Iterative methods for computing the weighted Minkowski inverses of matrices in Minkowski space

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Abstract—In this note, we consider a family of iterative formula $C \in M_{n,l}$, then for computing the weighted Minskowski inverses $A_{M,N}^{\bigoplus}$ in Minskowski space, and give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Keywords—iterative method, the Minskowski inverse, $A_{M,N}^{\bigoplus}$ inverse.

I. INTRODUCTION

In this paper, let $M_{m,n}$ denotes the set of all m - by - ncomplex matrices in Minskowski space. When m = n, M_n is instead of $M_{m,n}$. Let $A^*, ||A||, R(A), N(A), A^{\dagger}$ and $\sigma(A)$ stand for conjugate transpose, spectrum norm, range, null space, Moore-Penrose inverse and spectrum of matrix A.

In the following, we give some notations and lemmas for the Minskowski inverse in Minskowski space.

Let G be the Minskowski metric tensor defined by

$$Gu = (u_0, -u_1, -u_2, \cdots, -u_n).$$
(1)

where $u \in C^n$ is an element of the space of complex n-tuples.

For $G \in M_n$, it defined by

$$G = \begin{pmatrix} 1 & 0 \\ 0 & -I_{n-1} \end{pmatrix}; \quad G^* = G; \text{ and } G^2 = I_n.$$
 (2)

For $A \in M_{m,n} x$ and $y \in C^n$ in Minskowski space μ , using (1) we define the Minskowski conjugate matrix A^{\sim} of A as follow

$$(Ax, y) = [Ax, Gy] = [x, A^*Gy] = [x, G(GA^*G)y] = [x, GA^{\sim}y] = (x, A^{\sim}y)$$
(3)

where $A^{\sim} = GA^*G$ (see [4]).

Definition 1[4, Definition 2] For $A \in M_{m,n}$ in Minskowski space μ , the Minskowski conjugate matrix A^{\approx} of A is defined as

$$A^{\approx} = G_1 A^* G_2 \tag{4}$$

where G_1, G_2 are Minskowski metric matrices of $n \times n$ and $m \times m$, respectively. Obviously, (see [4])if $A, B \in M_{m,n}$ and

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$$\begin{split} (A+B)^{\approx} &= A^{\approx} + B^{\approx} \\ (AC)^{\approx} &= C^{\approx} A^{\approx}, \\ (A^{\approx})^{\approx} &= A^{\approx}, \\ (A^{\approx})^{*} &= (A^{*})^{\approx}. \end{split}$$

Analogous to Moore-Penrose inverse of A, we give the following definition of the Minskowski space $A_{M,N}^{\bigoplus}$ of A.

Definition 2 [4, Definition 1] Let $A \in M_{m,n}$, $M \in M_m$ and $N \in M_n$ be positive definite matrices. if there exists B such that

$$ABA = A, BAB = B,$$

MAB and NBA are M – symmetric.

then B is the weighted Minskowski inverse of A (denoted by $A_{M,N}^{\bigoplus}$). When $M = I_m$ and $N = I_n$, $A_{M,N}^{\bigoplus}$ reduces to the Minskowski inverse and denoted by $A \oplus$.

Lemma 1[4, Lemma 5] Let $A \in M_{m,n}$ be a matrix in μ , and let $M \in M_m$ and $N \in M_n$ be positive definite matrices. Then

$$A_{M,N}^{\bigoplus} = (A^{\infty})^{-1} A^{\approx}$$
(5)

where $A^{\approx} = N^{-1}G_1A^*G_2M$ and $A^{\propto} = A^{\approx}A|_{R(A^{\approx})}$ is the restriction of $A^{\approx}A$ on $R(A^{\approx})$.

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II. CONCLUSION

In this section, we will use a family of iterative formula which be defined in [3] for computing the Minkowski inverse A_{MN}^{\bigoplus} in Minskowski space. And also give two kinds of iterations and the necessary and sufficient conditions of the convergence of iterations.

Theorem 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^{\infty} \in$ $C^{n \times m}$ as follow

$$X_{k+1} = X_k (3I - 3AX_k + (AX_k)^2)$$
(6)

and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$||e_0|| = ||I - AX_0|| < 1 \tag{7}$$

then the sequence (6) converges to $A_{M,N}^{\bigoplus}$ if and only if

$$\rho(I - YA) < 1 \text{ (or}\rho(I - AY) < 1).$$

Furthermore, we have

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$$||X_{k+1} - X_k|| \le (1 + ||A||^2) \frac{q^{3^k}}{||A||}$$
(8)

where

$$q = \min\left\{\rho(I - YA) < 1, \rho(I - AY) < 1\right\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: Let $e_k = Y(I - AX_k)$, by the iteration (6), we have

$$e_{k+1} = Y(I - AX_{k+1})$$

= $Y(I - AX_k(3I - 3AX_k + (AX_k)^2))$
= ...
= $Y(I - AX_0)^{3^k} = Ye_0^{3^k}$ (9)

i.e. $Y=YAX_\infty$ when $k\to\infty$. In the following, we present

$$\lim_{k \to \infty} X_k = X_{\infty}.$$

Sufficient: From

$$\rho(I - YA) < 1,$$

we easily have

$$\rho(I - AY) < 1,$$

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

We also easily prove that YA is invertible on R(YA) and AY is invertible on R(AY).

From above, we can show that

$$X_{\infty} = Y(AY) \mid_{R(AY)}^{-1} = (YA) \mid_{R(YA)}^{-1} Y.$$
 (10)

we also get

$$\begin{aligned} AX_{\infty}A &= AY(AY)|_{R(AY)}^{-1}A = A, \\ X_{\infty}AX_{\infty} &= Y(AY)|_{R(AY)}^{-1}, \\ AY(AY)|_{R(AY)}^{-1} &= Y(AY)|_{R(AY)}^{-1}, \\ MAY(AY)|_{R(AY)}^{-1} &= I|_{R(AY)}, \\ N(YA)|_{R(AY)}^{-1}YA &= I_{R(YA)} \end{aligned}$$

i.e. we can prove $X_{\infty} = A_{M,N}^{\bigoplus}$. It show that (6) converges to $A_{M,N}^{\bigoplus}$.

Necessary: If (6) converges to $A_{M,N}^{\bigoplus}$. By (9), we have

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Finally we will consider the error of two adjacent iterations between X_{k+1} and X_k in the following.

$$AA_{M,N}^{\bigoplus} - AX_{k+1} = AA_{M,N}^{\bigoplus} - AA_{M,N}^{\bigoplus}AX_{k+1}$$
$$= AA_{M,N}^{\bigoplus}(I - AX_k)^3$$
$$= (AA_{M,N}^{\bigoplus} - AX_k)^3$$
$$= A^3(A_{M,N}^{\bigoplus} - X_k)^3$$
(11)

So we have

$$\|A_{M,N}^{\bigoplus} - X_k\| \le \frac{q^{3^k}}{\|A\|}$$
(12)

and

$$\|A_{M,N}^{\bigoplus} - X_{k+1}\| \le \|A\|^2 \|A_{M,N}^{\bigoplus} - X_k\|^3$$
(13)

From (11)-(13) we get

$$\begin{aligned} \|X_{k+1} - X_k\| &= \|X_{k+1} - A_{M,N}^{\bigoplus} + A_{M,N}^{\bigoplus} - X_k\| \\ &\leq \|A_{M,N}^{\bigoplus} - X_{k+1}\| + \|A_{M,N}^{\bigoplus} - X_k\| \\ &\leq (1 + \|A\|^2) \|A_{M,N}^{\bigoplus} - X_k\| \\ &\leq (1 + \|A\|^2) \frac{q^{3^k}}{\|A\|} \end{aligned}$$
(14)

By (14), it prove that (16) holds.

Corollary 1 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^{\infty} \in C^{n \times m}$ as (6) and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$||e_0|| = ||I - AX_0|| < 1$$
(15)

then the sequence (6) converges to A^{\bigoplus} if and only if

$$\rho(I - YA) < 1(\text{or } \rho(I - AY) < 1).$$

Furthermore, we have

$$|X_{k+1} - X_k\| \le (1 + ||A||^2) \frac{q^{3^{\kappa}}}{||A||}$$
(16)

where $q = \min \{ \rho(I - YA) < 1, \rho(I - AY) < 1 \}.$

Theorem 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\}_{k=0}^{\infty} \in C^{n \times m}$ as follow,

$$X_{k+1} = X_k[mI - \frac{m(m-1)}{2}AX_k + \dots + (-AX_m)^{m-1}],$$

$$k = 0, 1, \dots, m = 2, 3, \dots.$$
(17)

and if we take $X_0 = G_1 A^* G_2 = Y \in C^{n \times m}$ such that

$$\|e_0\| = \|I - AX_0\| < 1 \tag{18}$$

then (17) converge to $A_{M,N}^{\bigoplus}$ if and only

$$\rho(I - YA) < 1(or\rho(I - AY) < 1).$$

Furthermore, we have

$$|X_{k+1} - X_k|| \le (1 + ||A||^2) \frac{q^{m^{\kappa}}}{||A||}$$
(19)

where

$$q = \min\left\{\rho(I - YA) < 1, \rho(I - AY) < 1\right\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

In the following , we will consider another the iterative formula for computing the weighted Minskowski inverse $A_{M,N}^{\bigoplus}$ in Minskowski space.

Theorem 3 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n \times m}$ as

$$X_k = X_{k-1} + \beta Y (I_y - A X_{k-1}),$$

$$\beta \in C \setminus \{0\}, k = 1, 2, \cdots.$$
(20)

and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq YAX_0$ such that

$$||e_0|| = ||I - AX_0|| < 1$$
(21)

then (20) converges to $A_{M,N}^{\bigoplus}$ if and only if

$$\rho(I - YA) < 1$$
 (or $\rho(I - AY) < 1$).

Furthermore,

$$||X_{k+1} - X_k|| \le q^k |\beta| ||Y|| ||(I_y - AX_0)||$$
(22)

where

$$q = \min \left\{ \rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1 \right\}$$

and $M \in M_n, N \in M_n$ be positive definite matrices, respectively.

Proof: By iteration (27), we have

$$X_{k+1} = (I_n - \beta Y A)X_k + \beta Y \tag{23}$$

Hence

$$X_{k+1} - X_k = (I_n - \beta Y A)(X_k - X_{k-1})$$

= ...
= $(I_n - \beta Y A)^k (X - X_0)$
= $\beta (I_n - \beta Y A)^k Y (I_m - A X_0)$
= $\beta Y (I_n - \beta A Y)^k (I_m - A X_0)$ (24)

From (24), we obtain

$$YA(X_{k} - X_{0}) = \beta YA[(I_{n} - \beta YA)^{k-1} + \dots + (I_{n} - \beta YA) + I_{n}]Y(I_{y} - AX_{0})$$
$$= [I_{n} - (I_{n} - \beta AY)^{k}]Y(I_{m} - AX_{0}) \quad (25)$$

Similarly, we get

$$YA(X_k - X_0) = Y[I_n - (I_n - \beta AY)^k](I_m - AX_0)$$
 (26)

By (25)(26), we prove that (20) converges to $A_{M,N}^{\bigoplus}$ if and only if $\rho(I - \beta YA) < 1$ (or $\rho(I - \beta AY) < 1$), respectively. From

$$\rho(I - YA) < 1,$$

we have

$$\rho(I - AY) < 1,$$

Since

$$\sigma(YA) \cup \{0\} = \sigma(AY) \cup \{0\}.$$

As the proof in Theorem 1, we obtain

$$X_{\infty} = (YA)|_{R(YA)}^{-1}Y = Y(AY)|_{R(AY)}^{-1}.$$

Let $\lim_{k\to\infty} X_k = X_\infty$ and by (25)(26), we show that

$$\lim_{k \to \infty} X_k = (YA)|_{R(YA)}^{-1} Y(I_y - AX_0) + X_0$$

= $(YA)|_{R(YA)}^{-1} Y$

Using the definition of the weighted Minskowski inverse, we obtain

$$X_{\infty} = (YA)|_{R(YA)}^{-1}Y = Y(AY)|_{R(AY)}^{-1} = A_{M,N}^{\bigoplus}.$$

From (24), we can get

$$||X_{k+1} - X_k|| = |\beta| ||Y(I_x - \beta Y A)^k (I_y - A X_0)||$$

$$\leq q^k |\beta| ||Y|| ||(I_y - A X_0)||.$$

Corollary 2 Let $A \in M_{m,n}$, define the sequence $\{X_k\} \in C^{n \times m}$ as(20) and if we take $X_0 = Y \in C^{n \times m}$ and $Y \neq C^{n \times m}$ YAX_0 such that

$$||e_0|| = ||I - AX_0|| < 1$$
(27)

then (20) converges to $A \oplus$ if and only if

$$\rho(I - YA) < 1$$
 (or $\rho(I - AY) < 1$).

Furthermore,

$$||X_{k+1} - X_k|| \le q^k |\beta| ||Y|| ||(I_y - AX_0)||$$
(28)

where $q = \min \{\rho(I - \beta YA) < 1, \rho(I - \beta AY) < 1\}$.

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