{)}.$$

Furthermore,

$$\|X_{k+1} - X_k\| \leq q^k |\beta| \|Y\| \|(I_y - AX_0)\| \quad (28)$$

where $q = \min \{ \rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1 \}$.

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